On the Lanzhou index of graphs*

Chenxu Yang, Yaping Mao, Ivan Gutman, Qinghe Tong

Abstract: Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The Lanzhou index of a graph $G$ is defined as $Lz(G) = \sum_{v \in V(G)} d_G(v) d_G(v)^2$, where $d_G(v)$ denotes the degree of the vertex $v$ in $G$. In this paper, we determine extremal values of the Lanzhou index in terms of some graph parameters, as well as Nordhaus–Gaddum–type results. We also find relations between Lanzhou Index and other topological indices.

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1. Introduction

In chemical graph theory, many topological indices have been proposed, playing an important role in the modeling of chemical and physical properties of substances [10, 15]. One of the oldest among them is the first Zagreb index, introduced in the early 1970s [3, 9]. It is defined as

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2,$$

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where $d_G(v)$ denotes the degree of the vertex $v$ in $G$. Another, closely related such structure descriptor is the forgotten index [7],

$$F(G) = \sum_{v \in V(G)} d_G(v)^3.$$ \hfill (1)

Short time ago, Vukičević et al. [17] introduced a further topological index of this kind, the Lanzhou index, and showed that it behaves better than the existing ones in predicting chemically relevant properties. The Lanzhou index of a graph $G$ is defined as

$$Lz(G) = \sum_{v \in V(G)} d_G(v)^2 d_{\overline{G}}(v)$$

where $\overline{G}$ denotes the complement of the graph $G$, and $d_{\overline{G}}(v)$ is the degree of the vertex $v \in V(\overline{G})$.

Of the recently reported mathematical studies of the Lanzhou index, we point out the following.

Bera and Das [15] obtained bounds for $Lz(G)$ in terms of the first, second and third Zagreb indices, spectral radius, eccentric connectivity index, Schultz index, inverse sum indeg index, and symmetric division deg index. They also calculated the Lanzhou index of corona and join of graphs.

Dehgardi and Liu [5] proved that for any tree $T$ of order $n$ with maximum degree $\Delta$,

$$Lz(G) \geq (n - \Delta - 1)(4n + \Delta^2 - 12) + \Delta(n - 2)$$

and characterized the corresponding extremal trees. Wang et al. [18] studied the Lanzhou index of several chemical graph classes, and calculated it for some trees and Cartesian product graphs. They also reported Nordhaus–Gaddum–type results for the Lanzhou index.

In the present paper, in Section 2 we establish upper and lower bounds for $Lz(G)$ of bipartite graphs. In Section 3, we get additional Nordhaus–Gaddum–type results. In Section 4, we study the relations between the Lanzhou index and several other degree–based topological indices.

2. Bounds for the Lanzhou index of graphs

The following observation is immediate [17].

**Observation 2.1.** For a graph $G$, $Lz(G) = (n - 1)M_1(G) - F(G)$.

**Lemma 2.2.** Let $\ell_i \in \mathbb{N}$, $\sum_{i=1}^{k} \ell_i = n$ and $K_{\ell_1, \ell_2, \ldots, \ell_k}$ be the complete $k$-partite graph of order $n$ whose partition sets are of size $\ell_1, \ell_2, \ldots, \ell_k$. Then

$$Lz(K_{\ell_1, \ell_2, \ldots, \ell_k}) = \sum_{p=1}^{k} \ell_p(\ell_p - 1)(n - \ell_p)^2.$$  

**Proof.** Let $V(K_{\ell_1, \ell_2, \ldots, \ell_k}) = V_{\ell_1} \cup \cdots \cup V_{\ell_k}$. For $p \in \{1, 2, \ldots, k\}$, there are $\ell_p$ vertices in $V_{\ell_p}$, each of degree $n - \ell_p$. \hfill $\square$

**Theorem 2.3.** Let $K_{\ell_1, \ell_2}$ be a the complete 2-partite graph of order $n$ whose partition sets are of size $\ell_1, \ell_2$. Then

$$Lz(K_{\ell_1, \ell_2}) \leq \begin{cases} n^3(n - 1), & \text{if } n \text{ is even,} \\ \frac{1}{4}(n^2 - 1)(n - 2), & \text{if } n \text{ is odd.} \end{cases}$$

Equality holds if and only if $\ell_1 = \lfloor n/2 \rfloor$ and $\ell_2 = \lceil n/2 \rceil$. 


\begin{proof}
Let $K_{\ell_1, \ell_2}$ be a bipartite graph of order $n$ ($\ell_1 + \ell_2 = n$). Then
\[
Lz(K_{\ell_1, \ell_2}) = \sum_{p=1}^{2} \ell_p(\ell_p - 1)(n - \ell_p)^2
\]
\[
= [\ell_1^4 - (2n + 1)\ell_1^3 + (n^2 + 2n)\ell_1^2 - n^2\ell_1]
\]
\[
+ [\ell_2^4 - (2n + 1)\ell_2^3 + (n^2 + 2n)\ell_2^2 - n^2\ell_2]
\]
\[
= [\ell_1^4 - (2n + 1)\ell_1^3 + (n^2 + 2n)\ell_1^2 - n^2\ell_1]
\]
\[
+ [(n - \ell_1)^4 - (2n + 1)(n - \ell_1)^3 + (n^2 + 2n)(n - \ell_1)^2 - n^2(n - \ell_1)]
\]
\[
= -n^2\ell_1 + n\ell_1^2 + 2n^2\ell_1^2 - 4n\ell_1^3 + 2\ell_1^3.
\]
Consider a function
\[
f(x) = -n^2x + nx^2 + 2n^2x^2 - 4nx^3 + 2x^4, 1 \leq x \leq \left\lfloor \frac{n}{2} \right\rfloor.
\]
Since
\[
f'(x) = -n^2 + 2nx + 4n^2x - 12nx^2 + 8x^3 = (n - 2x)(-n + 4nx - 4x^2), \; 1 \leq x \leq \left\lfloor \frac{n}{2} \right\rfloor.
\]
It is easy to check that
\[
f'(x) = -4(n - 2x) \left( x - \frac{n - \sqrt{n^2 - n}}{2} \right) \left( x - \frac{n + \sqrt{n^2 - n}}{2} \right), \; 1 \leq x \leq \left\lfloor \frac{n}{2} \right\rfloor.
\]
and $\frac{n - \sqrt{n^2 - n}}{2} \leq \frac{1}{2}$ when $n \geq 1$.

It follows that $f(x)$ is an increasing function on $1 \leq x \leq \lfloor n/2 \rfloor$, and thus
\[
Lz(K_{\ell_1, \ell_2}) \leq \sum_{p=1}^{2} \ell_p(\ell_p - 1)(n - \ell_p)^2
\]
\[
\leq \left\lfloor \frac{n}{2} \right\rfloor \left( \left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \left( n - \left\lfloor \frac{n}{2} \right\rfloor \right)^2 + \left\lceil \frac{n}{2} \right\rceil \left( \left\lceil \frac{n}{2} \right\rceil - 1 \right) \left( n - \left\lceil \frac{n}{2} \right\rceil \right)^2.
\]
Moreover, equality holds if and only if $\ell_1 = \lfloor n/2 \rfloor$ and $\ell_2 = \lfloor n/2 \rfloor$. \qed
\end{proof}

Next we give upper and lower bounds for $Lz(G)$ in terms of $n$, and minimum and maximum vertex degree.

\begin{theorem}
Let $G$ be a graph of order $n$ with maximum degree $\Delta$ and minimum degree $\delta$. Then
\[
\frac{n\delta^2(n - 1 - \Delta)}{2} \leq Lz(G) \leq n\Delta^2(n - 1 - \delta),
\]
with equality (left and right) if and only if $G$ is a regular graph.
\end{theorem}

\begin{proof}
Since $\sum_{v_i \in V(G)} d_G(v_i) = 2m$, it holds
\[
\frac{n\delta}{2} \leq m \leq \frac{n\Delta}{2}
\]
with equality if and only if $d_G(v_i) = \delta = \Delta$ for any $v_i \in V(G)$. From the definition of the Lanzhou index, we then obtain
\[
Lz(G) = \sum_{v \in V(G)} d_G(v) d_G(v)^2 = \sum_{v \in V(G)} [n - d_G(v)] d_G(v)^2
\]
\[
\leq \sum_{v \in V(G)} (n - 1 - \delta) \Delta d(v) = 2m(n - 1 - \delta)\Delta.
\]
\end{proof}
and, analogously
\[ Lz(G) \geq \sum_{v \in V(G)} (n - 1 - \Delta)\delta^2 \geq n(n - 1 - \Delta)\delta^2. \] (3)

By inequalities (2) and (3),
\[ n\delta^2(n - 1 - \Delta) \leq Lz(G) \leq 2m(n - 1 - \delta)\Delta \leq n\Delta^2(n - 1 - \delta). \]

Equalities hold if and only if \( d_G(v_i) = \delta = \Delta \) for any \( v_i \in V(G) \).

**Theorem 2.5.** Let \( G \) be a graph of order \( n \) with maximum degree \( \Delta \), minimum degree \( \delta \) and \( (n-1) = 2\Delta \). Then
\[ Lz(G) \leq \begin{cases} \frac{n\Delta}{2} [\delta^2 + (n - 1 - \delta)^2], & \text{if } \delta + \Delta \leq n - 1, \\ \frac{n\Delta}{2} [\Delta^2 + (n - 1 - \Delta)^2], & \text{if } \delta + \Delta \geq n - 1, \end{cases} \] (4)
with equality holding if and only if \( G \) is \((n-1)/2\)-regular.

**Proof.** Consider the function \( f(x) = x^2 + (n - 1 - x)^2 \) for \( \delta \leq x \leq \Delta \). Then \( f'(x) = 2 - 2n + 4x = 4(x - \frac{n-1}{2}) \), and so \( f(x) \) is increasing on \( \frac{n-1}{2} \leq x \leq \Delta \) and is decreasing on \( \delta \leq x < \frac{n-1}{2} \). For each vertex \( u \in V(G) \),
\[ d_G(u)^2 + [n - 1 - d_G(u)]^2 \leq \begin{cases} \delta^2 + (n - 1 - \delta)^2, & \text{if } \delta + \Delta \leq n - 1, \\ \Delta^2 + (n - 1 - \Delta)^2, & \text{if } \delta + \Delta \geq n - 1, \end{cases} \]
which combined with the definition of the Lanzhou index
\[ Lz(G) = \sum_{u \in V(G)} [n - 1 - d_G(u)]d_G(v)^2 \leq \sum_{u \in V(G)} \frac{\Delta}{2} [(n - 1 - d_G(v))^2 + d_G(v)^2] \]
yields Eq. (4). \( \square \)

**Theorem 2.6.** Let \( G \) be a graph of order \( n \) and size \( m \) with maximum degree \( \Delta \) and minimum degree \( \delta \). Then
\[ 2m[\delta(n - 1) - \Delta^2] \leq Lz(G) \leq \frac{m(n - 1)^2}{2}, \]
with equality if and only if \( n \) is even and \( d_G(v) = d(u) \) holds for all \( u \in V(G) \).

**Proof.** Use the arithmetic–geometric mean inequality:
\[ \prod_{i=1}^{k} x_i \leq \left( \frac{1}{k} \sum_{i=1}^{k} x_i \right)^k, \]
with equality if and only if \( x_1 = \cdots = x_k \).

For \( k = 2 \), let \( x_1 = d_G(v) \) and \( x_2 = d_G(v) \). Then
\[ d_G(v)d_G(v) \leq \left( \frac{d_G(v) + d_G(v)}{2} \right)^2. \]
For any \( v \in V(G) \), \( d_G(v) + d_G(v) \leq n - 1 \), and thus
\[ d_G(v)d_G(v) \leq \left( \frac{d_G(v) + d_G(v)}{2} \right)^2 \leq \frac{(n - 1)^2}{4}. \]
and 
\[
d_G(v) d_{\Gamma}(v) = d_G(v)[n - 1 - d_G(v)] = d_G(v)(n - 1 - d_G(v)^2 \geq (n - 1)\delta - \Delta^2.
\]
Then we obtain
\[
(n - 1)\delta - \Delta^2 \leq d_G(v) d_{\Gamma}(v) \leq \frac{(n - 1)^2}{4},
\]
with equality if and only if \(d_G(v) = \delta = \Delta = d_{\Gamma}(v)\) for any \(v \in V(G)\).

From the definition of the Lanzhou index, we have
\[
Lz(G) = \sum_{v \in V(G)} [d_G(v) d_{\Gamma}(v)] d_G(v) \leq \frac{(n - 1)^2}{4} \sum_{v \in V(G)} d_G(v) \leq \frac{(n - 1)^2 m}{2}
\]
and
\[
Lz(G) = \sum_{v \in V(G)} [d_G(v) d_{\Gamma}(v)] d_G(v) \geq [(n - 1)\delta - \Delta^2] \sum_{v \in V(G)} d_G(v) \geq 2m[(n - 1)\delta - \Delta^2].
\]

**Theorem 2.7.** Let \(G\) be a graph such that the degree each vertex of \(G\) is either \(\Delta\) or \(\delta\). Then
\[
n(n - 1) - n\Delta^3 \leq Lz(G) \leq n(n - 1) - n\delta^3,
\]
with equality if and only if \(G\) is a regular graph.

**Proof.** For the sake of description, let \(\mathcal{S}_\delta = \{v_i : d(v_i) = \delta, v_i \in V(G)\}, \mathcal{S}_\Delta = \{v_i : d(v_i) = \Delta, v_i \in V(G)\}, m_1 = |\mathcal{S}_\delta|\) and \(m_2 = |\mathcal{S}_\Delta|\).

Clearly, \(m_1 + m_2 = n\), we have
\[
Lz(G) = \sum_{i=1}^{n} (n - 1 - d_i)d_i^2 = \sum_{v_i \in \mathcal{S}_\delta} (n - 1 - \delta)\delta^2 + \sum_{v_i \in \mathcal{S}_\Delta} (n - 1 - \Delta)\Delta^2
\]
\[
= m_1(n - 1 - \delta)\delta^2 + m_2(n - 1 - \Delta)\Delta^2
\]
\[
= (m_1 + m_2)(n - 1) - m_1\delta^3 - m_2\Delta^3 = n(n - 1) - m_1\delta^3 - m_2\Delta^3
\]
\[
\leq n(n - 1) - m_1\delta^3 - m_2\Delta^3 = n(n - 1) - (m_1 + m_2)\delta^3 = n(n - 1) - n\delta^3.
\]

Similarly, we can get the following result,
\[
Lz(G) = \sum_{i=1}^{n} (n - 1 - d_i)d_i^2 = \sum_{v_i \in \mathcal{S}_\delta} (n - 1 - \delta)\delta^2 + \sum_{v_i \in \mathcal{S}_\Delta} (n - 1 - \Delta)\Delta^2
\]
\[
= n(n - 1) - m_1\delta^3 - m_2\Delta^3 \geq n(n - 1) - m_1\Delta^3 - m_2\Delta^3 = n(n - 1) - n\Delta^3.
\]

with equality if and only if \(\delta = \Delta\).

### 3. Nordhaus–Gaddum type results

Let \(f(G)\) be a graph invariant and \(n\) a positive integer. The **Nordhaus–Gaddum Problem** is to determine sharp bounds for \(f(G) + f(G)\) and \(f(G) \cdot f(G)\), as \(G\) ranges over the class of all graphs of order \(n\), and to characterize the extremal graphs, i.e., graphs that achieve the bounds. Nordhaus–Gaddum type relations have received wide attention; see the recent survey by Aouchiche and Hansen [2] and the book chapter by Mao [13].
**Theorem 3.1.** Let $G$ be a simple graph of order $n$ and size $m$. Then

$$0 \leq Lz(G) + Lz(\overline{G}) \leq 2m(n-1)^2 - \frac{4m^2(n-1)}{n}.$$  

The upper bounds are attained in case of regular graphs and the lower bounds are attained in case of $G \cong K_n$, where $K_n$ is a complete graph of order $n$.

**Proof.** From the definition of Lanzhou index, we have

$$Lz(G) + Lz(\overline{G}) = \sum_{v \in V(G)} d_G(v)d_G^2(v) + \sum_{v \in V(\overline{G})} d_G(v)d_{\overline{G}}^2(v)$$

$$= \sum_{v \in V(G)} ((n-1 - d_G(v))d_G^2(v) + d_G(v)(n-1 - d_G(v))^2$$

$$= \sum_{v \in V(G)} ((n-1)^2d_G(v) - (n-1)d_G(v)^2)$$

$$= \sum_{v \in V(G)} (n-1)^2d_G(v) - \sum_{v \in V(G)} (n-1)d_G(v)^2$$

$$= 2m(n-1)^2 - (n-1) \sum_{v \in V(G)} d_G(v)^2$$

and

$$\sum_{v \in V(G)} d_G(v)^2 \geq n \left( \frac{\sum_{v \in V(G)} d_G(v)}{n} \right)^2 = n \left( \frac{2m}{n} \right)^2 = \frac{4m^2}{n}.$$ 

$$Lz(G) + Lz(\overline{G}) = 2m(n-1)^2 - (n-1) \sum_{v \in V(G)} d_G(v)^2 \leq 2m(n-1)^2 - \frac{4m^2(n-1)}{n}.$$ 

The above inequality obtains an equal if and only if $d_G(v_i) = d_G(v_j)$, which means $G$ is a regular graph.

On the other hand,

$$Lz(G) + Lz(\overline{G}) = \sum_{v \in V(G)} d_G(v)d_G^2(v) + \sum_{v \in V(\overline{G})} d_G(v)d_{\overline{G}}^2(v) \geq 0$$

we know $Lz(K_n) = Lz(K_n) = 0$, so, the lower bounds are attained in case of $G \cong K_n$, where $K_n$ is a complete graph of order $n$. 

The following corollary is immediate.

**Corollary 3.2.** Let $G$ be a simple graph of order $n$ and size $m$. Then

$$0 \leq Lz(G) Lz(\overline{G}) \leq \frac{m^2(n-1)^2(n-1 - 2m)}{n^2}.$$
4. Relations between Lanzhou and other indices

At this point we recall that the first Zagreb index and the forgotten index satisfy the relations [6]

\[ M_1(G) = \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] \] \quad (5)

and

\[ F(G) = \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2] \]. \quad (6)

4.1. First hyper-Zagreb/forgotten topological index

As an extension of Eq. (5), the hyper-Zagreb index of a graph \( G \) is defined as

\[ HM(G) = \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)]^2. \]


**Theorem 4.1.** Let \( G \) be a simple graph of order \( n \) and size \( m \). Then

\[ \frac{4(n-1)m^2}{n} - HM(G) \leq Lz(G) \leq \frac{(n-1)m}{2} + \frac{(n-3)}{2} HM(G). \]

Equality holds if and only if \( G \cong K_n \).

**Proof.** By Jensen’s inequality,

\[ \sum_{v_i, v_j \in E(G)} (d_G(v_i) + d_G(v_j)) = \sum_{v_i \in V(G)} d_G(v_i)^2 \geq n \left( \frac{1}{n} \sum_{v_i \in V(G)} d_G(v_i) \right)^2 = n \left( \frac{2m}{n^2} \right)^2 = \frac{4m^2}{n}. \]

For any two non-negative real numbers \( x \) and \( y \),

\[ x^2 + y^2 \leq (x + y)^2, \]

with equality if and only if \( xy = 0 \). For \( x = d(v_i) \) and \( x = d(v_j) \), this yields

\[ \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2] \leq \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)]^2, \]

and

\[ \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] \geq \frac{4m^2}{n}. \]

Combining this with the above results, we obtain

\[ Lz(G) = \sum_{v \in V(G)} d_G(v)^2 d_G(v)^2 = \sum_{v \in V(G)} [n - 1 - d_G(v)] d_G(v)^2 \]

\[ = (n - 1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] - \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2] \]

\[ \geq \frac{4m^2(n-1)}{n} - \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2] \geq \frac{4(n - 1)m^2}{n} - HM(G). \]
Since
\[
\sum_{v_i,v_j \in E(G)} [d_G(v_i) + d_G(v_j)] \times 1 \leq \sum_{v_i,v_j \in E(G)} \frac{1}{2} [(d_G(v_i) + d_G(v_j))^2 + 2^2]
\]
\[
\leq \frac{1}{2} \sum_{v_i,v_j \in E(G)} [d_G(v_i) + d_G(v_j)]^2 + \frac{1}{2} \sum_{v_i,v_j \in E(G)} 1
\]
\[
\leq \frac{1}{2} \sum_{v_i,v_j \in E(G)} [d_G(v_i) + d_G(v_j)]^2 + \frac{1}{2} m.
\]
it follows that
\[
(n-1) \sum_{v_i,v_j \in E(G)} [d_G(v_i) + d_G(v_j)] \leq \frac{(n-1)}{2} \sum_{v_i,v_j \in E(G)} [d_G(v_i) + d_G(v_j)]^2 + \frac{(n-1)m}{2},
\]
and hence
\[
Lz(G) = (n-1) \sum_{v_i,v_j \in E(G)} [d_G(v_i) + d_G(v_j)] - \sum_{v_i,v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2]
\]
\[
\leq \frac{(n-1)}{2} \sum_{v_i,v_j \in E(G)} [d_G(v_i) + d_G(v_j)]^2 + \frac{(n-1)m}{2} - \sum_{v_i,v_j \in E(G)} [d_G(v_i) + d_G(v_j)]^2
\]
\[
\leq \frac{(n-1)m}{2} + \frac{(n-3)}{2} \text{HM}(G).
\]

\[\Box\]

**Corollary 4.2.** Let $G$ be a connected graph with $n$ vertices and $m$ edges. Let $F(G)$ be the forgotten index, Eq. (1). Then
\[
Lz(G) \geq \frac{(n-m-2)}{m+1} F(G).
\]

Equality holds if and only if $d_G(v_i) + d_G(v_j) = m + 1$ for all $v_i,v_j \in E(G)$.

**Proof.** Bearing in mind Eq. (6), we have
\[
Lz(G) = (n-1) \sum_{v_i,v_j \in E(G)} [d_G(v_i) + d_G(v_j)] - \sum_{v_i,v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2]
\]
\[
= (n-1) \sum_{v_i,v_j \in E(G)} \frac{(d_G(v_i)^2 + d_G(v_j)^2)}{d_G(v_i) + d_G(v_j)} - \sum_{v_i,v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2]
\]
\[
\geq (n-1) \sum_{v_i,v_j \in E(G)} \frac{d_G(v_i)^2 + d_G(v_j)^2}{m + 1} - \sum_{v_i,v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2]
\]
\[
\geq \frac{n-m-2}{m+1} F(G).
\]

\[\Box\]
4.2. Sombor index

One of the present authors proposed recently a new vertex-degree-based topological index, called Sombor index [8]. It is defined via the term \[ \sqrt{\deg(u)^2 + \deg(v)^2}; \] see [8, 12, 14, 19] for more details.

Theorem 4.3. Let \( G \) be a connected graph of order \( n \) and size \( m \) having minimum vertex degree \( \delta \) and maximum vertex degree \( \Delta \). Then

\[ Lz(G) \geq (n-1)m \left( \frac{\delta}{2} + \frac{\delta^2}{2\sqrt{8\Delta^2 + \delta^2 + 4\Delta}} \right) + (n-1)(1 - \sqrt{2}) \text{SO}(G). \]

The bound is attained in case of regular graphs.

Proof. Since

\[ \sqrt{d_G(v_i)^2 + d_G(v_j)^2} \leq d_G(v_i) + d_G(v_j) - \frac{\delta}{2} - \frac{\delta^2}{2\sqrt{8\Delta^2 + \delta^2 + 4\Delta}}, \]

it follows that

\[ \sqrt{d_G(v_i)^2 + d_G(v_j)^2} + \frac{\delta}{2} + \frac{\delta^2}{2\sqrt{8\Delta^2 + \delta^2 + 4\Delta}} \leq d_G(v_i) + d_G(v_j). \]

Summing both sides of this inequality, we get

\[ (n-1) \sum_{v_i, v_j \in E(G)} \left( \sqrt{d_G(v_i)^2 + d_G(v_j)^2} + \frac{\delta}{2} + \frac{\delta^2}{2\sqrt{8\Delta^2 + \delta^2 + 4\Delta}} \right) \]

\[ \leq (n-1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)]. \]

Thus

\[ \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2] = \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2} \sqrt{d_G(v_i)^2 + d_G(v_j)^2} \]

\[ \geq \sqrt{2\delta} \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2} = \sqrt{2\delta} \text{SO}(G), \]

and

\[ \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2] = \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2} \sqrt{d_G(v_i)^2 + d_G(v_j)^2} \]

\[ \leq (n-1)\sqrt{2} \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2} \]

\[ \leq \sqrt{2}(n-1) \text{SO}(G). \]
Taking all this into account, we get
\[
\text{Lz}(G) = (n - 1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] - \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2]
\]
\[
\geq (n - 1) \sum_{v_i, v_j \in E(G)} \left( \sqrt{d_G(v_i)^2 + d_G(v_j)^2} + \frac{\delta}{2} + \frac{\delta^2}{2\sqrt{8\Delta^2 + \delta^2 + 4\Delta \sqrt{2}}} \right)
\]
\[
- (n - 1)\sqrt{2} \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2}
\]
\[
\geq (n - 1) \left( \frac{\delta}{2} + \frac{\delta^2}{2\sqrt{8\Delta^2 + \delta^2 + 4\Delta \sqrt{2}}} \right) + (n - 1)(1 - \sqrt{\delta}) \text{SO}(G).
\]

\[\square\]

**Theorem 4.4.** Let \( G \) be the same as in Theorem 4.3. Then
\[
\text{Lz}(G) \leq (n - 1 - \sqrt{\delta}) \sqrt{2} \text{SO}(G).
\]

Equality holds if and only if \( G \) is an empty (edgeless) graph.

**Proof.** For any two non-negative real numbers \( x \) and \( y \),
\[
x + y \leq \sqrt{2(x^2 + y^2)}
\]
with equality if and only if \( x = y \). Then for \( x = d_G(v_i) \) and \( y = d_G(v_j) \),
\[
\sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] \leq \sum_{v_i, v_j \in E(G)} \sqrt{2[d_G(v_i)^2 + d_G(v_j)^2]}
\]
and thus
\[
\text{Lz}(G) = (n - 1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] - \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2]
\]
\[
\leq (n - 1)\sqrt{2} \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2}
\]
\[
- \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2} \sqrt{d_G(v_i)^2 + d_G(v_j)^2}
\]
\[
\leq (n - 1)\sqrt{2} \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2} - \sqrt{2\delta} \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i)^2 + d_G(v_j)^2}
\]
\[
\leq \sqrt{2}[n - 1 - \sqrt{\delta}] \text{SO}(G).
\]

\[\square\]

### 4.3. Randić index

The *Randić index* is defined as [20]
\[
R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u) d_G(u)}}
\]
whereas the reduced reciprocal Randić index is
\[
RR(G) = \sum_{uv \in E(G)} \sqrt{d_G(u) d_G(u)}.
\] (8)

This latter index has the second-best correlating ability among many vertex–degree–based molecular structure descriptors; see [1] for more details.

**Theorem 4.5.** Let \( G \) be a simple graph of order \( n \) and size \( m \) having minimum degree \( \delta \) and maximum degree \( \Delta \). Then
\[
2[\delta^2(n - 1) - \Delta^3] \text{R}(G) \leq Lz(G) \leq [\Delta(m - 1)(n - 1) - \delta^3] \text{R}(G).
\]

Equalities hold if and only if \( G \) is regular and satisfies \( d_G(v_i) + d_G(v_j) = m + 1 \) for \( v_i, v_j \in E(G) \).

**Proof.** Note that
\[
Lz(G) = (n - 1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] - \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2]
\]
\[
= (n - 1) \sum_{v_i, v_j \in E(G)} \frac{1}{\sqrt{d_G(v_i)d_G(v_j)}} \sqrt{d_G(v_i)d_G(v_j)} [d_G(v_i) + d_G(v_j)]
\]
\[
- \sum_{v_i, v_j \in E(G)} \frac{1}{\sqrt{d_G(v_i)d_G(v_j)}} \sqrt{d_G(v_i)d_G(v_j)} [d_G(v_i)^2 + d_G(v_j)^2]
\]
\[
= (n - 1) \sum_{v_i, v_j \in E(G)} \frac{1}{\sqrt{d_G(v_i)d_G(v_j)}} \Delta(m + 1) - \sum_{v_i, v_j \in E(G)} \frac{1}{\sqrt{d_G(v_i)d_G(v_j)}} \delta^3
\]
\[
\leq [\Delta(m - 1)(n - 1) - \delta^3] \text{R}(G).
\]

The lower bound can be verified in an analogous manner. \( \square \)

**Theorem 4.6.** Let \( G \) be a connected graph of order \( n \) with \( m \) edges having minimum vertex degree \( \delta \) and maximum vertex degree \( \Delta \). Then
\[
\left(2(n - 1) - \frac{2\Delta^2}{\delta}\right) \text{RR}(G) \leq Lz(G) \leq \left(\sqrt{2}(n - 1)\left(\frac{\Delta}{\delta} + \frac{\delta}{\Delta}\right) - 2\right) \text{RR}(G),
\]
with equality if and only if \( G \) is regular.

**Proof.** Using an analogous reasoning as in the proof of Theorem 4.4, we have

\[
Lz(G) = (n - 1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] - \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2]
\]
\[
\leq (n - 1) \sqrt{2} \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i)d_G(v_j)} \sqrt{\frac{d_G(v_i)}{d_G(v_i)}} + \frac{d_G(v_j)}{d_G(v_i)} - \sum_{v_i, v_j \in E(G)} 2 \sqrt{d_G(v_i)d_G(v_j)}
\]
\[
\leq (n - 1) \sqrt{2} \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i)d_G(v_j)} \sqrt{\frac{\Delta}{\delta} + \frac{\delta}{\Delta}} - \sum_{v_i, v_j \in E(G)} 2 \sqrt{d_G(v_i)d_G(v_j)}.
\]
Similarly,
\[ Lz(G) = (n - 1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] - \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2] \]
\[ \geq (n - 1) \sum_{v_i, v_j \in E(G)} 2\sqrt{d_G(v_i) d_G(v_j)} - \frac{2\Delta^2}{\delta} \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i) d_G(v_j)} \]
\[ \geq \left(2(n - 1) - \frac{2\Delta^2}{\delta}\right) RR(G). \]

4.4. Symmetric division deg index

The symmetric division deg index is defined as
\[ SDD(G) = \sum_{v_i, v_j \in E(G)} \frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i) d_G(v_j)}. \]

This index, proposed several years ago by Vukičević et al. [16], was found to be useful in predicting physico-chemical properties of molecules.

**Theorem 4.7.** Let \( G \) be same as in Theorem 4.6. Then
\[ \left(\frac{2(n - 1)\Delta \delta^2}{\Delta^2 + \delta^2} - \Delta^2\right) SDD(G) \leq Lz(G) \leq \left(\frac{(n - 1)(m + 1)\Delta}{2\delta} - \delta^2\right) SDD(G), \]
with equality if and only if \( G \) is regular and satisfies \( d_G(v_i) + d_G(v_j) = m + 1 \) for all \( v_i, v_j \in E(G) \).

**Proof.**
\[ Lz(G) = (n - 1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] - \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2] \]
\[ = (n - 1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i) d_G(v_j)}\right) \left(\frac{d_G(v_i) d_G(v_j)}{d_G(v_i) d_G(v_j)}\right) \]
\[ - \sum_{v_i, v_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i) d_G(v_j)}\right) d_G(v_i) d_G(v_j) \]
\[ = (n - 1) \sum_{v_i, v_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i) d_G(v_j)}\right) \left(\frac{d_G(v_i) d_G(v_j)(d_G(v_i) + d_G(v_j))}{d_G(v_i)^2 + d_G(v_j)^2}\right) \]
\[ - \sum_{v_i, v_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i) d_G(v_j)}\right) d_G(v_i) d_G(v_j) \]
\[ = (n - 1) \sum_{v_i, v_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i) d_G(v_j)}\right) \left(\frac{d_G(v_i) + d_G(v_j)}{d_G(v_i)}\right) \left(\frac{d_G(v_i) + d_G(v_j)}{d_G(v_j)}\right). \]
Similarly, we obtain
\[
\begin{align*}
Lz(G) &= (n-1) \sum_{v_i, v_j \in E(G)} \left( \frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i) d_G(v_j)} \right) d_G(v_i) d_G(v_j) \\
&\leq (n-1) \sum_{v_i, v_j \in E(G)} \left( \frac{d_G(v_i)^2 + d_G(v_j)^2}{2 \delta} \right) - \sum_{v_i, v_j \in E(G)} \left( \frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i) d_G(v_j)} \right) \delta^2 \\
&\leq \left( \frac{(n-1)(m+1)}{2 \delta} - \delta^2 \right) \text{SDD}(G).
\end{align*}
\]

Similarly, we obtain
\[
\begin{align*}
Lz(G) &= (n-1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] - \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2] \\
&= (n-1) \sum_{v_i, v_j \in E(G)} \left( \frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i) d_G(v_j)} \right) \left( d_G(v_i) d_G(v_j) \right) \\
&= (n-1) \sum_{v_i, v_j \in E(G)} \left( \frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i) d_G(v_j)} \right) \left( d_G(v_i) d_G(v_j) \left( d_G(v_i) + d_G(v_j) \right) \right) \\
&= (n-1) \sum_{v_i, v_j \in E(G)} \left( \frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i) d_G(v_j)} \right) \left( \frac{d_G(v_i) + d_G(v_j)}{d_G(v_i) d_G(v_j)} \right) \\
&\geq (n-1) \sum_{v_i, v_j \in E(G)} \left( \frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i) d_G(v_j)} \right) \left( \frac{2 \delta}{\delta + \frac{4}{2}} \right) - \sum_{v_i, v_j \in E(G)} \left( \frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i) d_G(v_j)} \right) \Delta^2 \\
&\geq \left( \frac{(n-1)(2 \delta)}{\delta + \frac{4}{2}} - \Delta^2 \right) - \sum_{v_i, v_j \in E(G)} \left( \frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i) d_G(v_j)} \right) \\
&= \left( \frac{2(n-1) \Delta \delta^2}{\Delta^2 + \delta^2} - \Delta^2 \right) \text{SDD}(G).
\end{align*}
\]
4.5. Reciprocal sum-connectivity index

Motivated by the definition of the Randić index, Eq. (7), its variant, called sum-connectivity index was recently proposed [21], defined as

\[ \text{SC}(G) = \sum_{v_i, v_j \in E(G)} \frac{1}{\sqrt{d_G(v_i) + d_G(v_j)}}. \]

In parallel to Eq. (8), the reciprocal sum-connectivity index of a graph \( G \) is

\[ \text{RSC}(G) = \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i) + d_G(v_j)}. \]

**Theorem 4.8.** Let \( G \) be a connected graph of order \( n \) with \( m \) edges, having minimum vertex degree \( \delta \) and maximum vertex degree \( \Delta \). Then

\[ \left( \sqrt{2(n-1)\delta} - \frac{\sqrt{2}\Delta^2}{\delta} \right) \text{RSC}(G) \leq \text{Lz}(G) \leq \left( \sqrt{2(n-1)\Delta} - \frac{\sqrt{2}\delta^2}{\Delta} \right) \text{RSC}(G), \]

with equalities if and only if \( G \) is regular.

**Proof.** Starting with

\[ d_G(v_i) + d_G(v_i) = \sqrt{d_G(v_i) + d_G(v_i)} \]

we obtain

\[ \sqrt{2\delta} \sqrt{d_G(v_i) + d_G(v_i)} \leq \sqrt{d_G(v_i) + d_G(v_i)} \sqrt{d_G(v_i) + d_G(v_i)} \leq \sqrt{2\Delta} \sqrt{d_G(v_i) + d_G(v_i)}, \]

and thus

\[ \sqrt{2\delta}(n-1) \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i) + d_G(v_j)} \leq (n-1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] \]

\[ \leq \sqrt{2\Delta}(n-1) \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i) + d_G(v_j)}, \]

which means that

\[ \sqrt{2\delta}(n-1) \text{RSC}(G) \leq (n-1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] \leq \sqrt{2\Delta}(n-1) \text{RSC}(G). \]

Using the relation

\[ d_G(v_i)^2 + d_G(v_i)^2 = \frac{d_G(v_i)^2 + d_G(v_i)^2}{\sqrt{d_G(v_i) + d_G(v_j)}} \sqrt{d_G(v_i) + d_G(v_j)} \]

we have

\[ \frac{2\delta^2}{\sqrt{2}\Delta} \sqrt{d_G(v_i) + d_G(v_j)} \leq \frac{d_G(v_i)^2 + d_G(v_i)^2}{\sqrt{d_G(v_i) + d_G(v_j)}} \sqrt{d_G(v_i) + d_G(v_j)} \leq \frac{2\Delta^2}{\sqrt{2}\delta} \sqrt{d_G(v_i) + d_G(v_j)}, \]

from which it follows

\[ \frac{2\delta^2}{\sqrt{2}\Delta} \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i) + d_G(v_j)} \leq \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_i)^2] \]

\[ \leq \frac{2\Delta^2}{\sqrt{2}\delta} \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i) + d_G(v_j)} \]
and
\[ \text{Lz}(G) = (n - 1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] - \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2] \]
\[ \leq \sqrt{2\Delta}(n - 1) \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i) + d_G(v_j)} - \frac{2\Delta^2}{\sqrt{2\delta}} \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i) + d_G(v_j)} \]
\[ \leq \left( \sqrt{2\Delta}(n - 1) - \frac{\sqrt{2\delta^2}}{\Delta} \right) \text{RSC}(G) \]
as well as
\[ \text{Lz}(G) = (n - 1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] - \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2] \]
\[ \geq \sqrt{2\delta}(n - 1) \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i) + d_G(v_j)} - \frac{2\Delta^2}{\sqrt{2\delta}} \sum_{v_i, v_j \in E(G)} \sqrt{d_G(v_i) + d_G(v_j)} \]
\[ \geq \left( \sqrt{2}(n - 1)\delta - \frac{\sqrt{2\Delta^2}}{\delta} \right) \text{RSC}(G). \]

\[ \Box \]

**Theorem 4.9.** Let \( G \) be same as in Theorem 4.8. Then
\[ \left[ 2\sqrt{2}(n - 1)\delta^2 - 2\Delta^2 \sqrt{2(m+1)} \right] \text{SC}(G) \leq \text{Lz}(G) \leq \left[ (n-1)(m+1)^2 - 2\sqrt{2}\delta^2 \right] \text{SC}(G). \]
The above equalities hold if and only if \( G \) is regular and satisfies \( d_G(v_i) + d_G(v_j) = m + 1 \) for all \( v_i, v_j \in E(G) \).

**Proof.** Using the identity
\[ d_G(v_i) + d_G(v_j) = \frac{d_G(v_i) + d_G(v_j)}{\sqrt{d_G(v_i) + d_G(v_j)}} \sqrt{d_G(v_i) + d_G(v_j)} \]
we obtain
\[ \frac{2\delta \sqrt{2\delta}}{\sqrt{d_G(v_i) + d_G(v_j)}} \leq d_G(v_i) + d_G(v_j) \leq \frac{(m+1)\sqrt{m+1}}{\sqrt{d_G(v_i) + d_G(v_j)}}. \]
which after summation yields
\[ 2\delta \sqrt{2\delta} \sum_{v_i, v_j \in E(G)} \frac{1}{\sqrt{d_G(v_i) + d_G(v_j)}} \leq \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] \]
\[ \leq (m+1)\sqrt{m+1} \sum_{v_i, v_j \in E(G)} \frac{1}{\sqrt{d_G(v_i) + d_G(v_j)}}. \]
This means that
\[ 2\delta \sqrt{2\delta} \text{SC}(G) \leq \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] \leq (m+1)\sqrt{m+1} \text{SC}(G). \]
implying
\[(2\delta)^{\frac{3}{2}} \text{SC}(G) \leq \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] \leq 2\Delta^2 \sqrt{m+1} \text{SC}(G).\]

Similarly,
\[(2\delta^2)\sqrt{2}\delta \text{SC}(G) \leq \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2] \leq (2\Delta^2 \sqrt{m+1}) \text{SC}(G)\]

and thus
\[Lz(G) = (n-1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] - \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2]\]
\[\leq (m+1)(n-1)\sqrt{(m+1)} \text{SC}(G) - 2\sqrt{2}\delta^\frac{3}{2} \text{SC}(G)\]
\[= \left[(n-1)(m+1)^{\frac{3}{2}} - 2\sqrt{2}\delta^\frac{3}{2}\right] \text{SC}(G)\]

and
\[Lz(G) = (n-1) \sum_{v_i, v_j \in E(G)} [d_G(v_i) + d_G(v_j)] - \sum_{v_i, v_j \in E(G)} [d_G(v_i)^2 + d_G(v_j)^2]\]
\[\geq 2(n-1)\delta \sqrt{(2\delta)} \text{SC}(G) - 2\Delta^2 \sqrt{2(m+1)} \text{SC}(G)\]
\[= \left[2\sqrt{2}(n-1)\delta^\frac{3}{2} - 2\Delta^2 \sqrt{2(m+1)}\right] \text{SC}(G).\]

Combining the above two results, we arrive at Theorem 4.9

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**References**