On the isomorphism of unitary subgroups of noncommutative group algebras

Zsolt Adam Balogh

Abstract: Let $FG$ be the group algebra of a finite $p$-group $G$ over a field $F$ of characteristic $p$. Let $\oplus$ be an involution of the group algebra $FG$ which arises from the group basis $G$. The upper bound for the number of non-isomorphic $\oplus$-unitary subgroups is the number of conjugacy classes of the automorphism group $G$ with all the elements of order two. The upper bound is not always reached in the case when $G$ is an abelian group, but for non-abelian case the question is open. In this paper we present a non-abelian $p$-group $G$ whose group algebra $FG$ has sharply less number of non-isomorphic $\oplus$-unitary subgroups than the given upper bound.

2010 MSC: 16S34, 16U60

Keywords: Group ring, Group of units, Unitary subgroup

1. Introduction

Let $FG$ be the group algebra of the group $G$ over a field $F$. Let $\oplus$ be an involution of the group algebra $FG$. We say that the algebra involution $\oplus$ arises from the group $G$ when $\oplus$ is an antiautomorphism on $G$. This antiautomorphism of $G$ may also be called involution (for more details see in [19]). In this case the algebra involution $\oplus$ is the linear extension of the group involution $\oplus$ defined on $G$. A group algebra is always an algebra with involution, because the canonical $\ast$-involution of $FG$ (the linear extension of the involution on $G$ which sends each element of $G$ to its inverse) exists for every $F$ and $G$. The canonical involution $\ast$ on $FG$ is a simple example of an algebra involution that arises from the group basis $G$.

Let $V(FG)$ denote the normalized unit group of $FG$, that is, the subgroup of the unit group of $FG$ containing all units with augmentation 1. An element $u \in V(FG)$ is called $\oplus$-unitary if $u^{-1} = u^{\oplus}$. The set of all $\oplus$-unitary units of $FG$ forms a subgroup of $V(FG)$, which is called $\oplus$-unitary subgroup and is denoted by $V_{\oplus}(FG)$. Interest in the unitary subgroups arose in algebraic topology and unitary K-theory
introduced by Novikov [20]. The ε-unitary subgroup is an actively investigated subgroup and it plays an important role of studying the structure of $V(FG)$ for more details we refer the reader to Bovdi’s paper [9].

Let $L$ be a finite Galois extension of $F$ with Galois group $G$, where $F$ is a finite field of characteristic two. A relation between the self-dual normal basis of $L$ over $F$ and the ε-unitary subgroup of $FG$ was discovered by Serre [21]. It was shown in [2] that the ε-unitary subgroup of a group algebra determines the group basis $G$ when it is a finite abelian $p$-group and $F$ is a finite field of characteristic $p$. The structure of the unitary subgroups was studied in several papers (see [3], [4], [5], [12], [14], [15], [16], [17] and [22]).

Let $F$ be a field of characteristic $p$ and $G$ a nonabelian locally finite $p$-group. The groups $G$ when $V_*(FG)$ is normal in $V(FG)$ are listed in [12]. Bovdi and Szakács [10] described the structure of the group $V_*(FG)$ when $G$ is a finite abelian $p$-group and $F$ is a finite field of characteristic $p$. They also constructed a basis for $V_*(FG)$ in [11].

The order of the unitary subgroup $V_*(FG)$ is determined for finite $p$-groups and finite fields of characteristic $p$, if $p$ is an odd prime (see in [13]). The order of $V_*(FG)$ when $p = 2$ is an open question. It was determined only for some group classes (see in [1], [8] and [13]). The structure of $V_*(F2G)$, where $G$ is a 2-group of maximal class of order 8 or 16 and $F_2$ is the field of two elements has been established in [6]. Additionally, the structures of $V_*(FQ_8)$ and $V_*(FD_8)$ are established in [16] and [18] respectively, where $F$ is a finite field of characteristic 2, $Q_8$ is the quaternion group of order 8 and $D_8$ is the dihedral group of order 8.

In the case when $f$ is a homomorphism of $G$ to the multiplicative group of the commutative ring $k$ all the groups $G$ whose $f$-unitary subgroup coincides with the unit group of $KG$ are established in [9]. In [8] the invariants of the ε-unitary subgroup of $FG$ are presented, when $G$ is a finite abelian $p$-group, $F$ is a field of $p$ elements ($p$ is an odd prime) and ε is an involutory automorphism of $G$. In [3] an upper bound for the non-isomorphic ε-unitary subgroups is given, when ε arises from $G$. The upper bound coincides the number of conjugacy classes of the automorphism group $G$ with all the elements of order two including the identity map. In the case, when $G$ is an abelian $p$-group the upper bound is not always sharp. A counterexample can be found in [4]. For non-abelian groups this question is open. In this paper we gave an example for a non-abelian $p$-group whose group algebra $FG$ has less non-isomorphic ε-unitary subgroups than the given upper bound.

2. Involutions and unitary subgroups

Let $F$ be a finite field and $G$ is either the dihedral group of order 8 or the quaternion group of order 8. In this section we show that the number of non-isomorphic ε-unitary subgroups of $FG$ with respect to the involutions which arise from $G$ is equals to the upper bound mentioned in the introduction.

Let $Aut\ G\{2\}$ be the set of all automorphism of $G$ with the identity map. The composition of two anti-automorphisms ε and * of the group $G$ is an automorphism of order two. Therefore, ε can be considered as a composition of an automorphism of order two and the canonical involution, that is, ε $= \phi \circ ^*$, where $\phi \in Aut\ G\{2\}$. We say that the involutions ε $\phi_1 = \phi_1 \circ *$ and ε $\phi_2 = \phi_2 \circ*$ are similar if $\phi_1$ is conjugate to $\phi_2$ in $Aut\ G$. We need the following lemma.

Lemma 2.1. [3, Proposition 7] Let $G$ be a group and $F$ a field and let ε $\phi_1$ and ε $\phi_2$ be involutions of $FG$ which arise from $G$. If ε $\phi_1$ is similar to ε $\phi_2$, then $V_{\phi_1}(FG) \cong V_{\phi_2}(FG)$.

Let $G$ be a finite group and let $\Lambda_2$ denote the number of all distinct conjugacy classes of $Aut\ G\{2\}$. As a consequence of the previous lemma we have the following corollary.

Corollary 2.2. [3, Corollary 8] Let $G$ be a finite group and $F$ a field. The number of non-isomorphic unitary subgroups of $V(FG)$ with respect to the involutions which arise from $G$ is at most $\Lambda_2$. 

116
In this section we show that the upper bound $A_2$ is sharp for all the non-abelian groups of order 8. Moreover, we establish the structure of all non-isomorphic $\otimes$-unitary subgroups for these groups.

First, let us consider the dihedral group $D_8$ of order 8. It is well known that $D_8 \cong Aut D_8$ and $Aut G\{2\}$ is the union of four distinct conjugacy classes, that is, $\Lambda_2 = 4$. Throughout this section we will use Lemma 2.4 in [1] free.

**Lemma 2.3.** The number of non-isomorphic unitary subgroups of $FD_8$ with respect to the involutions which arise from $D_8$ is equals to $\Lambda_2$, where $|F| = 2^n \geq 2$.

**Proof.** It was shown in [18] that $V_6(FD_8) \cong C_2^{3n} \times C_2^n$.

According to Lemma 2.1 it is enough to establish the structure of $V_6(FD_8)$ when the involution $\otimes$ links to different conjugacy classes in $Aut G\{2\}$. Then $C_{\sigma_1} = \{\sigma_1\}$, $C_{\sigma_2}$, $C_{\sigma_3}$ and $C_{\sigma_4}$ are the distinct conjugacy classes of $Aut G\{2\}$, where $\sigma_1$ is the identity map and

$$
\sigma_2 : \begin{cases} 
  a \mapsto a^3 \\
  b \mapsto ab 
\end{cases} \quad \sigma_3 : \begin{cases} 
  a \mapsto a^3 \\
  b \mapsto a^2b 
\end{cases} \quad \sigma_4 : \begin{cases} 
  a \mapsto a \\
  b \mapsto a^2b 
\end{cases} 
$$

**Case** $\sigma_2$. Let $\alpha = \sum_{i=0}^{3} a^i (\alpha_i + \beta_i b) \in FD_8$, where $\alpha_i, \beta_i \in F$. Then $\alpha$ is $\otimes$-unitary if and only if $\alpha \alpha^\otimes = 1$. A straightforward computation shows that $\alpha \alpha^\otimes$ equals to

$$(\alpha_0 + \alpha_2)^2 + (\beta_1 + \beta_3)^2a + (\alpha_1 + \alpha_3)^2a^2 + \delta_1 (1 + a + b) + \delta_2 (a^2 + a^3)b,$$

where $\delta_1 = \alpha_0 (\beta_0 + \beta_1) + \alpha_1 (\beta_1 + \beta_2) + \alpha_2 (\beta_2 + \beta_3) + \alpha_3 (\beta_3 + \beta_0)$ and $\delta_2 = \alpha_0 (\beta_2 + \beta_3) + \alpha_1 (\beta_3 + \beta_0) + \alpha_2 (\beta_0 + \beta_1) + \alpha_3 (\beta_1 + \beta_2)$.

Clearly $\alpha \alpha^\otimes = 1$ if and only if $\alpha_0 + \alpha_2 = 1$, $\delta_0 = \beta_0$, $\alpha_1 = \alpha_3$ and $\beta_1 = \beta_3$. Therefore $\delta_1 = \delta_2 = \beta_0 + \beta_1 = 0$, that is, $\beta_0 = \beta_1$ and every $\otimes$-unitary element can be written as

$$\alpha_0 + \alpha_1 a + (1 + \alpha_0) a^2 + \alpha_1 a^3 + \beta_0 b + \beta_0 a^2 b + \beta_0 a^3 b.$$ 

Therefore $V_6(FD_8) \cong C_2^{3n}$.

**Case** $\sigma_3$. Let $\alpha = \sum_{i=0}^{3} a^i (\alpha_i + \beta_i b) \in FD_8$, where $\alpha_i, \beta_j \in F$. Then

$$\alpha \alpha^\otimes = (\alpha_0 + \alpha_2 + \beta_1 + \beta_3)^2 + (\alpha_1 + \alpha_3 + \beta_0 + \beta_2)^2a^2 + (1 + a + b)b,$$

where $\delta = (\alpha_0 + \alpha_2) (\beta_0 + \beta_2) + (\alpha_1 + \alpha_3) (\beta_1 + \beta_3)$. Clearly $\alpha \alpha^\otimes = 1$ if and only if $\alpha_0 + \alpha_2 + \beta_1 + \beta_3 = 1$, $\beta_0 = \beta_2$ and $\alpha_1 = \alpha_3$. Therefore every element of $V_6(FD_8)$ is central or it can be written in the form either $ab + x_1$ or $a^ib + x_2$, where $x_1, x_2 \in \zeta(V(FD_8))$. Since the exponent of $\zeta(V(FD_8))$ is two we have proved that $V_6(FD_8) \cong C_2^{3n}$.

**Case** $\sigma_4$. Let $\alpha = \sum_{i=0}^{3} a^i (\alpha_i + \beta_i b) \in FD_8$, where $\alpha_i, \beta_j \in F$. Then

$$\alpha \alpha^\otimes = (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)^2 + (\delta_1 + \delta_2)(a + a^2) + (\beta_0 + \beta_1 + \beta_2 + \beta_3)^2a^2 + (\delta_2 + \delta_3)(1 + a + b)b + (\delta_4 + \delta_5)(1 + a + b)ab,$$

where

$$\delta_1 = (\alpha_0 + \alpha_2)(\alpha_1 + \alpha_3), \quad \delta_2 = (\beta_0 + \beta_2)(\beta_1 + \beta_3),$$

$$\delta_3 = (\alpha_0 + \alpha_2)(\beta_0 + \beta_2), \quad \delta_4 = (\alpha_1 + \alpha_3)(\beta_1 + \beta_3),$$

$$\delta_5 = (\alpha_0 + \alpha_2)(\beta_0 + \beta_3).$$

Therefore, $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1$, $\beta_0 + \beta_1 + \beta_2 + \beta_3 = 0$, $\delta_0 + \delta_1 = 0$, $\delta_2 + \delta_3 = 0$, and $\delta_4 + \delta_5 = 0$. Since $\beta_1 + \beta_3 = \beta_0 + \beta_2$ we conclude that $\delta_4 = \delta_2$ and $\delta_5 = \delta_3$. Moreover, $0 = \delta_2 + \delta_3 = (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)^2$. Thus $V_6(FD_8) \cong C_2^{3n}$.
$\alpha_1(\beta_0 + \beta_2) = \beta_0 + \beta_2$ and we have that $\beta_0 = \beta_2$, $\beta_1 = \beta_3$, $\delta_1 = 0$ and $\delta_0 = 0$. Thus, every $\otimes$-unitary element can be written as either

$$a^3 + \alpha_0 \hat{C} + \alpha_1 \hat{C}a + \beta_0 \hat{C}b + \beta_1 \hat{C}ab, \quad \text{if } \alpha_2 = \alpha_0,$$

or

$$a^2 + \alpha_0 \hat{C} + \alpha_1 \hat{C}a + \beta_0 \hat{C}b + \beta_1 \hat{C}ab, \quad \text{if } \alpha_2 = 1 + \alpha_0.$$

Let us denote by $N$ the central elementary abelian subgroup $\{ 1 + \alpha_0 \hat{C} + \alpha_1 \hat{C}a + \beta_0 \hat{C}b + \beta_1 \hat{C}ab \ | \alpha_1, \beta_i \in F \}$. Evidently, $a^2 \in N$. Since $a, a^3$ belong to the $\otimes$-unitary subgroup we have proved that $V_{\otimes}(FD_8) \cong C_4 \times C_2^{4n-1}$.

It is well-known that $Aut Q_8 \cong S_4$, where $S_4$ is the symmetric group of order 24. It follows that $\Lambda_2 = 3$.

**Lemma 2.4.** The number of non-isomorphic unitary subgroups of $FQ_8$ with respect to the involutions which arise from $D_8$ equals $\Lambda_2$, where $|F| = 2^n \geq 2$.

**Proof.** Let $\sigma_1$ be the identity automorphism of $Q_8$. A straightforward computation shows that $Aut G[2] = C_{\sigma_1} \cup C_{\sigma_2} \cup C_{\sigma_3}$, where

$$\sigma_2 : \begin{cases} a \mapsto b \\ b \mapsto a \end{cases} \quad \sigma_3 : \begin{cases} a \mapsto a^3 \\ b \mapsto b \end{cases}.$$

It was shown in [16] that $V_{\otimes}(FQ_8) \cong Q_8 \times C_2^{4n-1}$. Let us consider the following two cases.

**Case $i = 2$.** Let $\alpha = \sum_{i=0}^{3} a^i(\alpha_i + \beta_i b) \in FQ_8$, where $\alpha_i, \beta_j \in F$. Then

$$a^\otimes a = (\alpha_0 + \alpha_2)^2 + \delta_1 a + (\beta_0 + \beta_2)^2 a^2 + \delta_2 a^3 + (\beta_1 + \beta_3)^2 b + \delta_1 ab + (\alpha_1 + \alpha_3) a^2 b + \delta_2 a^3 b,$$

where

$$\delta_1 = \alpha_0(\alpha_1 + \beta_1) + \alpha_2(\alpha_3 + \beta_3) + \beta_0(\alpha_1 + \beta_3) + \beta_2(\alpha_3 + \beta_1),$$

$$\delta_2 = \alpha_0(\alpha_3 + \beta_3) + \alpha_2(\alpha_1 + \beta_1) + \beta_0(\alpha_3 + \beta_1) + \beta_2(\alpha_1 + \beta_3).$$

Evidently, $a^\otimes a = 1$ if and only if $\alpha_0 + \alpha_2 = 1$, $\beta_0 = \beta_2$, $\alpha_1 = \alpha_3$ and $\beta_1 = \beta_3$. They imply that $\delta_1 = \delta_2 = \alpha_1 + \beta_1 = 0$, and so $\alpha_1 = \beta_1$.

Therefore every $\otimes$-unitary element can be written as

$$a^2 + \alpha_0 \hat{C}a^2 + \alpha_1 \hat{C}a + \beta_0 \hat{C}b + \alpha_1 \hat{C}ab.$$

Thus $V_{\otimes}(FQ_8) \cong C_2^{3n}$.

**Case $i = 3$.** Let $\alpha = \sum_{i=0}^{3} a^i(\alpha_i + \beta_i b) \in FQ_8$, where $\alpha_i, \beta_j \in F$. Then

$$a a^\otimes = (\alpha_0 + \alpha_2 + \beta_0 + \beta_2)^2 + (\alpha_1 + \alpha_3 + \beta_1 + \beta_3)^2 a^2 + \delta(1 + a^2)b,$$

where $\delta = (\alpha_0 + \alpha_2)(\beta_0 + \beta_2) + (\alpha_1 + \alpha_3)(\beta_1 + \beta_3)$. Let $S_{\otimes} = \{ \alpha^\otimes | \alpha \in V(FQ_8) \}$. Clearly, $S_{\otimes}$ is a subgroup of $\zeta(V(FQ_8))$, therefore $\psi : V(FQ_8) \rightarrow S_{\otimes}$ (given by $x \mapsto xx^\otimes$) is a homomorphism with kernel $V_{\otimes}(FQ_8)$. Thus

$$|V_{\otimes}(FQ_8)| = \frac{|V(FQ_8)|}{|S_{\otimes}|} = \frac{2^{7n}}{2^m} = 2^{5n}.$$
Let $n = 1$ and $G_\otimes = \{ g \in G \mid g^\otimes = g^{-1} \}$. It is easy to see that $G_\otimes = \langle b \rangle$ and $V_\otimes(FQ_8)$ is a subgroup of $G_\otimes \cdot N$, where $N$ is an elementary abelian group. Since $G_\otimes \cong C_4$, we get that $V_\otimes(FQ_8) \cong C_4 \times C_2^2$

Suppose that $n > 1$ and let $\omega_1$ and $\omega_2$ be elements of the unit group of $F$ satisfying that $\omega_1 \neq 1$ and $\omega_1 + \omega_2 = 1$. It is easy to see that $b$ and $\omega_1 + a + \omega_2 b + ab$ are elements of $V_\otimes(FQ_8)$, but they are not commute. Therefore $V_\otimes(FQ_8)$ is not an abelian group.

According to Theorem 2 in [7], the exponent of $V_\otimes(FQ_8)$ is 4. Since $b$ is a $\otimes$-unitary element with exponent 4 it follows that the exponent of $V_\otimes(FQ_8)$ is 4. Since $|\zeta(V_\otimes(FQ_8))| = 2^{4n}$ and $x^2 \in \zeta(V_\otimes(FQ_8))$ for all $x \in V_\otimes(FQ_8)$ we have proved that

$V_\otimes(FQ_8)/\zeta(V_\otimes(FQ_8)) \cong C_2^n.$

Therefore $V_\otimes(FQ_8)$ is a central extension of $C_2^n$ by $C_2^{4n}$.

\section{Isomorphic unitary subgroups of noncommutative group algebra with non similar involutions}

In this section we present a non-abelian group whose group algebra has sharply less number of non-isomorphic $\otimes$-unitary subgroups than the given upper bound given in Corollary 2.2.

Let $H_{16} = \langle a, c \mid a^4 = b^2 = c^2 = 1, (a, b) = 1, (a, c) = b, (b, c) = 1 \rangle$ be and let $F$ be a finite field with $|F| = 2^n$. The automorphism group of $H_{16}$ is isomorphic to the following group

$\langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1, (\sigma_1, \sigma_2) = \sigma_4, (\sigma_1, \sigma_3) = 1, (\sigma_2, \sigma_3) = 1 \rangle$,

where

$\sigma_1 = \begin{cases} a \mapsto a \\ b \mapsto a^2b \\ c \mapsto c \end{cases}$ \quad $\sigma_2 = \begin{cases} a \mapsto a \\ b \mapsto bc \\ c \mapsto c \end{cases}$ \quad $\sigma_3 = \begin{cases} a \mapsto ab \\ b \mapsto b \\ c \mapsto c \end{cases}$

Let us consider the following two automorphisms of order two in $\text{Aut} H_{16}$

$\tau_1 = \sigma_1 \sigma_2 \sigma_5 : \begin{cases} a \mapsto ac \\ b \mapsto a^2bc \\ c \mapsto c \end{cases}$ \quad and \quad $\tau_2 = (\sigma_1, \sigma_2) : \begin{cases} a \mapsto a^3c \\ b \mapsto bc \\ c \mapsto c \end{cases}$

The conjugacy class of $\tau_1$ is $C_{\tau_1} = \{ \sigma_1 \sigma_2 \sigma_5, \sigma_1 \sigma_2 \sigma_3 \}$ and $\tau_2$ is a central element of the automorphism group.

\textbf{Theorem 3.1.} Let $\otimes_1 = \tau_1 \circ *$ and $\otimes_2 = \tau_2 \circ *$ be involutions of $H_{16}$ and let $F$ be a finite field with $|F| = 2^n (n \geq 1)$. Then $\otimes_1$ is not similar to $\otimes_2$ and $V_{\otimes_1}(FH_{16}) \cong V_{\otimes_2}(FH_{16})$.

\textbf{Proof.} First, we establish the structure of $V_{\otimes_1}(FH_{16})$. Since every element of $FH_{16}$ can be written as

$x = a_0 + a_1 a + a_2 a^2 + a_3 a^3 + a_4 a + a_5 b + a_6 a^2 b + a_7 a^3 b +$

$+ a_8 a^2 b + a_9 a a_10 a^2 + a_11 a^3 + a_12 b + a_13 a b + a_14 a^2 b + a_15 a^3 b c$

we have

$x^{\otimes_1} = (a_0 + a_2 + a_8 + a_10)^2 + (a_5 + a_7 + a_13 + a_15)^2 a^2 + \delta_1 (a + a^3 c) +$

$+ \delta_2 (a^3 + ac) + \delta_3 (b + a^2 bc) + \delta_4 (ab + ab c) + \delta_5 (a^2 b + bc) + \delta_6 (a^3 b + a^3 bc) +$

$+ (a_1 + a_3 + a_9 + a_11)^2 c + (a_4 + a_6 + a_12 + a_14)^2 a^2 c,$

\hfill 119
where
\[ \delta_1 = (a_0 + a_{10})(a_1 + a_{11}) + (a_2 + a_8)(a_3 + a_9) + (a_4 + a_{14})(a_5 + a_{15}) + (a_6 + a_{12})(a_7 + a_{13}) \]
\[ \delta_2 = (a_0 + a_{10})(a_3 + a_9) + (a_2 + a_8)(a_1 + a_{11}) + (a_4 + a_{14})(a_7 + a_{13}) + (a_6 + a_{12})(a_5 + a_{15}) \]
\[ \delta_3 = (a_0 + a_{10})(a_4 + a_{14}) + (a_1 + a_{11})(a_5 + a_{15}) + (a_2 + a_8)(a_6 + a_{12}) + (a_9 + a_7)(a_7 + a_{13}) \]
\[ \delta_4 = (a_0 + a_8)(a_5 + a_{13}) + (a_2 + a_{10})(a_7 + a_{15}) + (a_4 + a_{12})(a_3 + a_{11}) + (a_6 + a_{14})(a_1 + a_9) \]
\[ \delta_5 = (a_0 + a_{10})(a_6 + a_{12}) + (a_1 + a_{11})(a_7 + a_{13}) + (a_3 + a_9)(a_5 + a_{15}) + (a_4 + a_{14})(a_2 + a_8) \]
\[ \delta_6 = (a_0 + a_8)(a_7 + a_{15}) + (a_2 + a_{10})(a_5 + a_{13}) + (a_1 + a_9)(a_4 + a_{12}) + (a_3 + a_{11})(a_6 + a_{14}) \].

Evidently, \( x \) belongs to \( V_{\tilde{\beta}_1}(FH_{10}) \) if and only if \( xx^{\otimes 1} = 1 \). Therefore \( a_0 + a_2 + a_8 + a_{10} = 1, a_5 + a_7 + a_{13} + a_{15} = 0, a_1 + a_3 + a_9 + a_{11} = 0, a_4 + a_6 + a_{12} + a_{14} = 0 \) and \( \delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5 = \delta_6 = 0 \).

Since \( a_2 + a_8 = 1 + a_9 + a_10 \) and \( a_4 + a_{14} = a_6 + a_{12} \) we have that
\[ \delta_1 = (a_3 + a_9) + (a_0 + a_{10})(a_1 + a_{11} + a_3 + a_9) + (a_4 + a_{14})(a_5 + a_{15} + a_7 + a_{13}) = a_3 + a_9, \]
\[ \delta_2 = (a_1 + a_{11}) + (a_0 + a_{10})(a_3 + a_9 + a_1 + a_{11}) + (a_4 + a_{14})(a_7 + a_{13} + a_5 + a_{15}) = a_1 + a_{11}, \]
\[ \delta_3 = (a_6 + a_{12}) + (a_0 + a_{10})(a_4 + a_{14} + a_6 + a_{12}) + (a_1 + a_{11})(a_5 + a_{15} + a_7 + a_{13}) = a_6 + a_{12}, \]
\[ \delta_4 = (a_7 + a_{15}) + (a_0 + a_8)(a_6 + a_{13} + a_7 + a_{15}) + (a_4 + a_{12})(a_3 + a_{11} + a_1 + a_9) = a_7 + a_{15}, \]
\[ \delta_5 = (a_{14} + a_{10})(a_6 + a_{12} + a_4 + a_{14}) + (a_1 + a_{11})(a_7 + a_{13} + a_5 + a_{15}) = a_4 + a_{14}, \]
\[ \delta_6 = (a_5 + a_{13}) + (a_0 + a_8)(a_7 + a_{15} + a_5 + a_{13}) + (a_1 + a_9)(a_4 + a_{12} + a_6 + a_{14}) = a_5 + a_{13}. \]

Therefore
\[ x = a_0 + a_2 a^2 + a_8 c + a_{10} a^2 c + a_1 \tilde{C} c a + a_4 \tilde{C} b + a_5 \tilde{C} a b, \]
where \( \tilde{C} = 1 + a^2 + c + a^2 c \). As a consequence \( V_{\tilde{\beta}_1}(FH_{16}) \) is a central subgroup of \( V(FH_{16}) \).

Let \( N = \{ 1 + \beta_1 \tilde{C} a, 1 + \beta_2 \tilde{C} b, 1 + \beta_3 \tilde{C} a b \mid \beta_i \in F \} \) be. Evidently, \( N \cong C_3^{\alpha} \). Since \( a^{\alpha} \tilde{C} = \alpha \tilde{C} = a^2 \tilde{C} \alpha \tilde{C} = \tilde{C} \) we conclude that \( N \cong a^2 N \cong a^2 c N \). Since \( a^2 N \cdot c N = a^2 c N \) and the pairwise intersections of \( N, a^2 N, a^2 c N \) are \{1\} we have proved that \( V_{\tilde{\beta}_1}(FG) \cong N \times a^2 N \times c N \). Thus \( V_{\tilde{\beta}_1}(FG) \cong C_2 \times C_3^{\alpha} \).

Now, we establish the structure of \( V_{\tilde{\beta}_2}(FH_{16}) \). Let \( x \in FH_{16} \) be. Using formula (1) we can compute the product
\[ xx^{\otimes} = (a_0 + a_2)(a_8 + a_{10}) + (a_5 + a_7 + a_{13} + a_{15}) a^2 \] + \[ \delta_1(a + ac) + \delta_2(a^3 + a^2 c) + \delta_3(b + bc) + \delta_4(ab + abc) + \delta_5(a^2 b + a^2 bc) + \delta_6(a^3 b + a^2 c) \] + \[ (a_4 + a_6 + a_{12} + a_{14}) a^2 c + (a + a_3 + a_9 + a_{11}) a^2 c, \]
where
\[ \delta_1 = (a_0 + a_8)(a_9 + a_1) + (a_2 + a_{10})(a_{11} + a_3) + (a_4 + a_{12})(a_5 + a_{13}) + (a_6 + a_{14})(a_7 + a_{15}), \]
\[ \delta_2 = (a_0 + a_8)(a_{11} + a_3) + (a_2 + a_{10})(a_9 + a_1) + (a_4 + a_{12})(a_7 + a_{15}) + (a_6 + a_{12})(a_5 + a_{13}), \]
\[ \delta_3 = (a_0 + a_8)(a_{12} + a_4) + (a_2 + a_{10})(a_{14} + a_6) + (a_1 + a_9)(a_{15} + a_7) + (a_3 + a_{11})(a_{13} + a_5), \]
\[ \delta_4 = (a_0 + a_8)(a_{13} + a_5) + (a_2 + a_{10})(a_{15} + a_7) + (a_4 + a_{12})(a_1 + a_9) + (a_6 + a_{14})(a_3 + a_{11}), \]
\[ \delta_5 = (a_0 + a_8)(a_{14} + a_6) + (a_1 + a_9)(a_{13} + a_5) + (a_2 + a_{10})(a_{12} + a_4) + (a_3 + a_{11})(a_{15} + a_7), \]
\[ \delta_6 = (a_0 + a_8)(a_{15} + a_7) + (a_2 + a_{10})(a_{13} + a_5) + (a_1 + a_9)(a_{14} + a_6) + (a_3 + a_{11})(a_{12} + a_4). \]

Keeping in mind that \( x \) belongs to \( V_{\tilde{\beta}_2}(FH_{16}) \), it follows that \( xx^{\otimes 2} = 1 \). Therefore \( a_0 + a_2 + a_8 + a_{10} = 1, a_5 + a_7 + a_{13} + a_{15} = 0, a_1 + a_3 + a_9 + a_{11} = 0, a_4 + a_6 + a_{12} + a_{14} = 0 \) and \( \delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5 = \delta_6 = 0 \).

Since \( a_0 + a_8 = 1 + a_2 + a_{10} \) and \( a_4 + a_{12} = a_6 + a_{14} \) we have that \( \delta_1 = a_3 + a_{11}, \delta_2 = a_1 + a_9, \delta_3 = a_6 + a_{14}, \delta_4 = a_7 + a_{15}, \delta_5 = a_4 + a_{12} \) and \( \delta_6 = a_5 + a_{13} \). Therefore \( a_1 = a_3 = a_9 = a_{11}, a_6 = a_4 = a_{12} = a_{14} \) and \( a_5 = a_7 = a_{13} = a_{15} \).
According to the above calculations we get that every $x \in V_2^\ast (FH_{16})$ can be written as
\[
x = \alpha_0 + \alpha_2a^2 + \alpha_8c + \alpha_{10}a^2c + \alpha_4\tilde{C}a + \alpha_4\tilde{C}b + \alpha_5\tilde{C}ab,
\]
where $\tilde{C} = 1 + a^2 + c + a^2c$, so $V_2^\ast (FH_{16})$ is a central subgroup of $V(FH_{16})$.

Let $N = \langle 1+\beta_1\tilde{C}a, 1+\beta_2\tilde{C}b, 1+\beta_3\tilde{C}ab \mid \beta_i \in F \rangle$ be. Clearly, $N \cong C^{3n}_{2}$ and $N \cong a^2N \cong cN \cong a^2cN$ because $a^2\tilde{C} = c\tilde{C} = a^2c\tilde{C} = \tilde{C}$. Since $a^2N \cdot cN = a^2cN$ and the pairwise intersections of $N, a^2N, a^2cN$ are $\{1\}$ we have proved that $V_2^\ast (FG) \cong N \times a^2N \times cN$. Therefore we have $V_2^\ast (FG) \cong V_2^\ast (FG) \cong C^{3n}_{2}$ and the proof is completed.

**Corollary 3.2.** The number of non-isomorphic unitary subgroups of $FH_{16}$ with respect to the involutions which arise from $H_{16}$ is less than $\Lambda_2 = 11$, where $|F| = 2^n \geq 2$.

**References**


