Abstract: The degree distance was introduced by Dobrynin, Kochetova and Gutman as a weighted version of the Wiener index. In this paper, we investigate the degree distance and Gutman index of complete, and strong product graphs by using the adjacency and distance matrices of a graph.

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1. Introduction

All graphs in this paper are assumed to be undirected, finite and simple. We refer to [2] for graph theoretical notation and terminology not specified here. For a graph $G$, let $V(G)$, $E(G)$ and $\overline{G}$ denote the set of vertices, the set of edges and the complement of $G$, respectively. If $G$ is a connected graph and $u, v \in V(G)$, then the distance $d(u, v)$ between $u$ and $v$ is the length of a shortest path connecting $u$ and $v$. If $v$ is a vertex of a connected graph $G$, then the eccentricity $e(v)$ of $v$ is defined by $e(v) = \max\{d(u, v) \mid u \in V(G)\}$. Furthermore, the diameter $\text{diam}(G)$ of $G$ is defined by $\text{diam}(G) = \max\{e(v) \mid v \in V(G)\}$.

Let $G$ be a finite, simple, connected, undirected graph with $p$ vertices and $q$ edges. In what follows, we say that $G$ is an $(p, q)$-graph. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $E(G) = \{e_1, e_2, \ldots, e_q\}$ be the vertex set and edge set of $G$, respectively. The adjacency matrix of $G$ is the $p \times p$ matrix $A = A(G)$ whose $(i, j)$ entry, denoted by $a_{ij}$, is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise}. \end{cases}$$

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The distance matrix of $G$ is the $p \times p$ matrix $D_G$ whose $(i, j)$ entry, denoted by $d_{ij}$, is defined by

$$d_{ij} = \begin{cases} 
  d_G(v_i, v_j) & \text{if } v_i \neq v_j \\
  0 & \text{otherwise},
\end{cases}$$

where $d_G(v_i, v_j)$ is the length of a shortest directed path in $G$ from $v_i$ to $v_j$.

The vertex $u$ is said to be a neighbor of $v$ if they are adjacent. The neighborhood of a vertex $v$, denoted by $N_G(v)$, is the set of all neighbors of $v$. The degree of a vertex $v$ in a graph $G$, denoted by $d_v = d_G(v)$, is the number of vertices in its neighborhood, that is, $d_G(v) = |N(v)|$. The common neighborhood graph $con(G)$ (in short congraph) of a graph $G$ is defined as the graph with $V(con(G)) = V(G)$ and two vertices in $con(G)$ are adjacent if they have a common neighbor in $G$. For every $x, y \in V(G)$,

$$xy \in E(con(G)) \text{ if and only if } N_G(x) \cap N_G(y) \neq \emptyset.$$ Some basic properties of congraphs have been established; see [1, 3].

The oldest and most studied degree-based structure descriptors are the first and second Zagreb indices [15], defined as

$$M_1(G) = \sum_{v \in V(G)} (d_G(v))^2 \quad \text{and} \quad M_2(G) = \sum_{u \in E(G)} (d_G(u))(d_G(v)).$$

It has been shown that the first Zagreb index obeys the identity [10]

$$M_1(G) = \sum_{u \in E(G)} (d_G(u) + d_G(v)).$$

The first investigation of the sum of distance between all pairs of vertices of a (connected) graph was done by Harold Wiener in 1947, who realized that there exists a correlation between the boiling points of paraffins and this sum [20]. Eventually, the distance–based graph invariant,

$$W(G) = \sum_{\{u, v\} \subseteq V(G)} d(u, v).$$

For more details, we refer to [8, 11, 13, 19].

The degree distance was introduced by Dobrynin and Kochetova [9] and Gutman [14] as a weighted version of the Wiener index. The degree distance $DD(G)$ of a graph $G$ is defined as

$$DD(G) = \sum_{\{u, v\} \subseteq V(G)} d_G(u)d_G(v) + d_G(v)[d_G(u) + d_G(v)] = \frac{1}{2} \sum_{u, v \in V(G)} d_G(u)[d_G(u) + d_G(v)]$$

with the summation runs over all pairs of vertices of $G$. The degree distance is also known as the Schultz index in chemical literature; see [21]. In [14], Gutman showed that if $G$ is a tree on $n$ vertices, then $DD(G) = 4W(G) - n(n - 1)$; see [5, 6] and [9]. In [7], Gutman index $Gut(G)$ of a graph $G$ is defined as

$$Gut(G) = \sum_{\{u, v\} \subseteq V(G)} d_G(u)d_G(v)d(u, v).$$

For more details on Gutman index, we refer to [4, 7, 12].

The relations between the degree distance, Gutman index and Wiener index are shown in the following Table 1.

The join and strong products are defined as follows.

The join or complete product $G \vee H$ of two disjoint graphs $G$ and $H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H) \}$. 


Table 1. Three distance parameters

<table>
<thead>
<tr>
<th>Wiener index</th>
<th>$W(G) = \sum_{(u,v) \subseteq V(G)} d_G(u,v)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degree distance</td>
<td>$DD(G) = \sum_{(u,v) \subseteq V(G)} [d_G(u) + d_G(v)]$</td>
</tr>
<tr>
<td>Gutman index</td>
<td>$Gut(G) = \sum_{(u,v) \subseteq V(G)} d_G(u) d_G(v)$</td>
</tr>
</tbody>
</table>

The strong product $G \boxtimes H$ of graphs $G$ and $H$ has the vertex set $V(G) \times V(H)$. Two vertices $(u, v)$ and $(u', v')$ are adjacent whenever $uu' \in E(G)$ and $v = v'$, or $u = u'$ and $vv' \in E(H)$, or $uu' \in E(G)$ and $vv' \in E(H)$.

Paulraja and Agnes [16] studied the degree distance of Cartesian and lexicographic products. Later, they [17] investigated the Gutman index of Cartesian and lexicographic products. In this paper, we investigate the degree distance and Gutman index of strong and complete product graphs.

2. Preliminary

We define,

$$N_1(G) = \sum_{v \in V(G)} d_G(v) d_{\text{con}(G)}(v) \quad \text{and} \quad N_2(G) = \sum_{uv \in E(\text{con}(G))} d_G(u) d_G(v).$$

**Definition 2.1.** Let $A = [a_{ij}]_{m \times n}$. Then, we define

$$S(A) = \sum_{1 \leq i \leq m, 1 \leq j \leq n} a_{ij}.$$

The following lemma is immediate.

**Lemma 2.2.** Let $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$. Then

1. $S(A^T) = S(A)$ and $S(\alpha A) = \alpha S(A)$ for every $\alpha \in \mathbb{R}$;
2. $S(A + B) = S(A) + S(B)$.

**Lemma 2.3.** Let $G$ be a $(p,q)$-graph, and let $\text{con}(G)$ be a $(p,q')$-graph. Let $A, B, K$ be the adjacency matrices of $G, \text{con}(G), K_p$, respectively. Then

1. $S(A) = 2q$;
2. $S(A^2) = M_1(G)$;
3. $S(AB) = N_1(G)$;
4. $S(A^3) = 2M_2(G)$;
5. $S(AK) = 2q(p - 1)$;
6. $S(ABA) = 2N_2(G)$.

**Proof.** For (1), we have

$$S(A) = \sum_{1 \leq i,j \leq p} a_{ij} = \sum_{i=1}^{p} \sum_{j=1}^{p} a_{ij} = \sum_{i=1}^{p} d_{vi} = 2q.$$
For (2), we have

\[
S(A^2) = \sum_{1 \leq i, j \leq p} a_{ij}^{(2)} = \sum_{1 \leq i, j \leq p} \sum_{k=1}^{p} a_{ik} a_{kj}
\]

\[
= \sum_{k=1}^{p} \sum_{i=1}^{p} a_{ik} \sum_{j=1}^{p} a_{kj} = \sum_{k=1}^{p} d_{vk} d_{vh} = M_1(G).
\]

For (3), we have

\[
S(AB) = \sum_{1 \leq i, j \leq p} \sum_{k=1}^{p} a_{ik} b_{kj}
\]

\[
= \sum_{k=1}^{p} \sum_{i=1}^{p} a_{ik} \sum_{j=1}^{p} b_{kj} = \sum_{k=1}^{p} d_{vk} d_{conG} v_k = N_1(G).
\]

For (4), we have

\[
M_2(G) = \sum_{v_i, v_j \in E(G)} d_{v_i} d_{v_j} = \frac{1}{2} \sum_{1 \leq i, j \leq p} d_{v_i} d_{v_j} a_{ij}
\]

\[
= \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \left( \sum_{k=1}^{p} a_{ki} \right) \left( \sum_{s=1}^{p} a_{js} \right) a_{ij}
\]

\[
= \frac{1}{2} \sum_{k=1}^{p} \sum_{i=1}^{p} \sum_{j=1}^{p} a_{ki} a_{ij}.
\]

Since \(\sum_{i=1}^{p} a_{ki} a_{ij}\) is the entry \(t_{kj}\) of matrix \(A^2\), it follows that

\[
M_2(G) = \frac{1}{2} \sum_{k=1}^{p} \sum_{j=1}^{p} \sum_{i=1}^{p} a_{js} a_{ki} a_{ij}
\]

\[
= \frac{1}{2} \sum_{k=1}^{p} \sum_{s=1}^{p} (p - 1) = \frac{1}{2} S(A^3).
\]

For (5), we have

\[
S(AK) = \sum_{1 \leq i, j \leq p} a_{ir} k_{rj}
\]

\[
= \sum_{r=1}^{p} \sum_{i=1}^{p} a_{ir} \sum_{j=1}^{p} k_{rj} = \sum_{r=1}^{p} d_{vr} (p - 1) = 2q(p - 1).
\]

For (6), we have

\[
N_2(G) = \sum_{v_i, v_j \in E(\text{con}G)} d_{v_i} d_{v_j} = \frac{1}{2} \sum_{1 \leq i, j \leq p} d_{v_i} d_{v_j} b_{ij}
\]

\[
= \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \left( \sum_{k=1}^{p} a_{ki} \right) \left( \sum_{s=1}^{p} a_{sj} \right) b_{ij}
\]

\[
= \frac{1}{2} \sum_{k=1}^{p} \sum_{i=1}^{p} \sum_{j=1}^{p} a_{ki} a_{ij}.
\]
Since, \( \sum_{i=1}^{p} a_{ki} b_{ij} \) is the entry \( t_{kj} \) of matrix \( AB \), hence
\[
N_2(G) = \frac{1}{2} \sum_{k=1}^{p} \sum_{s=1}^{p} \sum_{j=1}^{p} a_{kj} \sum_{i=1}^{p} a_{ki} b_{ij} = \frac{1}{2} \sum_{k=1}^{p} \sum_{s=1}^{p} \sum_{j=1}^{p} t_{kj} a_{js} = \frac{1}{2} \sum_{k=1}^{p} \sum_{s=1}^{p} \sum_{j=1}^{p} f_{ks} = \frac{1}{2} S(ABA),
\]
where \( \sum_{i=1}^{p} t_{kj} a_{js} \) is the entry \( f_{ks} \) of matrix \( ABA \).

The following result for classical distance are from the book [18].

**Lemma 2.4.** [18] Let \((u,v)\) and \((u',v')\) be two vertices of \( G_1 \otimes G_2 \). Then
\[
d_{G_1 \otimes G_2}((u,v),(u',v')) = \max\{d_{G_1}(u,u'),d_{G_2}(v,v')\}.
\]

For \( G_2 = K_p \), the following result is immediate.

**Corollary 2.5.** Let \( K_p \) be a complete graph, and let \((u,v)\) and \((u',v')\) be two vertices of \( G \otimes K_p \). Then
\[
d_{G \otimes K_p}((u,v),(u',v')) = \begin{cases} d_{G}(u,u') & \text{if } u \neq u', \\ 1 & \text{if } u = u' \text{ and } v \neq v', \\ 0 & \text{if } u = u' \text{ and } v = v'. \end{cases}
\]

## 3. Main results

In this section, we give our main results and their proofs.

### 3.1. Relation between degree distance and Gutman index

We first define a matrix, which will be used later.

**Definition 3.1.** Let \( G(V,E) \) be a graph with order \( n \) and \( m \) edges. For \( k = 1, 2, \cdots, \alpha \) where \( \alpha \) denotes the diameter of graph \( G \), we define
\[
A_k = [a_{ij}^k]_{n \times n},
\]
where \( a_{ij}^k = \begin{cases} 1 & d(v_i, v_j) = k, \\ 0 & \text{otherwise}. \end{cases} \)

The following results are easily seen.

**Observation 3.1.** Let \( A \) and \( D_G \) be the adjacency matrix and the distance matrix of a graph \( G \), respectively. Then
\begin{enumerate}
\item \( A_1 = A \);
\item \( D_G = A_1 + 2A_2 + \cdots + \alpha A_\alpha \);
\item \( A_1 + A_2 + \cdots + A_\alpha = K \), where \( K \) is the adjacency matrix complete graph \( K_n \);
\item if \( \text{diam}(G) = 2 \) then \( D_G = A + 2A \).
\end{enumerate}
Lemma 3.2. Let $G$ be a graph containing no triangles, and let $A, B, K$ be the adjacency matrix of $G, \text{con}(G), K_n$, respectively. Then

(1) for every $u, v \in V(G)$, $d_G(u, v) = 2$ if and only if $uv \in E(\text{con}(G));$

(2) if $\text{diam}(G) = 3$ then $D_G = 3K - 2A - B$.

**Proof.** (1) Suppose $d_G(u, v) = 2$. Then there exists a vertex $x \in V(G)$ such that $x \notin \{u, v\}$ and $ux, xv \in E(G)$, and hence $N(u) \cap N(v) \neq \emptyset$. Therefore, we have $uv \in E(\text{con}(G))$. Conversely, suppose $uv \in E(\text{con}(G))$. Then $N(u) \cap N(v) \neq \emptyset$, and hence there exists a vertex $x \in N(u) \cap N(v)$. Note that $ux, xv \in E(G)$. Therefore, $d(u, v) \leq 2$. If $d(u, v) = 1$, then we have a triangle, a contradiction. So $d_G(u, v) = 2$, as desired.

(2) From Observation 3.1, we have $D_G = A_1 + 2A_2 + 3A_3$. Note that $A_1 = A$, $A_2 = B$ and $A + B + A_3 = K$. Therefore, $D_G = 3K - 2A - B$. \hfill $\Box$

Lemma 3.3. Let $G(p, q)$ be a graph, and let $A, D_G$ be the adjacency matrix and the distance matrix of a graph $G$, respectively. Then

(1) $S(AD_G) = DD(G)$;

(2) if $\text{diam}(G) = 2$ then $DD(G) = 4(p - 1)q - M_1(G)$;

(3) if $\text{diam}(G) = 3$ and $G$ has no triangles, then

$$DD(G) = 6q(p - 1) - 2M_1(G) - N_1(G).$$

**Proof.** (1) Since

$$S(AD_G) = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{k=1}^{p} a_{ik} d_{kj} = \sum_{1 \leq j, k \leq p} a_{ik} d(v_k, v_j) = \sum_{1 \leq j, k \leq p} d(v_k) d(v_j),$$

and

$$S(D_G A) = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{k=1}^{p} d_{ik} a_{kj} = \sum_{1 \leq i, k \leq p} d(v_i, v_k) \sum_{j=1}^{p} a_{kj} = \sum_{1 \leq i, k \leq p} d(v_i, v_k) d(v_k) = \sum_{1 \leq i, k \leq p} d(v_i, v_j) d(v_j),$$

it follows that

$$2S(AD_G) = S(AD_G) + S((AD_G)^T) = S(AD_G) + S(D_G A) = \sum_{1 \leq i, k \leq p} d(v_k, v_j) [d(v_k) + d(v_j)] = 2DD(G).$$

For (2), we have

$$DD(G) = S(AD_G) = S(A(A + 2\overline{A})) = 2S(A(A + \overline{A})) - S(A^2) = 2S(\overline{A}K) - S(A^2) = 4(p - 1)q - M_1(G).$$
For (3), we have
\[
\begin{align*}
DD(G) &= S(AD_G) \\
&= S(A(3K - 2A - B)) = 3S(AK) - 2S(A^2) - S(AB) \\
&= 6q(p - 1) - 2M_1(G) - N_1(G).
\end{align*}
\]

Lemma 3.4. Let $G$ be a $(p, q)$-graph and $A$ be the adjacency matrix of $G$. Then $Gut(G) = \frac{1}{2}S(AD_G)$.  

Proof.
\[
S(AD_G) = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{k=1}^{p} a_{ik} d_{kj},
\]
\[
= \sum_{k=1}^{p} \sum_{i=1}^{p} \sum_{j=1}^{p} a_{ik} a_{kj} d_{ks},
\]
\[
= \sum_{k=1}^{p} d_G(v_k) \cdot d_G(v_s) \cdot d_G(v_k, v_s) = 2Gut(G).
\]

Corollary 3.5. Let $G(p, q)$ be a graph, then
\[
\frac{\delta}{2} \leq \frac{Gut(G)}{DD(G)} \leq \frac{\Delta}{2}.
\]

Proof. Since, $S(AD_G) = DD(G)$ and $Gut(G) = \frac{1}{2}S(AD_G)$, hence
\[
\begin{align*}
2Gut(G) - DD(G) &= S(AD_G) - S(AD_G) \\
&= S(AD_G(A - I)) \\
&= \sum_{1 \leq i, j \leq p} t_{ik}(a_{kj} - 1_{kj}) \\
&= \sum_{1 \leq i, k \leq p} t_{ik}(d_G(v_k) - 1).
\end{align*}
\]
Therefore,
\[
(\delta - 1) \sum_{1 \leq i, k \leq p} t_{ik} \leq 2Gut(G) - DD(G) \leq (\Delta - 1) \sum_{1 \leq i, k \leq p} t_{ik}.
\]
Hence,
\[
(\delta - 1)S(AD_G) \leq 2Gut(G) - DD(G) \leq (\Delta - 1)S(AD_G).
\]

Thus,
\[
\delta DD(G) \leq 2Gut(G) \leq \Delta DD(G),
\]
that is
\[
\frac{\delta}{2} \leq \frac{Gut(G)}{DD(G)} \leq \frac{\Delta}{2}.
\]
3.2. For degree distance

In this subsection, we study the degree distance of strong product graphs. We first begin with an easy case.

**Theorem 3.6.** Let \( G \) be a connected graph with \( p_1 \) vertices and \( q_1 \) edges, and \( K_p \) be a complete graph with order \( p \). Then

\[
DD(G \Box K_p) = p^3 DD(G) + 2p^2 (p - 1)[W(G) + q_1] + p_1 p(p - 1)^2.
\]

**Proof.** Let \( V(G) = V_1 \) and \( V(K_p) = V_2 \). From the definition of strong product and Corollary 2.5, we have

\[
DD(G \Box K_p) = \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} [d_{G \Box K_p}(a,b) + d_{G \Box K_p}(c,d)]d_{G \Box K_p}[(a,b),(c,d)]
\]

\[
= \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a \neq c} [d_G(a) + d_{K_p}(b) + d_G(a) + d_G(a) + d_G(c) + d_{K_p}(d) + d_G(c) + d_{K_p}(d)]
\]

\[
= \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a \neq c} [p(d_G(a) + d_G(c)) + 2(p - 1)]d_G(a,c)
\]

\[
+ \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a = c} [2pd_G(a) + 2(p - 1)]
\]

\[
= p \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a \neq c} [d_G(a) + d_G(c)]d_G(a,c) + 2(p - 1) \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a \neq c} d_G(a,c)
\]

\[
+ 2p \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a = c} d_G(a) + 2(p - 1) \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a = c} 1
\]

\[
= p^3 DD(G) + 2p^2 (p - 1)W(G) + 2p \cdot \frac{p(p - 1)}{2} \cdot 2q_1 + 2p_1 (p - 1) \cdot \frac{p(p - 1)}{2}
\]

\[
= p^3 DD(G) + 2p^2 (p - 1)[W(G) + q_1] + p_1 p(p - 1)^2.
\]

\(\square\)

For the strong product of two general graphs, we have the following.

**Theorem 3.7.** Let \( G_1 \) be a connected graph with \( p_1 \) vertices and \( q_1 \) edges, and \( G_2 \) be a connected graph with \( p_2 \) vertices and \( q_2 \) edges. Then

\[
DD(G_1 \Box G_2) = p_1^3 DD(G_1) + 2p_1^2 (p_1 - 1)[W(G_1) + q_1] + p_2 p_1(p_1 - 1)^2.
\]
with \( p_2 \) vertices and \( q_2 \) edges. Then
\[
\max \left\{ DD(G_1)[2p_2q_2 + p_2^2] + 4p_2q_2W(G_1) + 2p_2(p_2 - 1)q_1W(G_2) + DD(G_2)(2q_1 + p_1),
\right.
\[
DD(G_1)[2p_1q_1 + p_1^2] + 4p_1q_1W(G_2) + 2p_1(p_1 - 1)q_2W(G_1) + DD(G_1)(2q_2 + p_2) \right\}
\]
\[
\leq DD(G_1 \boxtimes G_2) \leq (2q_2 + p_2)(p_2 + 1)DD(G_1) + q_2(4p_2 + 2p_1^2 - 2p_1)W(G_1)
\]
\[
+ (2q_1 + p_1)(p_1 + 1)DD(G_2) + q_1(4p_1 + 2p_2^2 - 2p_2)W(G_2).
\]

Moreover, the lower bound is sharp.

In particular, if \( G \) be a connected graph with \( p \) vertices and \( q \) edges, then
\[
(2q + p)(p + 1)DD(G) + 2pq(p + 1)W(G)
\]
\[
\leq DD(G \boxtimes G) \leq 2\{(2q + p)(p + 1)DD(G) + 2pq(p + 1)W(G)\}. 
\]

**Proof.** From Lemma 2.4 and the definition of degree distance, we have
\[
DD(G_1 \boxtimes G_2) = \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} \left[d_{G_1 \boxtimes G_2}(a,b) + d_{G_1 \boxtimes G_2}(c,d)\right]d_{G_1 \boxtimes G_2}((a,b),(c,d)]
\]
\[
= \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} \left[d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)\]
\]
\[
\cdot \max\{d_{G_1}(a,c),d_{G_2}(b,d)\}
\]
\[
\geq \max \left\{ \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} \left[d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)\right]d_{G_1}(a,c) \right.
\]
\[
+ \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} \left[d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)\right]d_{G_2}(b,d),
\]
\[
\sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} \left[d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)\right]d_{G_2}(b,d)
\]
\[
\geq \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} \left[d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)\right]d_{G_1}(a,c) \right\}
\[
\begin{align*}
&= \max \left\{ \sum_{\{(a, b), (c, d)\} \subseteq V_1 \times V_2} |d_{G_1}(a) + d_{G_1}(c)|d_{G_1}(a, c) + \sum_{\{(a, b), (c, d)\} \subseteq V_1 \times V_2} |d_{G_2}(b) + d_{G_2}(d)|d_{G_1}(a, c) \\
&+ \sum_{\{(a, b), (c, d)\} \subseteq V_1 \times V_2} [d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a, c) \\
&+ \sum_{\{(a, b), (c, d)\} \subseteq V_1 \times V_2} [2d_{G_1}(a) + (d_{G_1}(a) + 1)(d_{G_2}(b) + d_{G_2}(d))]d_{G_2}(b, d), \\
&\sum_{\{(a, b), (c, d)\} \subseteq V_1 \times V_2, b \neq d} [d_{G_1}(a) + d_{G_1}(c)]d_{G_2}(b, d) + \sum_{\{(a, b), (c, d)\} \subseteq V_1 \times V_2, b \neq d} [d_{G_2}(b) + d_{G_2}(d)]d_{G_2}(b, d) \\
&+ \sum_{\{(a, b), (c, d)\} \subseteq V_1 \times V_2, b \neq d} [d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b, d) \\
&+ \sum_{\{(a, b), (c, b)\} \subseteq V_1 \times V_2} [2d_{G_2}(b) + (d_{G_2}(b) + 1)(d_{G_1}(a) + d_{G_1}(c))]d_{G_1}(a, c) \right\} \\
&= \max \left\{ p_2^2 DD(G_1) + 4p_2q_2W(G_1) + \sum_{\{(a, b), (c, d)\} \subseteq V_1 \times V_2} [d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a, c) \\
&+ 2p_2(p_2 - 1)q_1W(G_2) + DD(G_2)(2q_1 + p_1), \\
p_2^2 DD(G_2) + 4p_1q_1W(G_2) + \sum_{\{(a, b), (c, d)\} \subseteq V_1 \times V_2} [d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b, d) \\
+ 2p_1(p_1 - 1)q_2W(G_1) + DD(G_1)(2q_2 + p_2) \right\} \\
\end{align*}
\]
\[
\begin{aligned}
&= \sum_{\{a,c\} \subseteq V_1} d_{G_1}(a)d_{G_1}(a,c) \cdot \sum_{\{b,d\} \subseteq V_2} d_{G_2}(b) + \sum_{\{a,c\} \subseteq V_1} d_{G_1}(c)d_{G_1}(a,c) \cdot \sum_{\{b,d\} \subseteq V_2} d_{G_2}(d) \\
&= 2p_2q_2 \cdot \sum_{\{a,c\} \subseteq V_1} d_{G_1}(a)d_{G_1}(a,c) + 2p_2q_2 \cdot \sum_{\{a,c\} \subseteq V_1} d_{G_1}(c)d_{G_1}(a,c) \\
&= 2p_2q_2 \left( \sum_{\{a,c\} \subseteq V_1} [d_{G_1}(a) + d_{G_1}(c)]d_{G_1}(a,c) \right) \\
&= 2p_2q_2 \cdot DD(G_1)
\end{aligned}
\]

and similarly
\[
\begin{aligned}
&= \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} [d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b,d) \\
&= 2p_1q_1 \left( \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} [d_{G_2}(b) + d_{G_2}(d)]d_{G_2}(b,d) \right) = 2p_1q_1 DD(G_2),
\end{aligned}
\]

it follows that
\[
\begin{aligned}
DD(G_1 \boxtimes G_2) \\
&\geq \max \left\{ DD(G_1)[2p_2q_2 + p_2^2] + 4p_2q_2 W(G_1) + 2p_2(p_2 - 1)q_1 W(G_2) + DD(G_2)(2q_1 + p_1), \\
&DD(G_2)[2p_1q_1 + p_1^2] + 4p_1q_1 W(G_2) + 2p_1(p_1 - 1)q_2 W(G_1) + DD(G_1)(2q_2 + p_2) \right\}.
\end{aligned}
\]

Also, we have
\[
\begin{aligned}
DD(G_1 \boxtimes G_2) \\
&= \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} [d_{G_1 \boxtimes G_2}(a,b) + d_{G_1 \boxtimes G_2}(c,d)]d_{G_1 \boxtimes G_2}([(a,b),(c,d)]) \\
&= \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} [d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)] \\
&\cdot \max\{d_{G_1}(a,c), d_{G_2}(b,d)\} \\
&\leq \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} [d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a,c) \\
&+ \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} [d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b,d) \\
&= (2q_2 + p_2)(p_2 + 1)DD(G_1) + q_2(4p_2 + 2p_2^2 - 2p_1) W(G_1) \\
&+ (2q_1 + p_1)(p_1 + 1)DD(G_2) + q_1(4p_1 + 2p_2^2 - 2p_2) W(G_2).
\end{aligned}
\]

To show the sharpness of the lower bounds of Theorem 3.7, we consider the following example.

**Example 1.** Let \( G \) be a complete graph of order \( n \). If \( n = 2 \), then \( G = K_2 \) and \( G \boxtimes G = K_4 \), and hence \( DD(G \boxtimes G) = 36 = (2q + p)(p + 1)DD(G) + 2pq(p + 1)W(G) \). If \( n = 3 \), then \( G = K_3 \) and \( G \boxtimes G = K_9 \), and hence \( DD(G \boxtimes G) = 576 = (2q + p)(p + 1)DD(G) + 2pq(p + 1)W(G) \). From the proof of Theorem 3.7, one can check that \( K_\infty \boxtimes K_\infty \) is an sharp example of the lower bound.
3.3. For Gutman index

In this subsection, we study the Gutman index of strong product graphs. We first begin with an easy case.

**Theorem 3.8.** Let $G$ be a connected graph with $p_1$ vertices and $q_1$ edges, and $K_p$ be a complete graph with $p$ vertices. Then

$$
Gut(G \boxtimes K_p) = p^3 \text{Gut}(G) + p^3(p - 1)DD(G) + p^2(p - 1)^2 \cdot W(G)
+ \frac{p^3(p - 1)}{2} M_1(G) + \frac{p(p - 1)^3}{2} p_1 + 2p^2(p - 1)^2 q_1.
$$

**Proof.** Let $V(G) = V_1$ and $V(K_p) = V_2$. From the definition of strong product and Corollary 2.5, we have

$$
\sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} d_{G \boxtimes K_p}(a,b) \cdot d_{G \boxtimes K_p}(c,d) \cdot d_{G \boxtimes K_p}[(a,b),(c,d)]
= \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} [d_G(a) + d_{K_p}(b) + d_G(a)d_{K_p}(b)] \cdot [d_G(c) + d_{K_p}(d) + d_G(c)d_{K_p}(d)]
\cdot d_{G \boxtimes K_p}[(a,b),(c,d)]
= \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a \neq c} [pd_G(a) + p - 1] \cdot [pd_G(c) + p - 1] \cdot d_G(a,c)
+ \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a = c} [pd_G(a) + p - 1] \cdot [pd_G(a) + (p - 1)] \cdot 1
= p^2 \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a \neq c} d_G(a)d_G(c)d_G(a,c)
+ p(p - 1) \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a \neq c} [d_G(a) + d_G(c)]d_G(a,c)
+ (p - 1)^2 \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a \neq c} d_G(a,c) + p^2 \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a = c} d_G^2(a)
+ \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a = c} (p - 1)^2 + 2p(p - 1) \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a = c} d_G(a)
= p^3 \text{Gut}(G) + p^3(p - 1)DD(G) + p^2(p - 1)^2 \cdot W(G)
+ \frac{p^3(p - 1)}{2} M_1(G) + \frac{p(p - 1)^3}{2} p_1 + 2p^2(p - 1)^2 q_1.
$$

For the strong product of two general graphs, we have the following.
Theorem 3.9. Let $G_1$ be a connected graph with $p_1$ vertices and $q_1$ edges, and $G_2$ be a connected graph with $p_2$ vertices and $q_2$ edges. Then

$$\max \left\{ Gut(G_1)(p_1^2 + 4p_2q_2 + 4q_2^2) + (2p_2q_2 + 4q_2^2)DD(G_1) + 4q_2^2W(G_1) \right. $$

$$+ M_1(G_1)W(G_2) + [2q_1 + M_1(G_1)]DD(G_2) + [p_1 + 4q_1 + M_1(G_1)]Gut(G_2), \right.$$  

$$Gut(G_2)(p_1^2 + 4p_1q_1 + 4q_1^2) + (2p_1q_1 + 4q_1^2)DD(G_2) + 4q_1^2W(G_2) \left.+ M_2(G_2)W(G_1) + [2q_2 + M_2(G_2)]DD(G_1) + [p_2 + 4q_2 + M_2(G_2)]Gut(G_1) \right\}$$

$$\leq \ Gut(G_1 \boxtimes G_2)$$

$$\leq \ Gut(G_1)[p_1^2 + 4p_2q_2 + 4q_2^2 + p_2 + 4q_2 + M_2(G_2)] + [2p_2q_2 + 4q_2^2 + 2q_2 + M_2(G_2)]DD(G_1) + [4q_2^2 + M_2(G_2)]W(G_1) + (2p_1q_1 + 4q_1^2 + 2q_1 + M_1(G_1)) \]$$

In particular, if $G$ be a connected graph with $p$ vertices and $q$ edges, then

$$\ Gut(G)[p^2 + 4pq + 4q^2 + p + 4q] + [2pq + 4q^2 + 2q + M_1(G)]DD(G) + [4q^2 + M_1(G)]W(G) \right.$$  

$$\leq \ Gut(G \boxtimes G)$$

$$\leq \ 2 \left\{ Gut(G)[p^2 + 4pq + 4q^2 + p + 4q] + [2pq + 4q^2 + 2q + M_1(G)]DD(G) + [4q^2 + M_1(G)]W(G) \right\}$$

Proof. Let $V(G_1) = V_1$ and $V(G_2) = V_2$. From the definition of strong product and Lemma 2.4, we have

$$Gut(G_1 \boxtimes G_2)$$

$$= \sum_{(a,b),(c,d) \subseteq V_1 \times V_2} d_{G_1 \boxtimes G_2}(a,b) \cdot d_{G_1 \boxtimes G_2}(c,d) \cdot d_{G_1 \boxtimes G_2}[(a,b),(c,d)]$$

$$= \sum_{(a,b),(c,d) \subseteq V_1 \times V_2} [d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b)] \cdot [d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)]$$

$$\cdot \max \{d_{G_1}(a,c), d_{G_2}(b,d)\}$$

$$\geq \ max \left\{ \sum_{(a,b),(c,d) \subseteq V_1 \times V_2} [d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b)] \cdot [d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a,c) \right.$$  

$$+ \sum_{(a,b),(c,d) \subseteq V_1 \times V_2} [d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b)] \cdot [d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b,d),$$

$$\sum_{(a,b),(c,d) \subseteq V_1 \times V_2} [d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b)] \cdot [d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b,d) \right.$$  

$$+ \sum_{(a,b),(c,d) \subseteq V_1 \times V_2} [d_{G_2}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b)] \cdot [d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a,c) \right\}$$
\[
= \max \left\{ \sum_{\{a,b\}, \{c,d\} \subseteq V_1 \times V_2} \left[ d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(c)d_{G_2}(d) \right] \cdot \left[ d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d) \right] \cdot d_{G_1}(a, c) \right. \\
+ \sum_{\{a,b\}, \{c,d\} \subseteq V_1 \times V_2} \left[ d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) \right] \cdot \left[ d_{G_1}(a) + d_{G_2}(d) + d_{G_1}(a)d_{G_2}(d) \right] \cdot d_{G_2}(b, d) \\
+ \sum_{\{a,b\}, \{c,d\} \subseteq V_1 \times V_2} \left[ d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) \right] \cdot \left[ d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d) \right] \cdot d_{G_2}(b, d) \\
+ \sum_{\{a,b\}, \{c,d\} \subseteq V_1 \times V_2} \left[ d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) \right] \cdot \left[ d_{G_1}(c) + d_{G_2}(b) + d_{G_1}(c)d_{G_2}(b) \right] \cdot \left[ d_{G_1}(a, c) \right] \right\}
\]

= \max\{X_1 + X_2, Y_1 + Y_2\},
\]

where

\[
X_1 = \sum_{\{a,b\}, \{c,d\} \subseteq V_1 \times V_2} \left[ d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) \right] \cdot \left[ d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d) \right] \cdot d_{G_1}(a, c),
\]

\[
X_2 = \sum_{\{a,b\}, \{c,d\} \subseteq V_1 \times V_2} \left[ d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) \right] \cdot \left[ d_{G_1}(a) + d_{G_2}(d) + d_{G_1}(a)d_{G_2}(d) \right] \cdot d_{G_2}(b, d),
\]

\[
Y_1 = \sum_{\{a,b\}, \{c,d\} \subseteq V_1 \times V_2} \left[ d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) \right] \cdot \left[ d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d) \right] \cdot d_{G_2}(b, d),
\]

and

\[
Y_2 = \sum_{\{a,b\}, \{c,d\} \subseteq V_1 \times V_2} \left[ d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) \right] \cdot \left[ d_{G_1}(c) + d_{G_2}(b) + d_{G_1}(c)d_{G_2}(b) \right] \cdot \left[ d_{G_1}(a, c) \right].
\]

Note that

\[
X_1 = \sum_{\{a,b\}, \{c,d\} \subseteq V_1 \times V_2} d_{G_1}(a) \cdot d_{G_1}(c) \cdot d_{G_1}(a, c) + \sum_{\{a,b\}, \{c,d\} \subseteq V_1 \times V_2} d_{G_1}(a) \cdot d_{G_2}(d) \cdot d_{G_1}(a, c) + \sum_{\{a,b\}, \{c,d\} \subseteq V_1 \times V_2} d_{G_1}(a) \cdot d_{G_1}(a)d_{G_2}(d) \cdot d_{G_1}(a, c) + \sum_{\{a,b\}, \{c,d\} \subseteq V_1 \times V_2} d_{G_2}(b) \cdot d_{G_1}(c) \cdot d_{G_1}(a, c).
\]
\[
\begin{align*}
&+ \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} d_{G_2}(b)d_{G_2}(d)d_{G_1}(a,c) + \sum_{a \neq c} d_{G_2}(b)d_{G_1}(c)d_{G_2}(d)d_{G_1}(a,c) \\
&+ \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} d_{G_1}(a)d_{G_2}(b)d_{G_1}(a,c) + \sum_{a \neq c} d_{G_1}(a)d_{G_2}(b)d_{G_2}(d)d_{G_1}(a,c) \\
&+ \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} d_{G_1}(a)d_{G_2}(b)d_{G_1}(c)d_{G_2}(d)d_{G_1}(a,c)
\end{align*}
\]

\[= p_2^2 \Gut(G_1) + 4p_2q_2 \Gut(G_1) + 4q_2^2 \Gut(G_1) + 2p_2q_2 DD(G_1) + 4q_2^2 DD(G_1) + 4q_2^2 W(G_1)
\]

and

\[X_2 = \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} d_{G_1}(a)^2 \cdot d_{G_2}(b,d) + \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} d_{G_1}(a) \cdot d_{G_2}(d) \cdot d_{G_2}(b,d) \]

\[+ \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} d_{G_1}(a)^2 d_{G_2}(d) d_{G_2}(b,d) + \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} d_{G_2}(b)d_{G_1}(a) d_{G_2}(b,d) \]

\[+ \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} d_{G_2}(b) d_{G_2}(d) d_{G_2}(b,d) + \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} d_{G_2}(b) d_{G_2}(b) d_{G_2}(d) d_{G_2}(b,d) \]

\[+ \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} d_{G_1}(a)^2 d_{G_2}(b) d_{G_2}(d) d_{G_2}(b,d) \]

\[= M_1(G_1) W(G_2) + 2q_1 DD(G_2) + M_1(G_1) DD(G_2) + p_1 \Gut(G_2) + 4q_1 \Gut(G_2) \]

\[+ M_1(G_1) \Gut(G_2) \]

\[= M_1(G_1) W(G_2) + [2q_1 + M_1(G_1)] DD(G_2) + [p_1 + 4q_1 + M_1(G_1)] \Gut(G_2). \]

Similarly, we have

\[Y_1 = \Gut(G_2)(p_2^2 + 4p_1 q_1 + 4q_2^2) + (2p_1 q_1 + 4q_2) DD(G_2) + 4q_2^2 W(G_2) \]

and

\[Y_2 = M_2(G_2) W(G_1) + [2q_2 + M_2(G_2)] DD(G_1) + [p_2 + 4q_2 + M_2(G_2)] \Gut(G_1). \]

Then

\[\Gut(G_1 \boxtimes G_2) \geq \max \left\{ \Gut(G_1)(p_2^2 + 4p_2q_2 + 4q_2^2) + (2p_2q_2 + 4q_2^2) DD(G_1) + 4q_2^2 W(G_1) \right. \]

\[+ M_1(G_1) W(G_2) + [2q_1 + M_1(G_1)] DD(G_2) + [p_1 + 4q_1 + M_1(G_1)] \Gut(G_2), \]

\[\Gut(G_2)(p_2^2 + 4p_1 q_1 + 4q_2^2) + (2p_1 q_1 + 4q_2) DD(G_2) + 4q_2^2 W(G_2) \]

\[+ M_2(G_2) W(G_1) + [2q_2 + M_2(G_2)] DD(G_1) + [p_2 + 4q_2 + M_2(G_2)] \Gut(G_1) \right\} \]
and
\[
\text{Gut}(G_1 \boxtimes G_2) = \sum_{(a,b),(c,d) \subseteq V_1 \times V_2} d_{G_1 \boxtimes G_2}(a, b) \cdot d_{G_1 \boxtimes G_2}(c, d) \cdot d_{G_1 \boxtimes G_2}[(a, b), (c, d)]
\]
\[
= \sum_{(a,b),(c,d) \subseteq V_1 \times V_2} [d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b)] \cdot [d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)]
\]
\[
\cdot \max \{d_G(a, c), d_{G_2}(b, d)\}
\]
\[
\leq \sum_{(a,b),(c,d) \subseteq V_1 \times V_2} [d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b)] \cdot [d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a, c)
\]
\[
+ \sum_{(a,b),(c,d) \subseteq V_1 \times V_2} [d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b)] \cdot [d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b, d)
\]
\[
= X_1 + X_2 + Y_1 + Y_2
\]
\[
\leq \text{Gut}(G_1)[p_2^2 + 4p_2q_2 + 4q_2^2 + p_2 + 4q_2 + M_2(G_2)] + [2p_2q_2 + 4q_2^2 + 2q_2 + M_2(G_2)]DD(G_1)
\]
\[
+ [4q_2^2 + M_2(G_2)]W(G_1) + \text{Gut}(G_2)[p_1^2 + 4p_1q_1 + 4q_1^2 + p_1 + 4q_1 + M_1(G_1)]
\]
\[
+ [2p_1q_1 + 4q_1^2 + 2q_1 + M_1(G_1)]DD(G_2) + [4q_1^2 + M_1(G_1)]W(G_2).
\]

To show the sharpness of the lower bounds of Theorem 3.9, we consider the following example.

**Example 1.** Let $G$ be a complete graph of order $n$. If $n = 2$, then $G = K_2$ and $G \boxtimes G = K_4$, and hence $\text{Gut}(G \boxtimes G) = 54 = \text{Gut}(G)[p^2 + 4pq + 4q^2 + p + 4q] + [2pq + 4q^2 + 2q + M_1(G)]DD(G) + [4q^2 + M_1(G)]W(G)$. If $n = 3$, then $G = K_3$ and $G \boxtimes G = K_9$, and hence $\text{Gut}(G \boxtimes G) = 2304 = \text{Gut}(G)[p^2 + 4pq + 4q^2 + p + 4q] + [2pq + 4q^2 + 2q + M_1(G)]DD(G) + [4q^2 + M_1(G)]W(G)$. From the proof of Theorem 3.9, one can check that $K_n \boxtimes K_n$ is an sharp example of the lower bound.

### 3.4. For complete product

We first give the following lemma.

**Lemma 3.10.** (1) If $A = [a_{ij}]_{m \times n}$ be any matrix and $I = [1]_{p \times n}$, then $S(I A) = pS(A)$;

(2) If $A = [a_{ij}]_{n \times n}$ and $I = [1]_{n \times p}$, then $S(A I) = pS(A)$;

(3) If $A = [a_{ij}]_{p \times m}$, $I = [1]_{m \times n}$ and $B = [b_{ij}]_{n \times q}$, then $S(A I B) = S(A) \cdot S(B)$. In particular, if $A = [a_{ij}]_{n \times n}$ then $S(A I A) = S(A)^2$.

**Proof.** For (1), we have
\[
S(I A) = \sum_{i=1}^{p} \sum_{j=1}^{m} \sum_{k=1}^{n} 1_{ik}a_{kj} = \sum_{i=1}^{p} \sum_{j=1}^{m} a_{kj}
\]
\[
= \sum_{i=1}^{p} S(A) = pS(A).
\]
For (2), we have

\[ S(AI) = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik}1_{kj} = \sum_{j=1}^{p} S(A) = pS(A). \]

For (3), we have

\[ S(AIB) = \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{m} a_{ik} \sum_{s=1}^{n} 1_{ks}b_{sj} = \sum_{i=1}^{p} \sum_{k=1}^{m} a_{ik} \sum_{j=1}^{q} \sum_{s=1}^{n} b_{sj} = S(A) \cdot S(B). \]

**Corollary 3.11.** Let \( G \) be a \((p,q)\)-graph and \( A \) and \( K \) be the adjacency matrix of \( G \) and \( K_p \) respectively. Let \( I = [1]_{p \times p} \) and \( I_p \) be the identity matrix. Then \( S(AKA) = 4q^2 - M_1(G) \).

**Proof.** By Lemma 3.10, we have

\[ S(AKA) = S(A(I - I_p)A) = S(A IA) - S(A^2) = S(A)^2 - S(A^2) = 4q^2 - M_1(G). \]

**Theorem 3.12.** Let \( G \) be a \((p,q)\)-graph, then

1. If \( \text{diam}(G) = 2 \), then \( \text{Gut}(G) = 4q^2 - M_1(G) - M_2(G) \).
2. If \( \text{diam}(G) = 3 \) and \( G \) has no cycles of size 3 then
   \[ \text{Gut}(G) = 6q^2 - \frac{3}{2}M_1(G) - 2M_2(G) - N_2(G). \]

**Proof.**

(1) By Lemma 3.4 and Observation 3.1, we have:

\[ 2\text{Gut}(G) = S(ADG)A = S(A(A + 2\overline{A})A) = S(A^3) + 2S(4A \overline{A} A) = 2S(A(A + \overline{A})A) - S(A^3) = 2S(4AKA) - S(A^3) = 8q^2 - 2M_1(G) - 2M_2(G). \]

(2) By Lemma 3.4 and Observation 3.1, we have:

\[ 2\text{Gut}(G) = S(ADG)A = S(A(3K - 2A - B)A) = 3S(AKA) - 2S(A^3) - S(ABA) = 12q^2 - 3M_1(G) - 4M_2(G) - 2N_2(G). \]
Remark 3.13. Let $A_1 = [a_{ij}]_{n_1 \times n_1}$ and $A_2 = [b_{ij}]_{n_2 \times n_2}$ be the adjacency matrix of $G_1$ and $G_2$, respectively. Let $D_G$ be distance matrix of graph $G = G_1 \lor G_2$. Let $I_1 = [1]_{n_1 \times n_1}$, $I_2 = [1]_{n_2 \times n_2}$, $I'_1 = [1]_{n_1 \times n_2}$, $I'_2 = [1]_{n_2 \times n_1}$ and $I_n$ be identity matrices. Then
\[
D_G = \begin{pmatrix}
2I_1 - A_1 - 2I_{1n_1} & I'_1 \\
I'_2 & 2I_2 - A_2 - 2I_{n_2}
\end{pmatrix}
\]
is distance matrix of $G_1 \lor G_2$.

Theorem 3.14. Let $G_1$ be a graph with order $n_1$ and $m_1$ edges and $G_2$ be a graph with order $n_2$ and $m_2$ edges. Then
\[
DD(G_1 \lor G_2) = 4(n_1 + n_2 - 1)(m_1 + m_2 + n_1n_2) - M(G_1 \lor G_2).
\]

Proof. Since $diam(G_1 \lor G_2) = 2$, it follows from Lemma 3.3 that
\[
DD(G_1 \lor G_2) = 4(n_1 + n_2 - 1)(m_1 + m_2 + n_1n_2) - M_1(G_1 \lor G_2).
\]
For computing $M_1(G_1 \lor G_2)$, let $A$ be the adjacency matrix of graph $G = G_1 \lor G_2$. Then
\[
M_1(G_1 \lor G_2) = S(A^2) = S \begin{pmatrix}
A_1 & I'_1 \\
I'_2 & A_2
\end{pmatrix}
\begin{pmatrix}
A_1 & I'_1 \\
I'_2 & A_2
\end{pmatrix}
\]
\[
= S \begin{pmatrix}
A_1^2 + I'_1I'_2 & A_1I'_1 + I'_1A_2 \\
I'_2A_1 + A_2I'_2 & I'_2I'_1 + A_2^2
\end{pmatrix}
\]
\[
= S(A_1^2) + S(I'_1I'_2) + S(A_1I'_1) + S(I'_1A_2)
+ S(I'_2A_1) + S(A_2I'_2) + S(I'_2I'_1) + S(A_2^2)
= M_1(G_1) + n_1^2n_2 + 4n_2m_1 + 4n_1m_2 + n_2^2n_1 + M_1(G_2).
\]

Theorem 3.15. Let $G_1$ be an $(n_1, m_1)$-graph and let $G_2$ be an $(n_2, m_2)$-graph. Then
\[
Gut(G_1 \lor G_2) = 4(m_1 + m_2 + n_1n_2)^2 - M_1(G_1 \lor G_2) - M_2(G_1 \lor G_2).
\]

Proof. Let $A_1 = [a_{ij}]_{n_1 \times n_1}$ and $A_2 = [b_{ij}]_{n_2 \times n_2}$ be the adjacency matrix of $G_1$ and of $G_2$ respectively. Let $D_G$ be distance matrix of graph $G = G_1 \lor G_2$. If we set $I_1 = [1]_{n_1 \times n_1}$, $I_2 = [1]_{n_2 \times n_2}$, $I'_1 = [1]_{n_1 \times n_2}$, $I'_2 = [1]_{n_2 \times n_1}$ and $I_n$ be identity matrix, then it follows from Theorem 3.12 that
\[
Gut(G_1 \lor G_2) = 4(m_1 + m_2 + n_1n_2)^2 - M_1(G_1 \lor G_2) - M_2(G_1 \lor G_2),
\]
since $diam(G_1 \lor G_2) = 2$.

For computing $M_2(G_1 \lor G_2)$, let $A$ be the adjacency matrix of graph $G = G_1 \lor G_2$. Then
From Lemma 3.10, we have

\[
2M_2(G_1 \vee G_2) = S(A^3) = S \left[ \begin{pmatrix} A_1 & I_1 \\ I_2 & A_2 \end{pmatrix} \begin{pmatrix} A_1 & I_1 \\ I_2 & A_2 \end{pmatrix} \begin{pmatrix} A_1 & I_1 \\ I_2 & A_2 \end{pmatrix} \right] = S \left[ \begin{pmatrix} A_1^2 + I_1 & I_1 \\ I_2 A_1 + A_2 I_2 & I_2^2 + A_2 \end{pmatrix} \begin{pmatrix} A_1 & I_1 \\ I_2 & A_2 \end{pmatrix} \right] = S \left[ \begin{pmatrix} A_1^3 + I_1 I_2 A_1 + A_1 I_1 I_2 + I_1 A_2 I_2 + A_1^2 I_1 I_2 + I_1^2 I_1 + A_1 I_1 A_2 + I_1 A_2^2 \\ I_2 A_2 I_1 + A_2 I_2 I_2 + A_2^2 I_2 + I_2 A_1 I_2 + A_2 I_2 I_2 + I_2^2 I_2 + A_2^3 \end{pmatrix} \right]\]

\[= S(A_1^3) + S(I_1 I_2 A_1) + S(A_1 I_1 I_2) + S(I_1 A_2 I_2) + S(A_1^2 I_1) + S(I_1^2 I_2 I_1) + S(A_1 I_1 A_2) + S(I_2^2 I_2) + S(I_2 A_2 I_1) + S(A_2 I_2 I_2) + S(A_2^2 I_2) + S(I_2 A_2 I_1) + S(A_2 I_2 I_2) + S(A_2^2 I_2).\]

From Lemma 3.10, we have

\[
2M_2(G_1 \vee G_2) = 2M_2(G_1) + 4n_1 n_2 m_1 + 2n_2 M_1(G_1) + 2n_1^2 m_2 + 8m_1 m_2 + 2n_1^2 m_2 + 2n_1 M_1(G_2) + 2n_2 m_1 + 4n_1 n_2 m_2 + 2M_2(G_2),
\]

and hence

\[
M_2(G_1 \vee G_2) = M_2(G_1) + M_2(G_2) + n_2 M_1(G_1) + n_1 M_1(G_2) + 2n_1 n_2 m_2 + 2n_2 m_1 + n_1^2 m_2 + 4m_1 m_2 + n_1^2 m_2 + n_2^2 m_1 = M_2(G_1) + M_2(G_2) + n_2 M_1(G_1) + n_1 M_1(G_2) + (n_1 n_2 + 2m_2)(n_1 n_2 + 2m_1) + n_1^2 m_2 + n_2^2 m_1.
\]

References