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Zero forcing, graphs on k parallel paths, and linear preservers

Research Article

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Abstract: The zero forcing number of a simple loopless undirected graph, being an upper bound on the path cover number and the maximum nullity of the graph, is an important parameter in the study of the minimum rank problem. In this article, we show that the minimum k for which a graph G is a graph on k parallel paths is an upper bound on the zero forcing number of G, and hence an upper bound on the path number and maximum nullity of G. We also determine an upper bound on the possible size (number of edges) of a graph on k parallel paths. Finally we show that the only linear operators that preserve the zero forcing number of a graph are the vertex permutations.

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1. Introduction

Let $\mathcal{G}_n$ denote the set of all undirected simple loopless graphs on the vertex set $V = \{v_1, v_2, \dots, v_n\}$. Let G and H be two graphs in $\mathcal{G}_n$. The union of G and H is the graph whose edge set is the union of the edge set of G and the edge set of H. That is, $G \cup H = (V, E(G) \cup E(H))$. The graph G *contains* the graph H if the edge set of H is a subset of the edge set of G and is denoted $H \sqsubseteq G$ or $G \sqsupseteq H$. If $G \sqsupseteq H$ then we write $G \setminus H$ to denote the graph whose edge set consists of the edges of G that are not edges of H.

Let $G \in \mathcal{G}_n$ and let $\mathbb{R}$ be the real field. The *minimum rank* of G, $mr(G)$, is the minimum rank of any symmetric $n \times n$ matrix, A, with entries in $\mathbb{R}$ and with the restrictions that for $i \neq j$, $a_{i,j} \neq 0$ if and only if (v_i, v_j) is an edge in G, and without restrictions on the entries on the diagonal. The *maximum nullity* of G, $M(G)$, is the maximum nullity of any symmetric $n \times n$ matrix, A, with entries in $\mathbb{R}$ and with the restrictions that for $i \neq j$, $a_{i,j} \neq 0$ if and only if (v_i, v_j) is an edge in G, and without restrictions on the entries on the diagonal. Clearly, $mr(G) + M(G) = n$. The zero forcing number discussed below is an upper bound on the maximum nullity of G. See [1, 3].

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Let $G \in \mathcal{G}_n$. Let $C(G)$ be a two coloring of the vertices of G, say a blue-white coloring. The *color change rule* is: given u a vertex colored blue, and v a vertex colored white, if u is the only vertex colored blue that is a neighbor of v, change the color of u to white. We then say that v *forces* u. Let Z be the set of vertices of G that are colored white by $C(G)$. If, following the color change rule as far as possible, all vertices of G are colored white, then Z is called a *zero forcing set*. The *zero forcing number* of G, $Z(G)$, is the minimum number of vertices in any zero forcing set, that is $Z(G) = \min\{|Z| : Z \text{ is a zero forcing set for } G\}$. Note, if G is not connected, say $G = G_1 \cup G_2 \cup \cdots \cup G_k$ where $G_1, G_2, \cdots, G_k$ are the connected components of G, then $Z(G) = \sum_{i=1}^{k} Z(G_i)$. The only graphs with zero forcing number one are the Hamiltonian paths (paths incident to all n vertices). The only graph with zero forcing number n is the empty graph, consisting of n isolated vertices.

A *zero forcing chain* (for a particular chronological list of forces) is a sequence of vertices, $(v_{i_1}, v_{i_2}, \ldots, v_{i_\ell})$ such that v_{i_j} colored white forces $v_{i_{j+1}}$ to change from blue to white. A *zero forcing chain cover* is a set of zero forcing chains that cover all the vertices of G. Note that a zero forcing chain cover of G is a path cover of G.

For more on zero forcing numbers and sets, path cover numbers, maximum nullity and minimum rank see [1].

Let K_n denote the complete loopless graph and O_n denote the empty graph (also called the edgeless graph).

2. Graphs on k parallel paths

Above we have given a characterization of graphs of zero forcing number 1 and n. In [5], Darren Row characterized the graphs in $\mathcal{G}_n$ with zero forcing number 2:

Definition 2.1. [5] *A graph $G \in \mathcal{G}_n$ is said to be a* graph on two parallel paths*, if there exist two independent induced paths of G that cover all the vertices of G and such that the graph can be drawn in the plane in such a way that the paths are parallel and edges drawn as segments, not curves, between the two paths do not cross.*

Theorem 2.2. [5, Theorem 2.3] *Let $G \in \mathcal{G}_n$. Then $Z(G) = 2$ if and only if G is a graph on two parallel paths.*

The following is a generalization of Definition 2.1.

Definition 2.3. *A graph, $G \in \mathcal{G}_n$, is said to be a* graph on k parallel paths *if there exist k independent induced paths that cover all the vertices of G and such that the graph can be drawn in the plane in such a way that these paths lie on parallel lines and the edges that do not lie in one of the paths (i.e., those connecting any two distinct paths), drawn as segments, not as curves, do not cross.*

Note that in a standard drawing of a graph on k parallel paths two edges, neither of which is an edge of one of the k paths may not intersect except at endpoints, however, an edge between two paths may intersect an edge in a third path. See Figure 1 or 2.

Theorem 2.4. [2, Theorem 3] *For a graph G with maximum degree of any vertex at most three, $Z(G)$=3 if and only if G is a graph on 3 parallel paths. In particular, The left most vertices of the parallel paths in any standard drawing of G form a zero forcing set.*

In [2] it was observed that for $i \geq 6$, K_i is not a graph on k parallel paths for any k. In Figure 1 a drawing of K_5 as a graph on 4 parallel paths is shown. That K_4 is a graph on 3 parallel paths and that K_3 is a graph on 2 parallel paths are easily shown. In Figure 2 we see a drawing of $K_{3,3}$ as a graph on 4 parallel paths.

One of the basic graph parameters is that of planarity or genera. A question at this point seems appropriate:

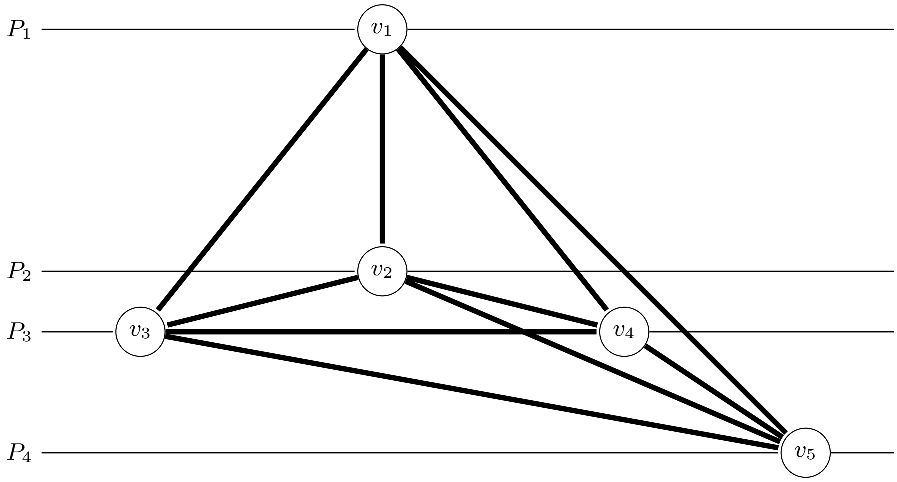

Figure 1. K_5 **as a graph on 4 parallel paths.**

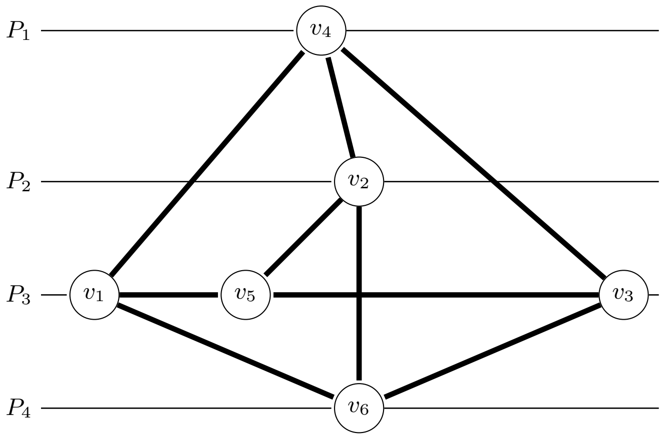

Figure 2. $K_{3,3}$ **as a graph on 4 parallel paths.**

Question 2.5. *What is the smallest k for which some non planar graph is a graph on k parallel paths. What is the smallest k such that there is a non planar graph with zero forcing number k?*

We conjecture that the answer to the above question is four. The two smallest non planar graphs K_5 and $K_{3,3}$ are both graphs on 4 parallel paths and have zero forcing number four. Generalizing the above question we also pose:

Question 2.6. *What is the smallest k for which some graph of genus g is a graph on k parallel paths. What is the smallest k such that there is a graph of genus g with zero forcing number k?*

We now consider the question of the size (number of edges) of a graph on k parallel paths. Let $G \in \mathcal{G}_n$, with edge set $E(G)$. Then define $\varepsilon(G) = |E(G)|$, the number of edges in G.

Theorem 2.7. *Let $G \in \mathcal{G}_n$ be a graph on k parallel paths. Then $n - k \leq \varepsilon(G) \leq \binom{n}{2} + n - k - k\binom{\lfloor \frac{n}{k} \rfloor}{2}$.*

Proof. Suppose that G is a graph on k parallel paths. If there are no edges connecting one path to another, so that the graph is the graph on k disjoint paths, then $\varepsilon(G) = n - k$

Let G be a largest graph on k parallel paths, say the paths are $P_1, P_2, \cdots, P_k$ with m_i edges and $n_i = m_i + 1$ vertices in path P_i. If the vertices of path P_i is V_i, then the complete graph on vertices V_i has $\binom{n_i}{2}$ edges. Now, the largest such graph on k parallel paths ($P_1, P_2, \cdots, P_k$) is a subgraph of the complete graph on n vertices with the edges of each complete graph on vertex set V_i removed, then replacing the edges on the path. So that $\varepsilon(G) \le \binom{n}{2} - \sum_{i=1}^{k} \binom{n_i}{2} + n - k$. The largest graph on k parallel paths, say the paths are $P_1, P_2, \cdots, P_k$ is attained when n_i is either $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$, so that $\varepsilon(G) \le \binom{n}{2} + n - k - k\binom{\lfloor \frac{n}{k} \rfloor}{2}$. □

Note that the bound in the above theorem is far from sharp, since it counts many edges between parallel paths that must cross. In fact the maximum number of edges in a graph on 2 parallel paths is $2n-3$. So a 10 vertex graph on two paths has at most 17 edges, while the above bound, $\binom{n}{2} + n - k - k\binom{\lfloor \frac{n}{k} \rfloor}{2}$ at $n = 10$ and $k = 2$, is $\binom{10}{2} + 10 - 2 - 2\binom{\lfloor \frac{10}{2} \rfloor}{2} = 33$.

We now show that the the minimum k for which a graph G is a graph on k parallel paths is an upper bound on the zero forcing number of G.

Theorem 2.8. *For any integer $k \ge 1$, if G is a graph on k parallel paths in standard form then the left most vertices of G form a zero forcing set for G.*

Proof. We proceed by induction on k. The statement is true for $k = 2$ so assume the statement is true for any $\ell < k$.

Suppose there exists a graph, L, on k parallel paths in standard form such that the left most vertices of L do not form a zero forcing set for L.

Let G be the smallest graph (vertex wise) that is a graph on k parallel paths, such that G is in standard form and such that the k left most vertices do not form a zero forcing set for G.

First observe that none of the parallel paths consists of a single vertex, otherwise, deleting that vertex for G would yeild a graph whose left most vertices form a zero forcing set for it by induction and since coloring the vertex in the path of one vertex white would not change the zero forcing chains so that G would have the left most vertices of G forming a zero forcing set for G, a contradiction to the choice of G. Thus, every path in the k parallel paths has at least two vertices.

Now, color the left most vertex in each path white and the remaining vertices blue. If the neighborhood of any left most vertex has only one vertex colored blue that vertex would be the second vertex in that path and hence, the first vertex, call it x would force the second to be colored white. Since the left most vertices of $G - x$ is a zero forcing set for $G - x$ and x forces the second vertex in that path, the left most vertices in G form a zero forcing set for G, a contradiction. Thus, every left most vertex in any path has at least two neighbors colored blue.

Begin with the bottom path, its first vertex, x, must be adjacent to a vertex, y, which is not the first vertex in any path. Next consider the first path above it whose left most vertex lies to the left of the segment $\overline{xy}$ joining the two vertices mentioned. The left most vertex of this path must be adjacent to a vertex colored white, and since it cannot be adjacent to any vertex colored blue in a path below it (G is a graph on k parallel paths in standard form) it must be adjacent to a vertex which is not the first in a path that lies above it. Continue this process until the left most vertex in the top path is the only possible vertex remaining. Then the first vertex in the top path must be adjacent to a vertex colored blue, but all vertices colored blue that are not in the top path are below it and cannot be adjacent as the segment connecting it would have to cross one of the previous segments. We have arrived at a contradiction, and hence, the left most vertices in any graph on k parallel paths in standard form of any graph form a zero forcing set. □

As an immediate consequence of Theorem 2.8 we have

Corollary 2.9. *If G is a graph on k parallel paths then G has zero forcing number at most k. The left most vertices of the parallel paths in any standard drawing of G form a zero forcing set.*

3. Linear preservers

A mapping $T : \mathcal{G}_n \to \mathcal{G}_n$ is said to be linear if the image of a union of graphs is the union of the images and the image of the empty graph is the empty graph. The operator T is said to *preserve* the zero forcing number if $Z(T(G)) = Z(G)$ for every $G \in \mathcal{G}_n$. T preserves the zero forcing number k if $Z(G) = k$ implies that $Z(T(G)) = k$, and T *strongly* preserves the zero forcing number k if $Z(G) = k$ if and only if $Z(T(G)) = k$. Preserver theory is an area of interest not only to linear algebraists but also to group theorists studying stable sets.

An edge graph is a graph whose edge set is a singleton.

Theorem 3.1. *Let $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that either*

1. *preserves zero forcing number 1 and zero forcing number 2, or*
2. *T strongly preserves zero forcing number 1,*

then T is bijective on the set of edge graphs.

Proof. Let E be an edge graph and suppose that $T(E) = O_n$. Let P be a Hamilton path containing E, say $P = E \cup H$, where $H \in \mathcal{G}_n$ does not contain E. Then, H is not connected, and hence, $Z(H) \geq 2$, and in fact $Z(H) = 2$. Then, $T(P) = T(E \cup H) = T(H)$, a contradiction since $Z(P) = 1$ and $Z(H) = 2$. Thus, T is non singular. (Which means only that $T(X) = O_n$ implies that $X = O_n$, it does not imply invertibility. See [4].)

Suppose that $T(E)$ contains two edges, say $T(E) \sqsupseteq S \cup T$ where S and T are edge graphs, $S \neq T$. Let P be a Hamilton path containing E. Then $T(P)$ must be a Hamiltonian path containing $S \cup T$. But then $T(P) = T(P \setminus Q)$ for some edge graph Q contained in P. Since $Z(P) = 1$ and $Z(P \setminus Q) = 2$, we again have a contradiction. Thus the image of an edge graph is an edge graph.

Suppose that $T(E) = T(F)$ for two distinct edge graphs. Then for P a Hamiltonian path containing $E \cup F$, we have that $T(P)$ contains at most $n - 2$ edges, a contradiction since the only graphs of zero forcing number 1 are Hamiltonian paths with $n - 1$ edges.

Thus, T is bijective on the set of edge graphs. □

Let $G \in \mathcal{G}_n$. Recall that $\varepsilon(G) = |E(G)|$, the number of edges in G. A *2 -star* (or *2-path*) is a graph of two edges sharing a vertex, A *parallel pair* is a graph of two edges without a shared vertex.

Lemma 3.2. *Let $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator. If T is bijective on the set of edge graphs then*

1. *T preserves ε the number of edges in a graph G,*
2. *For any k, T preserves the set of graphs each having precisely k edges,*
3. *In particular, T preserves the set of graphs on two edges.*

Proof. If T did not preserve graphs of size k then the image of a graph on k edges would be mapped to a graph on ℓ edges. If $k < \ell$ then some edge graph would be mapped into a graph of more than one edge, a contradiction. If $k > \ell$ then, since T is bijective on the set of edge graphs, a graph of more that one edge would be mapped to an edge graph, also a contradiction. Thus T preserves ε. The rest follows routinely. □

Lemma 3.3. *Let $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that either*

1. *preserves zero forcing number 1 and zero forcing number 2, or*
2. *T strongly preserves zero forcing number 1,*

then T maps 2-stars to 2-stars.

Proof. By Theorem 3.1, T is bijective on the set of edge graphs and hence, by Lemma 3.2, T is bijective on the set of graphs on two edges. The set of graphs on two edges consists only of 2-stars and parallel pairs. Thus, T maps 2-stars to 2-stars if and only if T maps parallel pairs to parallel pairs. We show that T maps parallel pairs to parallel pairs.

Suppose that the image of a parallel pair is a 2-star, say $T(E \cup F) = S \cup T$ where E, F, S and T are edge graphs, $E \cup F$ is a parallel pair and $S \cup T$ is a 2-star. Let U be an edge graph such that $T(U) \cup S \cup T$ is a 3-cycle. This is possible since T is bijective on the set of edge graphs. Since E and F form a parallel pair, $E \cup F \cup U$ is contained in a Hamilton path, P. But then, $T(P)$ contains a 3-cycle so that $Z(T(P)) \neq 1$, while $Z(P) = 1$, a contradiction. Thus T maps parallel pairs to parallel pairs and hence 2-stars to 2-stars. □

Lemma 3.4. [4, Lemma 2.2] *If T is a linear operator on $\mathcal{G}_n$ that preserves ε and the set of 2-stars, then T is a vertex permutation.*

Theorem 3.5. *Let $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator, then the following are equivalent:*

1. *T preserves zero forcing number 1 and zero forcing number 2,*
2. *T strongly preserves zero forcing number 1,*
3. *T is a vertex permutation.*

Proof. An easy observation is that a relabeling of the vertex set will not change the zero forcing number of a graph. Thus, vertex permutations preserve zero forcing numbers.

Now, if T preserves zero forcing number 1 and zero forcing number 2, or if T strongly preserves zero forcing number 1, then by Theorem 3.1, T is bijective on the set of edge graphs and hence preserves ε. Now by Lemma 3.3, T maps 2-stars to 2-stars, and by Lemma 3.4 T is a vertex permutation. □

Since a linear operator that preservers zero forcing number must strongly preserve zero forcing mumber k for any k we have

Corollary 3.6. *Let $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator. Then the following are equivalent:*

1. *T preserves the zero forcing number,*
2. *T preserves zero forcing number 1 and zero forcing number 2,*
3. *T strongly preserves zero forcing number 1,*
4. *T strongly preserves zero forcing number 2,*
5. *T is a vertex permutation.*

From above, an obvious corollary is

Corollary 3.7. *Let $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator. Then the following are equivalent:*

1. *T preserves the set of graphs on k parallel paths for all $k \leq n$,*
2. *T preserves paths and the set of graphs on 2 parallel paths,*
3. *T strongly preserves the set of graphs on 2 parallel paths,*
4. *T strongly preserves paths.*
5. *T is a vertex permutation.*

References

[1] AIM minimum rank–special graphs work group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetkovič, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanovič, H. van der Holst, K. Vander Meulen, A. Wangsness), Zero forcing sets and the minimum rank of graphs, Linear Algebra Appl. 428(7) (2008) 1628–1648.

[2] M. Alishahi, E. Rezaei-Sani, E. Sharifi, Maximum nullity and zero forcing number on graphs with maximum degree at most three, Disc. Appl. Math. 284 (2020) 179–194.

[3] F. Barioli, W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, H. van der Holst h, Zero forcing parameters and minimum rank problems, Linear Algebra Appl. 433(2) (2010) 401–411.

[4] L. B. Beasley, N. J. Pullman, Linear operators preserving properties of graphs, Congr. Numer. 70 (1990) 105–112.

[5] D. D. Row, A technique for computing the zero forcing number of a graph with a cut-vertex, Linear Algebra Appl. 436(12) (2012) 4423–4432.