



An upper bound on the total double Roman domination number

Research Article

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Abstract: Let $G = (V, E)$ be a graph with no isolated vertex. A total double Roman dominating function (TDRDF) is a DRDF such that the vertices of positive weight under function are not isolated. In this paper, we initiate to study the bound of TDRD number of a graph G from above, and we improve this bound. In particular, we show that $\gamma_{tdR}(G) \leq \frac{4n}{3}$, whenever G is a graph of order n and minimum degree at least 2.

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1. Introduction and Preliminaries

Let $G = (V, E)$ be a graph of order $n = n(G) = |V(G)|$ with $V = V(G)$ and $E = E(G)$. The open neighborhood of a vertex $v \in V(G)$ is the set $N(v) = \{u : uv \in E(G)\}$. The closed neighborhood of a vertex $v \in V(G)$ is $N[v] = N(v) \cup \{v\}$. The open neighborhood of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$. The closed neighborhood of a set $S \subseteq V$ is the set $N[S] = N(S) \cup S = \cup_{v \in S} N[v]$. We denote the degree of v by $d_G(v) = |N(v)|$. By $\Delta = \Delta(G)$ and $\delta = \delta(G)$, we denote the maximum degree and minimum degree of a graph G , respectively. We write K_n , P_n and C_n for the complete graph, path and cycle of order n , respectively. A tree T is an acyclic connected graph. (see [13]).

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A set $S \subseteq V$ in a graph G is called a dominating set if $N[S] = V$. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set in G , and a dominating set of G of cardinality $\gamma(G)$ is called a γ -set of G . A subset $S \subseteq V$ is a k -dominating set if every vertex of $V - S$ has at least k neighbors in S . The k -domination number $\gamma_k(G)$ is the minimum cardinality of a k -dominating set of G . A total dominating set, abbreviated TD -set, of G is a set S of vertices of G such that every vertex of G is adjacent to a vertex in S . The total domination number of G , denoted by $\gamma_t(G)$, is the minimum cardinality of a TD -set of G . A TD -set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$ -set. The concept of total domination in graphs is now well studied ([4, 5]).

Given a graph G and a positive integer m , assume that $f : V(G) \rightarrow \{0, 1, 2, \dots, m\}$ is a function, and suppose that $(V_0, V_1, V_2, \dots, V_m)$ is the ordered partition of V induced by f , where $V_i = \{v \in V : f(v) = i\}$ for $i \in \{0, 1, \dots, m\}$. A Roman dominating function (RDF) on graph G is a function $f : V \rightarrow \{0, 1, 2\}$ such that if $v \in V_0$ for some $v \in V$, then there exists a vertex $w \in N(v)$ with $w \in V_2$. The weight of a Roman dominating function ($\omega(f)$) is the value $f(V) = \sum_{v \in V(G)} f(v)$. The Roman domination number $\gamma_R(G)$ of a graph G is the minimum weight of a RDF on G [2].

This concept was formally defined by Cockayne *et al.* [2] motivated, in some sense, by the article of Ian Stewart entitled "Defend the Roman Empire!" ([11]). The idea is that the values 1 and 2 represent the number of Roman legions stationed at a location v . A location $u \in N(v)$ is considered to be *unsecured* if no legion is stationed there ($f(u) = 0$). The unsecured location u can be secured by sending a legion to u from an adjacent location v . But a legion cannot be sent from a location v if doing so leaves that location unsecured (if $f(v) = 1$). Thus, two legions must be stationed at a location ($f(v) = 2$) before one of the legions can be sent to an adjacent location. Once the original paper [2] was published, many researchers attended to this topics. Some of these variations of Roman domination have been outlined in [7]. For instance, a total Roman domination version of such parameter which was presented in [7].

The total Roman dominating function ($TRDF$) on a graph G with no isolated vertex is an RDF f on G with the additional property that the subgraph of G induced by the set $\{v \in V : f(v) \neq 0\}$ has no isolated vertices. The total Roman domination number $\gamma_{tR}(G)$ is the minimum weight of a $TRDF$ on G . A $TRDF$ on G with weight $\gamma_{tR}(G)$ is called a $\gamma_{tR}(G)$ -function.

On the other hand, Beeler *et al.* [1] introduced the concept of double Roman dominating function. A double Roman dominating function ($DRDF$) on a graph G is a function $f : V \rightarrow \{0, 1, 2, 3\}$ such that the following conditions are met:

- (a) if $f(v) = 0$, then vertex v must have at least two neighbors in V_2 or one neighbor in V_3 .
- (b) if $f(v) = 1$, then vertex v must have at least one neighbor in $V_2 \cup V_3$.

The weight of a double Roman dominating function ($\omega(f)$) is the value $f(V) = \sum_{v \in V(G)} f(v)$. The double Roman domination number $\gamma_{dR}(G)$ of a graph G is the minimum weight of a $DRDF$ on G . A $DRDF$ of G with weight $\gamma_{dR}(G)$ is called a $\gamma_{dR}(G)$ -function of G . We note that $\omega(f) = |V_1| + 2|V_2| + 3|V_3|$. Double Roman domination was studied in [1, 6] and elsewhere.

A variation of double Roman domination is (outer) independent double Roman domination [8, 9], total double Roman domination [3, 10, 12]. Here, we want to progress in total double Roman domination more. The total double Roman dominating function ($TDRDF$) on a graph G with no isolated vertex is a $DRDF$ f on G with the additional property that the subgraph of G induced by the set $\{v \in V : f(v) \neq 0\}$ has no isolated vertices. The total double Roman domination number $\gamma_{tdR}(G)$ is the minimum weight of a $TDRDF$ on G . A $TDRDF$ on G with weight $\gamma_{tdR}(G)$ is called a $\gamma_{tdR}(G)$ -function [3].

We make use of the following results in this paper.

Proposition 1.1. ([3] Proposition 3) For $n \geq 2$,

$$\gamma_{tdR}(P_n) = \begin{cases} 6, & \text{if } n = 4, \\ \lceil \frac{6n}{5} \rceil, & \text{otherwise.} \end{cases}$$

Proposition 1.2. ([3] Proposition 2) For $n \geq 3$,

$$\gamma_{tdR}(C_n) = \left\lceil \frac{6n}{5} \right\rceil.$$

Remark For P_3 and C_3 , clearly $\gamma_{tdR}(P_3) = \gamma_{tdR}(C_3) = 4 = \frac{4n}{3}$. Also for P_6 and C_6 , $\gamma_{tdR}(P_6) = \gamma_{tdR}(C_6) = 8 = \frac{4n}{3}$. Therefore If G is a union of some P_3 , C_3 , P_6 and C_6 , then $\gamma_{tdR}(G) = \frac{4n}{3}$.

From Propositions 1.1 and 1.2 and the Remark, we have.

Corollary 1.3. For any cycle C_n with $6 \neq n \geq 4$, $\gamma_{tdR}(C_n) < \frac{4n}{3}$.
For any path P_n with $6 \neq n \geq 5$, $\gamma_{tdR}(P_n) < \frac{4n}{3}$.

From Propositions 1.1 and 1.2 and Corollary 1.3 we have.

Proposition 1.4. Let G be a graph with a Hamiltonian path of order $n \neq 2, 4$ or Hamiltonian cycle of order n . Then $\gamma_{tdR}(G) \leq \frac{4n(G)}{3}$.

In this paper we initiate to study the bounds of $\gamma_{tdR}(G)$ of G from above, and we improve this bound. Furthermore we show that for any graph G of order n with $\delta(G) \geq 2$, $\gamma_{tdR}(G) \leq \frac{4n}{3}$.

2. Main results

In this section we show that $\gamma_{tdR}(G) \leq \frac{4n(G)}{3}$ for any graph G of order n with $\delta(G) \geq 2$. Before presenting the proof of the main result, we give some lemmas that are useful for investigation of the main result. For integers m and k where $m \geq 3$ and $k \geq 1$, let $C_{m,k}$ be the graph obtained from a cycle $C_m = x_1x_2 \cdots x_mx_1$ and a path $y_1y_2 \cdots y_k$ by adding the edge x_1y_1 . Let $\mathcal{Q}$ be the family of all connected graphs G of order n with $\delta(G) \geq 2$ and $\gamma_{tdR}(G) \leq \frac{4n}{3}$. Let A denotes the set of vertices of degree at least 3 in G , and let $B = V(G) - A$. A path P of G is called maximal if $V(P) \subseteq B$ and each end-vertex of P is adjacent to a vertex of A .

For each $i \geq 1$, let $\mathcal{P}_i = \{P \mid P \text{ is a maximal path with } |V(P)| = i\}$. Let $\mathcal{P} = \bigcup_{i \geq 1} \mathcal{P}_i$. For $P \in \mathcal{P}$, let $X_P = \{u \in A \mid u \text{ is adjacent to an end-vertex of } P\}$.

Lemma 2.1. Let $Q \in \mathcal{Q}$ and $u \in V(Q)$. If G is a graph obtained from Q and $C_{m,k}$ for some integers $m \geq 3$ and $k \geq 1$, by adding the edge uy_k , then $\gamma_{tdR}(G) \leq \frac{4n(G)}{3}$.

Proof. Let $C_{m,k}$ be the graph obtained from a cycle $C_m = x_1x_2 \cdots x_mx_1$ and a path $y_1y_2 \cdots y_k$ by adding the edge x_1y_1 . Let f be a $\gamma_{tdR}(Q)$ -function and g be a $\gamma_{tdR}(C_{m,k})$ -function. Then the function h defined by $h(x) = f(x)$ for $x \in V(Q)$ and $h(x) = g(x)$ for $x \in V(C_{m,k})$, is a $TDRDF$ of G . If $m = 3$ and $k = 1$, then we assign 3 to x_1 , 0 to x_2, x_3 and 1 to y_1 . Thus $\omega(C_{m,k}) \leq \frac{4(m+k)}{3}$ for $m = 3$ and $k = 1$. According to the above and since $C_{m,k}$ is a graph with a hamiltonian path, by Proposition 1.4 we have $\gamma_{tdR}(C_{m,k}) \leq \frac{4(m+k)}{3}$ when $m \geq 3$ and $k \geq 1$. The fact $Q \in \mathcal{Q}$ and $\gamma_{tdR}(C_{m,k}) \leq \frac{4(m+k)}{3}$ imply that $\gamma_{tdR}(G) \leq \omega(f) + \omega(g) \leq \frac{4n(Q)}{3} + \frac{4(m+k)}{3} = \frac{4n(G)}{3}$. $\square$

Lemma 2.2. Let $Q \in \mathcal{Q}$ and $u \in V(Q)$. If G is a graph obtained from Q and a cycle $C_m = x_1x_2 \cdots x_mx_1$, by adding the edge ux_1 , then $\gamma_{tdR}(G) \leq \frac{4n(G)}{3}$.

Proof. Let f be a $\gamma_{tdR}(Q)$ -function and let g be a $\gamma_{tdR}(C_m)$ -function. Then the function h defined by $h(x) = f(x)$ for $x \in V(Q)$ and $h(x) = g(x)$ otherwise, is a $TDRDF$ of G . By Proposition 1.4, $\gamma_{tdR}(C_m) \leq \frac{4m}{3}$. The fact $Q \in \mathcal{Q}$ and by $\gamma_{tdR}(C_m) \leq \frac{4m}{3}$ imply that $\gamma_{tdR}(G) \leq \omega(f) + \omega(g) \leq \frac{4n(Q)}{3} + \frac{4m}{3} = \frac{4n(G)}{3}$. $\square$

Lemma 2.3. Let $Q \in \mathcal{Q}$ and $u \in V(Q)$. If G is a graph obtained from Q and a cycle $C_m = x_1x_2 \cdots x_mx_1$ where $m \notin \{3, 5\}$, by identifying vertices u and x_1 , then $\gamma_{tdR}(G) \leq \frac{4n(G)}{3}$.

Proof. Let z denote the vertex resulting by identifying u and x_1 . Let f be a $\gamma_{tdR}(Q)$ -function and let g be a $\gamma_{tdR}(P)$ -function of the path P of order $m - 1$ induced by $x_2x_3 \cdots x_m$. By Proposition 1.4, $\gamma_{tdR}(P) \leq \frac{4(m-1)}{3}$. Then the function h defined by $h(x) = f(x)$ for $x \in V(Q) - \{u\}$, $h(z) = f(u)$ and $h(x) = g(x)$ otherwise, is a $TDRDF$ of G and

$$\gamma_{tdR}(G) \leq \gamma_{tdR}(Q) + \gamma_{tdR}(P) \leq \frac{4n(Q)}{3} + \frac{4(m-1)}{3} = \frac{4n(G)}{3}.$$

□

Theorem 2.4. Let G be a simple graph of order n with $\delta(G) \geq 2$. Then

$$\gamma_{tdR}(G) \leq \frac{4n}{3}.$$

This bound is sharp for union of the cycles C_3 or union of the cycles C_6 or union of both of them.

Proof. The proof is given by induction on n . We assume without loss of generality that G is a connected graph. The result is immediate for $n < 6$. Suppose $n \geq 6$ and the result hold for all graphs of order less than n . Let G be a graph of order $n \geq 6$. If $A = \emptyset$, then $G = C_m$, and it follows from Proposition 1.4 that, $\gamma_{tdR}(G) \leq \frac{4n}{3}$. Therefore, from now on we consider $A \neq \emptyset$.

First, we assume that G is a graph obtained from $Q \in \mathcal{Q}$ and a cycle $C_m = x_1x_2 \cdots x_mx_1$, for $m \in \{3, 5\}$, by identifying a vertex $u \in V(Q)$ with x_1 , and let z be the resulting vertex. Suppose first that there exists a vertex $y \in N_G(z) \cap Q$ such that $\deg(y) = 2$ and y has a neighbor $y' \in N_G(z) \cap Q$. Then, let G' be the graph obtained from G by removing the vertices $z, x_2, \dots, x_m$ and joining y to each vertex $x \in N_G(z) - N_G[y]$ and joining y' to each vertex $x \in N_G(z) - N_G[y']$. On the other hand, Suppose that any vertex $y \in N_G(z) \cap Q$ satisfies either $\deg(y) \geq 3$ or y has a neighbor not in $N_G(z) \cap Q$ other than z . In this case, let G' be the graph obtained from G by removing the vertices $z, x_2, \dots, x_m$ and joining a vertex $y \in N_G(z) \cap Q$ to each vertex $x \in N_G(z) - N_G[y]$. For every $v \in V(G')$, $\deg(v) \geq 2$. By the induction hypothesis, there exists a $TDRDF$ f of G' such that $\omega(f) \leq \frac{4n(G')}{3}$.

1. Assume that $m = 3$. Then define the function g by $g(x_2) = 1$, $g(x_3) = 0$, $g(z) = 3$ and $g(x) = f(x)$ otherwise. Clearly, g is a $TDRDF$ of G , and

$$\gamma_{tdR}(G) \leq \omega(f) + 4 \leq \frac{4(n-3)}{3} + 4 = \frac{4n}{3}.$$

2. Assume that $m = 5$. Suppose first that, there exists a vertex $y \in N_G(z) \cap Q$ such that $f(y) \geq 1$. Then we define the function g by $g(z) = 3$, $g(x_2) = g(x_5) = 0$, $g(x_3) = 2$, $g(x_4) = 1$, and $g(x) = f(x)$ otherwise. On the other hand, suppose that $f(y) = 0$ for each $y \in N_G(z) \cap Q$. In this case, we define the function g by $g(z) = g(x_3) = 2$, $g(x_2) = 0$, $g(x_4) = g(x_5) = 1$, and $g(x) = f(x)$ otherwise. Clearly, g is a $TDRDF$ of G and

$$\gamma_{tdR}(G) \leq \frac{4(n-5)}{3} + 6 < \frac{4n}{3}.$$

Now, it is assumed that the graph is not necessary obtained from Q and a cycle $C_m = x_1x_2 \cdots x_mx_1$, by identifying a vertex $u \in V(Q)$ with x_1 . According to the above and Lemmas 2.1 to 2.3 and since $\gamma_{tdR}(G) \leq \gamma_{tdR}(G - e)$ for every edge e , we consider the following cases.

Case 1. There exists a vertex $u \in A$ and a maximal path $P_1 \in \mathcal{P}_k$ such that u is adjacent to the path P_1 , and $\{ux_1, ax_k\} \subseteq E(G)$ where $k \geq 3$, $a \in A$.

Suppose that $u = a$. Then $C_{k+1} = ux_1x_2 \cdots x_ku$ is a cycle.

* Assume that $\deg(u) > 3$. Let $Q = G - \{x_1, \dots, x_k\}$. For every $v \in V(Q)$, $\deg(v) \geq 2$. By the

induction hypothesis, there exists a *TDRDF* f of Q such that $\omega(f) \leq \frac{4n(Q)}{3}$. We extend Q to G by adding $ux_1x_2 \cdots x_ku$.

According to content we proved at the begining of this Theorem and Lemma 2.3 conclude that $\gamma_{tdR}(G) \leq \frac{4n}{3}$.

* Assume that $\deg(u) = 3$ and u is adjacent to a maximal path $P = y_1y_2 \cdots y_{k'} \in \mathcal{P}_{k'}$ and $uy_1, zy_{k'} \in E(G)$ where $z \in A - \{u\}$. Let $Q = G - \{u, x_1, \dots, x_k, y_1, \dots, y_{k'}\}$. For every $v \in V(Q)$, $\deg(v) \geq 2$. By the induction hypothesis, there exists a *TDRDF* f of Q such that $\omega(f) \leq \frac{4n(Q)}{3}$. Let $C_{k+1,k'}$ be the graph obtained from the cycle C_{k+1} and the path $y_1y_2 \cdots y_{k'}$ by adding the edge uy_1 . We extend Q to G by adding $C_{k+1,k'}$, the edge $zy_{k'}$. According to Lemma 2.1, conclude that $\gamma_{tdR}(G) \leq \frac{4n}{3}$.

* Assume that $\deg(u) = 3$ and $uz \in E(G)$ where $z \in A$. Let $Q = G - \{u, x_1, \dots, x_k\}$. For every $v \in V(Q)$, $\deg(v) \geq 2$. By the induction hypothesis, there exists a *TDRDF* f of Q such that $\omega(f) \leq \frac{4n(Q)}{3}$. We extend Q to G by adding $ux_1x_2 \cdots x_ku$ and the edge uz . According to Lemma 2.2, conclude that $\gamma_{tdR}(G) \leq \frac{4n}{3}$.

Now assume that $u \neq a$. Then let G' be the graph obtained from G by removing the vertices x_1, x_2, x_3 when $k = 3$ and by removing the vertices x_1, x_2, x_3 and joining u to x_4 when $k \geq 4$. Then for every $v \in V(G')$, $\deg(v) \geq 2$. By the induction hypothesis, there exists a *TDRDF* f of G' such that $\omega(f) \leq \frac{4(n-3)}{3}$.

1. Assume that $k = 3$. We define the function g by $g(x_1) = 1$, $g(x_2) = 3$, $g(x_3) = 0$ and $g(x) = f(x)$ otherwise.

2. Assume that $k > 3$. If $f(u) = 3$, $f(x_4) > 0$, then define the function g by $g(x_1) = g(x_2) = 1$, $g(x_3) = 2$, and $g(x) = f(x)$ otherwise.

If $f(u) = 3$, $f(x_4) = 0$, then define the function g by $g(x_1) = 0$, $g(x_2) = 1$, $g(x_3) = 3$, and $g(x) = f(x)$ otherwise.

If $f(u) = 1, 2$, $f(x_4) \geq 1$, then define the function g by $g(x_1) = g(x_3) = 2$, $g(x_2) = 0$, and $g(x) = f(x)$ otherwise.

If $f(u) = 2$, $f(x_4) = 0$, then define the function g by $g(x_1) = 0$, $g(x_2) = g(x_3) = 2$, and $g(x) = f(x)$ otherwise.

If $f(u) = 1$, $f(x_4) = 0$, then define the function g by $g(x_1) = 0$, $g(x_2) = 3$, $g(x_3) = 1$, and $g(x) = f(x)$ otherwise.

If $f(u) = 0$, $f(x_4) = 3$, then define the function g by $g(x_1) = 3$, $g(x_2) = 1$, $g(x_3) = 0$, and $g(x) = f(x)$ otherwise.

If $f(u) = 0$, $f(x_4) = 2$, then define the function g by $g(x_1) = g(x_2) = 2$, $g(x_3) = 0$ and $g(x) = f(x)$ otherwise.

If $f(u) = 0$, $f(x_4) \leq 1$, then define the function g by $g(x_1) = 1$, $g(x_2) = 3$, $g(x_3) = 0$, and $g(x) = f(x)$ otherwise.

Clearly, g is a *TDRDF* of G , and

$$\gamma_{tdR}(G) \leq \omega(g) \leq \omega(f) + 4 \leq \frac{4(n-3)}{3} + 4 = \frac{4n}{3}.$$

Therefore, the result is valid for Case 1.

Assume now that $|A| = 2$ and $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. We define the function g by $g(x) = 2$ for $x \in A$, $g(x) = 1$ for $x \notin A$. Clearly, g is a *TDRDF* of G and by a simple investigation we can see that

$$\gamma_{tdR}(G) \leq \frac{4n}{3}.$$

Henceforth, we assume that $|A| \geq 3$.

Case 2. There exists a vertex $u \in A$ and a maximal path $P_1 = x_1x_2 \in \mathcal{P}_2$ such that u is adjacent to the path P_1 , and $\{ux_1, ax_2\} \subseteq E(G)$ where $a \in A$.

Suppose that $u = a$. Then $C_3 = ux_1x_2u$ is a cycle.

* Assume that $\deg(u) > 3$. Let $Q = G - \{x_1, x_2\}$. For every $v \in V(Q)$, $\deg(v) \geq 2$. By the induction

hypothesis, there exists a *TDRDF* f of Q such that $\omega(f) \leq \frac{4n(Q)}{3}$. We extend Q to G by adding ux_1x_2u . According to content we proved at the beginning of this Theorem conclude that $\gamma_{tdR}(G) \leq \frac{4n}{3}$.

* Assume that $\deg(u) = 3$ and u is adjacent to a maximal path $P = y_1y_2 \cdots y_{k'} \in \mathcal{P}_{k'}$ and $uy_1, zy_{k'} \in E(G)$ where $z \in A - \{u\}$. Let $Q = G - \{u, x_1, x_2, y_1, \dots, y_{k'}\}$. For every $v \in V(Q)$, $\deg(v) \geq 2$. By the induction hypothesis, there exists a *TDRDF* f of Q such that $\omega(f) \leq \frac{4n(Q)}{3}$. Let $C_{3,k'}$ be the graph obtained from the cycle C_3 and the path $y_1y_2 \cdots y_{k'}$ by adding the edge uy_1 . We extend Q to G by adding $C_{3,k'}$, the edge $zy_{k'}$. According to Lemma 2.1, conclude that $\gamma_{tdR}(G) \leq \frac{4n}{3}$.

* Assume that $\deg(u) = 3$ and $uz \in E(G)$ where $z \in A$. Let $Q = G - \{u, x_1, \dots, x_k\}$. For every $v \in V(Q)$, $\deg(v) \geq 2$. By the induction hypothesis, there exists a *TDRDF* f of Q such that $\omega(f) \leq \frac{4n(Q)}{3}$. We extend Q to G by adding $ux_1x_2 \cdots x_ku$ and the edge uz . According to Lemma 2.2, conclude that $\gamma_{tdR}(G) \leq \frac{4n}{3}$.

Suppose that $u \neq a$. Let Q is the graph obtained from G by removing the vertices u, x_1, x_2 and joining a to each vertex $z \in N_G(u) - \{x_1\} - N_G[a]$.

1. Suppose that $\delta(Q) > 1$. Then let $G' = Q$. For every $v \in V(G')$, $\deg(v) \geq 2$. By the induction hypothesis, there exists a *TDRDF* f of G' such that $\omega(f) \leq \frac{4(n-3)}{3}$.

• Assume that $f(a) = 0$. We consider the following cases.

Suppose that $f(y) \neq 2, 3$ for every vertex $y \in N_G(a)$. Then define the function g by $g(u) = 0$, $g(x_1) = 1$, $g(x_2) = 3$ and $g(x) = f(x)$ otherwise.

Suppose that there exist a vertex $y \in N_G(a)$ such that $f(y) = 3$ or there exist two vertices $y_1, y_2 \in N_G(a)$ such that $f(y_1) = f(y_2) = 2$. Then define the function g by $g(u) = 0$, $g(x_1) = 3$, $g(x_2) = 1$, and $g(x) = f(x)$ otherwise.

Suppose that there exists a vertex $y_1 \in N_G(a)$ such that $f(y_1) = 2$ and $f(y) < 2$ for every vertex $y \in N_G(a) - \{y_1\}$. Then define the function g by $g(u) = 0$, $g(x_1) = g(x_2) = 2$, and $g(x) = f(x)$ otherwise.

• Assume that $f(a) = 1$.

Suppose that there exists a vertex $y \in N_G(a)$ such that $f(y) \geq 2$. Then define the function g by $g(u) = 1$, $g(x_1) = 3$, $g(x_2) = 0$, and $g(x) = f(x)$ otherwise.

On the other hand, assume that $f(y) < 2$ for every vertex $y \in N_G(a)$. Then define the function g by $g(u) = 1$, $g(x_1) = 1$, $g(x_2) = 2$, and $g(x) = f(x)$ otherwise.

• Assume that $f(a) = 2$. Then define the function g by $g(u) = 2$, $g(x_1) = g(x_2) = 1$, and $g(x) = f(x)$ otherwise.

• Assume that $f(a) = 3$.

Suppose that there exists a vertex $y \in N_G(a)$ such that $f(y) \geq 1$. Then define the function g by $g(u) = 3$, $g(x_1) = 1$, $g(x_2) = 0$, and $g(x) = f(x)$ otherwise.

On the other hand, suppose that $f(y) = 0$ for every vertex $y \in N_G(a)$. Then define the function g by $g(u) = 3$, $g(x_1) = 0$, $g(x_2) = 1$, and $g(x) = f(x)$ otherwise.

Clearly, g is a *TDRDF* of G , and

$$\gamma_{tdR}(G) \leq \omega(f) + 4 \leq \frac{4(n-3)}{3} + 4 = \frac{4n}{3}.$$

2. Suppose that $\delta(Q) = 1$. Then there exists a maximal path $P_2 = y$ in G where $uy, ya \in E(G)$.

2.1. Assume that u is adjacent to a maximal path $P_3 = y'$ where $y' \neq y$. Then let G' be the graph obtained from G by removing the vertex y' . For every $v \in V(G')$, $\deg(v) \geq 2$. By the induction hypothesis, there exists a *TDRDF* f of G' such that $\omega(f) \leq \frac{4(n-1)}{3}$.

The weights that vertices u, a, y, x_1, x_2 can have under the function f are as follows.

$f(u) = 0, f(a) = f(x_1) = 2, f(x_2) = f(y) = 1. f(u) = 1, f(a) = f(x_1) = 2, f(y) = 1, f(x_2) = 0.$

$f(u) = 2, f(a) = 2, f(x_1) = f(x_2) = 1, f(y) = 0. f(u) = f(a) = 3, f(x_1) = f(x_2) = f(y) = 0.$

$f(u) = f(a) = 3, f(y) = 1, f(x_1) = f(x_2) = 0.$

Therefore, in the graph G' , $f(u) + f(a) + f(y) + f(x_1) + f(x_2) \geq 6$. Define the function g by $g(u) = g(a) = 3$, $g(y) = 1$, $g(y') = g(x_1) = g(x_2) = 0$, and $g(x) = f(x)$ otherwise. Clearly, g is a *TDRDF* of G and

$$\gamma_{tdR}(G) \leq \omega(f) + 1 \leq \frac{4(n-1)}{3} + 1 < \frac{4n}{3}.$$

2.2. Assume that u is adjacent to a maximal path $P_3 = y'_1y'_2$, where $P_3 \neq P_1$.

2.2.1. Suppose that $y'_2b \in E(G)$, $b \in A - \{u, a\}$. Then let Q' is the graph obtained from G by removing the vertices u, y'_1, y'_2 and joining b to each vertex $z \in N_G(u) - \{y'_1\} - N_G[b]$.

If $\delta(Q') > 1$, then according to 1 in this case, this bound for $\gamma_{tdR}(G)$ is proved.

If $\delta(Q') = 1$, then there exists a maximal path $P_4 = x$ in G where $ux, xb \in E(G)$. Then because u is adjacent to the maximal path $P_2 = y$, according to the part 2.1 in this Case, this bound for $\gamma_{tdR}(G)$ is proved.

2.2.2. Suppose that $y'_2a \in E(G)$. In this case, let G' be the graph obtained from G by removing the vertices u, y, y'_1, y'_2 , and joining a to each vertex $z \in N_G(u) - \{y, y'_1\} - N_G[a]$. Then for every $v \in V(G')$, $\deg(v) \geq 2$. By the induction hypothesis, there exists a *TDRDF* f of G' such that $\omega(f) \leq \frac{4n'}{3}$, where $n' = n(G')$. In the graph G' , $f(a) + f(x_1) + f(x_2) \geq 3$. Define the function g by $g(u) = g(a) = 3$, $g(y) = 1$, $g(x_1) = g(x_2) = g(y'_1) = g(y'_2) = 0$, and $g(x) = f(x)$ otherwise. Clearly, g is a *TDRDF* of G and

$$\gamma_{tdR}(G) \leq \frac{4n'}{3} + \frac{4(n - n')}{3} \leq \frac{4n}{3}.$$

2.3. Suppose that u is adjacent to b_j s where $b_j \in A$ for every j , $1 \leq j \leq t$, t is the number of all vertices in A adjacent to the vertex u .

Now, let G' be the graph obtained from G by removing the edges incident with u except for ux_1, uy . According to content we proved at the begining of this Theorem there exists a *TDRDF* f of G' such that $\gamma_{tdR}(G') \leq \frac{4n}{3}$ and since $\gamma_{tdR}(G) \leq \gamma_{tdR}(G - e)$ for every $e \in E(G)$,

$$\gamma_{tdR}(G) \leq \gamma_{tdR}(G') \leq \frac{4n}{3}.$$

Case 3. There exists a vertex $u \in A$ and two maximal paths $P_1, P_2 \in \mathcal{P}_1$ such that u is adjacent to the paths P_1, P_2 .

Subcase 3.1. Suppose that u is adjacent to maximal paths $P_i = x_i \in \mathcal{P}_1$ such that P_i s have no common vertex except in u where $1 \leq i \leq k$, $k \geq 2$ and k is the number of all maximal paths in $\mathcal{P}_1$ adjacent to u . Let C be the set of vertices in $A - \{u\}$ that are adjacent to one of x_i s, $1 \leq i \leq k$, $k \geq 2$.

Let D be the set of vertices in C that are adjacent to u .

Subcase 3.1.1. Assume that $D = \emptyset$. In this case, let G' be the graph obtained from G by removing $\bigcup_{i \geq 1} V(P_i)$, u . For every $v \in V(G')$, $\deg(v) \geq 2$. By the induction hypothesis, there exists a *TDRDF* f of G' such that $\omega(f) \leq \frac{4n(G')}{3}$. Define the function g by $g(u) = 3$, $g(x_1) = 1$, $g(x_i) = 0$ for $i \geq 2$, and $g(x) = f(x)$ otherwise. Clearly, g is a *TDRDF* of G , and

$$\gamma_{tdR}(G) \leq \omega(f) + 4 \leq \frac{4n(G')}{3} + 4 \leq \frac{4n}{3}.$$

Subcase 3.1.2. Assume that $D \neq \emptyset$, $|D| = t$. Then let G' be the graph obtained from G by removing the edges yd_j for each $d_j \in D$ where $1 \leq j \leq t$, $t \geq 1$. By Subcase 3.1.1, $\gamma_{tdR}(G') \leq \frac{4n}{3}$ and since $\gamma_{tdR}(G) \leq \gamma_{tdR}(G - e)$ for every $e \in E(G)$,

$$\gamma_{tdR}(G) \leq \gamma_{tdR}(G') \leq \frac{4n}{3}.$$

Subcase 3.2. Suppose there is a vertex $a \in A - \{u\}$ such that u, a are adjacent to $P_i = x_i \in \mathcal{P}_1$ where $1 \leq i \leq k$, $k \geq 2$ and k is the number of all maximal paths in $\mathcal{P}_1$ adjacent to u, a .

Suppose first that u, a are adjacent to a vertex $c \in A$ where $\deg(c) = 3$. Assume that c is adjacent to a maximal path $P = x \in \mathcal{P}_1$. Then let G' be the graph obtained from G by removing edge ac . For every $v \in V(G')$, $\deg(v) \geq 2$. By Case 2, $\gamma_{tdR}(G') \leq \frac{4n}{3}$ and since $\gamma_{tdR}(G) \leq \gamma_{tdR}(G - e)$ for every $e \in E(G)$, $\gamma_{tdR}(G) \leq \gamma_{tdR}(G') \leq \frac{4n}{3}$.

Now assume that $\deg(y) \geq 3$ for every $y \in N_G(c)$. In this case, let G' be the graph obtained from G by removing the vertex c . For every $v \in V(G')$, $\deg(v) \geq 2$. By the induction hypothesis, there exists a *TDRDF* f of G' such that $\omega(f) \leq \frac{4n(G')}{3}$. Since the two vertices x_1, x_2 are not adjacent, for these two vertices to be optimally dominated by function f , the vertices u, a must dominate them. So, $f(u) \geq 3$ or $f(a) \geq 3$ or $f(u) = f(a) = 2$.

We extend f to g on G by assigning 1 to c and $f(x) = g(x)$ otherwise.

On the other hand, suppose that vertices u and a are not adjacent to a common vertex of degree 3.

In this case, let H be the graph obtained from G by removing the vertices $u, \bigcup_{i \geq 1} V(P_i)$ and joining a to each vertex $z \in N_G(u) - \{x_i\} - N_G[a]$ for $1 \leq i \leq k, k \geq 2$.

1. Suppose that $\deg_H(a) > 1$. Then let $G' = H$. For every $v \in V(G')$, $\deg(v) \geq 2$. By the induction hypothesis, there exists a *TDRDF* f of G' such that $\omega(f) \leq \frac{4n(G')}{3}$.

If $f(a) = 0$.

Assume that there exists a vertex $y \in N_G(a)$ such that $f(y) = 3$ or there exist two vertices $y_1, y_2 \in N_G(a)$ such that $f(y_1) = f(y_2) = 2$. Then define the function g by $g(u) = 3, g(x_1) = 1, g(x_i) = 0$ for $i > 1$, and $g(x) = f(x)$ otherwise.

Assume that there is only one vertex $y \in N_G(a)$ such that $f(y) = 2$. Then define the function g by $g(u) = g(a) = 2, g(x_i) = 0$ for $i \geq 1$, and $g(x) = f(x)$ otherwise.

Assume that $f(y) = 0, 1$ for every vertex $y \in N_G(a)$. Then define the function g by $g(u) = 0, g(a) = 3, g(x_1) = 1, g(x_i) = 0$ for $i > 1$ and $g(x) = f(x)$ otherwise.

If $f(a) = 1$.

Assume that there exists a vertex $y \in N_G(a)$ such that $f(y) \geq 2$. Then define the function g by $g(u) = 3, g(x_1) = 1, g(x_i) = 0$ for $i > 1$, and $g(x) = f(x)$ otherwise.

Assume that $f(y) = 0, 1$ for every vertex $y \in N_G(a)$. Then define the function g by $g(a) = 3, g(u) = g(x_1) = 1, g(x_i) = 0$ for $i > 1$, and $g(x) = f(x)$ otherwise.

If $f(a) = 2$, then define the function g by $g(u) = 2, g(x_1) = 1, g(x_i) = 0$ for $i > 1$, and $g(x) = f(x)$ otherwise.

If $f(a) = 3$, then define the function g by $g(u) = 3, g(x_1) = 1, g(x_i) = 0$ for $i > 1$, and $g(x) = f(x)$ otherwise.

2. Suppose that $\deg_H(a) = 1$.

Assume that a is adjacent to a maximal path $P' = x'$. Then let G' be the graph obtained from G by removing the vertices $\bigcup_{i \geq 1} V(P_i), u, a, x'$. For every $v \in V(G')$, $\deg(v) \geq 2$. By the induction hypothesis, there exists a *TDRDF* f of G' such that $\omega(f) \leq \frac{4n(G')}{3}$. Then we extend f to g on G by assigning 2 to u, a , 1 to x_1, x' , 0 to x_i s for $i > 1$ and $f(x) = g(x)$ otherwise.

Assume that a is adjacent to a vertex $x \in A - \{u\}$. Then let G' be the graph obtained from G by removing the vertices $\bigcup_{i \geq 1} V(P_i), u, a$. For every $v \in V(G')$, $\deg(v) \geq 2$. By the induction hypothesis, there exists a *TDRDF* f of G' such that $\omega(f) \leq \frac{4n(G')}{3}$. Then we extend f to g on G by assigning 2 to u, a , 1 to x_1 , 0 to x_i s for $i > 1$ and $f(x) = g(x)$ otherwise.

Let $n' = n(G')$. Clearly, g is a *TDRDF* of G and

$$\gamma_{tdR}(G) \leq \omega(f) + \frac{4(n - n')}{3} \leq \frac{4n'}{3} + \frac{4(n - n')}{3} = \frac{4n}{3}.$$

Case 4. There exists a vertex $u \in A$ and a path $P_1 \in \mathcal{P}_1$ such that u is adjacent to the path P_1 , and a vertex $b \in A$.

Assume that $P_1 = x$ and $\{ux, ax, ub\} \subseteq E(G)$ where $a \in A$. Then let G' be the graph obtained from G by removing the edges incident with b except for bv, bu where v is a vertex in $N_G(b)$. According to Case 3, $\gamma_{tdR}(G') \leq \frac{4n}{3}$ and since $\gamma_{tdR}(G) \leq \gamma_{tdR}(G - e)$ for every $e \in E(G)$,

$$\gamma_{tdR}(G) \leq \gamma_{tdR}(G') \leq \frac{4n}{3}.$$

Case 5. $V(G) = A$

Assume that $x, y, z \in V(G)$ such that y is adjacent to x, z . Then let G' be the graph obtained from G by removing the edges incident with y except for xy, yz . By Case 4, $\gamma_{tdR}(G') \leq \frac{4n}{3}$ and since $\gamma_{tdR}(G) \leq \gamma_{tdR}(G - e)$ for every $e \in E(G)$,

$$\gamma_{tdR}(G) \leq \gamma_{tdR}(G') \leq \frac{4n}{3}.$$

□

3. Conclusion

We showed an upper bound for total double Roman domination number for any graph. Characterize graphs G which attain this bound. Finally, from the results that we had in this paper so far, we make a Problem as follows.

Problem Let G be a connected graph of order $n \geq 3$ and $n \notin \{3, 6\}$. If $\delta(G) \geq 2$, then $\gamma_{tdR}(G) < \frac{4n(G)}{3}$.

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Declarations

Author contributions

All authors contributed to the manuscript and approved the final version.

Conflict of interest

The author(s) declare that they have no conflict of interest.

Data availability

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