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On the Lanzhou index of graphs*

Research Article

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Abstract: Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The Lanzhou index of a graph G is defined as $\mathrm{Lz}(G) = \sum_{u \in V(G)} d_{\overline{G}}(v)\, d_G(v)^2$, where $d_G(v)$ denotes the degree of the vertex v in G. In this paper, we determine extremal values of the Lanzhou index in terms of some graph parameters, as well as Nordhaus–Gaddum–type results. We also find relations between Lanzhou Index and other topological indices.

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1. Introduction

In chemical graph theory, many topological indices have been proposed, playing an important role in the modeling of chemical and physical properties of substances [10, 15]. One of the oldest among them is the *first Zagreb index*, introduced in the early 1970s [3, 9]. It is defined as

$$\mathrm{M}_1(G) = \sum_{v \in V(G)} d_G(v)^2 \,,$$

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where $d_G(v)$ denotes the degree of the vertex v in G. Another, closely related such structure descriptor is the *forgotten index* [7],

$$\mathrm{F}(G) = \sum_{v \in V(G)} d_G(v)^3 \,. \tag{1}$$

Short time ago, Vukičević et al. [17] introduced a further topological index of this kind, the *Lanzhou index*, and showed that it behaves better than the existing ones in predicting chemically relevant properties. The Lanzhou index of a graph G is defined as

$$\mathrm{Lz}(G) = \sum_{v \in V(G)} d_{\overline{G}}(v)\, d_G(v)^2$$

where $\overline{G}$ denotes the complement of the graph G, and $d_{\overline{G}}(v)$ is the degree of the vertex $v \in V(\overline{G})$.

Of the recently reported mathematical studies of the Lanzhou index, we point out the following.

Bera and Das [?] obtained bounds for $\mathrm{Lz}(G)$ in terms of the first, second and third Zagreb indices, spectral radius, eccentric connectivity index, Schultz index, inverse sum indeg index, and symmetric division deg index. They also calculated the Lanzhou index of corona and join of graphs.

Dehgardia and Liu [5] proved that for any tree T of order n with maximum degree Δ,

$$\mathrm{Lz}(G) \geq (n - \Delta - 1)(4n + \Delta^2 - 12) + \Delta(n - 2)$$

and characterized the corresponding extremal trees. Wang et al. [18] studied the Lanzhou index of several chemical graph classes, and calculated it for some trees and Cartesian product graphs. They also reported Nordhaus–Gaddum–type results for the Lanzhoul index.

In the present paper, in Section 2 we establish upper and lower bounds for $\mathrm{Lz}(G)$ of bipartite graphs. In Section 3, we get additional Nordhaus–Gaddum–type results. In Section 4, we study the relations between the Lanzhou index and several other degree–based topological indices.

2. Bounds for the Lanzhou index of graphs

The following observation is immediate [17].

Observation 2.1. *For a graph G,* $\mathrm{Lz}(G) = (n-1)\,\mathrm{M}_1(G) - F(G)$.

Lemma 2.2. *Let $\ell_i \in \mathbb{N}$, $\sum_{i=1}^{i=k} \ell_i = n$ and $K_{\ell_1,\ell_2,\cdots,\ell_k}$ be the complete k-partite graph of order n whose partition sets are of size $\ell_1, \ell_2, \cdots, \ell_k$. Then*

$$\mathrm{Lz}(K_{\ell_1,\ell_2,\cdots,\ell_k}) = \sum_{p=1}^{k} \ell_p(\ell_p - 1)(n - \ell_p)^2.$$

Proof. Let $V(K_{\ell_1,\ell_2,\cdots,\ell_k}) = V_{\ell_1} \cup \cdots \cup V_{\ell_k}$. For $p \in \{1, 2, \cdots, k\}$, there are ℓ_p vertices in V_{ℓ_p}, each of degree $n - \ell_p$. □

Theorem 2.3. *Let K_{ℓ_1,ℓ_2} be a the complete 2-partite graph of order n whose partition sets are of size ℓ_1, ℓ_2. Then*

$$\mathrm{Lz}(K_{\ell_1,\ell_2}) \leq \begin{cases} n^3(n-1), & \text{if } n \text{ is even,} \\ \frac{1}{4}(n^2-1)(n-2), & \text{if } n \text{ is odd.} \end{cases}$$

Equality holds if and only if $\ell_1 = \lceil n/2 \rceil$ and $\ell_2 = \lfloor n/2 \rfloor$.

Proof. Let K_{ℓ_1,ℓ_2} be a bipartite graph of order n ($\ell_1+\ell_2=n$). Then

$$\begin{aligned}\mathrm{Lz}(K_{\ell_1,\ell_2}) &= \sum_{p=1}^{2}\ell_p(\ell_p-1)(n-\ell_p)^2\\ &= \left[\ell_1^4-(2n+1)\ell_1^3+(n^2+2n)\ell_1^2-n^2\ell_1\right]\\ &+\left[\ell_2^4-(2n+1)\ell_2^3+(n^2+2n)\ell_2^2-n^2\ell_2\right]\\ &= \left[\ell_1^4-(2n+1)\ell_1^3+(n^2+2n)\ell_1^2-n^2\ell_1\right]\\ &+\left[(n-\ell_1)^4-(2n+1)(n-\ell_1)^3+(n^2+2n)(n-\ell_1)^2-n^2(n-\ell_1)\right]\\ &= -n^2\ell_1+n\ell_1^2+2n^2\ell_1^2-4n\ell_1^3+2\ell_1^4.\end{aligned}$$

Consider a function

$$f(x)=-n^2x+nx^2+2n^2x^2-4nx^3+2x^4, 1\le x\le\left\lfloor\frac{n}{2}\right\rfloor.$$

Since

$$f'(x)=-n^2+2nx+4n^2x-12nx^2+8x^3=(n-2x)(-n+4nx-4x^2)\ ,\ 1\le x\le\left\lfloor\frac{n}{2}\right\rfloor.$$

It is easy to check that

$$f'(x)=-4(n-2x)\left(x-\frac{n-\sqrt{n^2-n}}{2}\right)\left(x-\frac{n+\sqrt{n^2-n}}{2}\right)\ ,\ 1\le x\le\left\lfloor\frac{n}{2}\right\rfloor.$$

and $\frac{n-\sqrt{n^2-n}}{2}\le\frac{1}{2}$ when $n\ge 1$.

It follows that $f(x)$ is an increasing function on $1\le x\le\lfloor n/2\rfloor$, and thus

$$\begin{aligned}\mathrm{Lz}(K_{\ell_1,\ell_2}) &\le \sum_{p=1}^{2}\ell_p(\ell_p-1)(n-\ell_p)^2\\ &\le \left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right)^2+\left\lceil\frac{n}{2}\right\rceil\left(\left\lceil\frac{n}{2}\right\rceil-1\right)\left(n-\left\lceil\frac{n}{2}\right\rceil\right)^2.\end{aligned}$$

Moreover, equality holds if and only if $\ell_1=\lceil n/2\rceil$ and $\ell_2=\lfloor n/2\rfloor$. □

Next we give upper and lower bounds for $\mathrm{Lz}(G)$ in terms of n, and minimum and maximum vertex degree.

Theorem 2.4. *Let G be a graph of order n with maximum degree Δ and minimum degree δ. Then*

$$n\delta^2(n-1-\Delta)\le\mathrm{Lz}(G)\le n\Delta^2\,(n-1-\delta),$$

with equality (left and right) if and only if G is a regular graph.

Proof. Since $\sum_{v_i\in V(G)}d_G(v_i)=2m$, it holds

$$\frac{n\delta}{2}\le m\le\frac{n\Delta}{2}$$

with equality if and only if $d_G(v_i)=\delta=\Delta$ for any $v_i\in V(G)$. From the definition of the Lanzhou index, we then obtain

$$\begin{aligned}\mathrm{Lz}(G) &= \sum_{v\in V(G)}d_{\overline{G}}(v)\,d_G(v)^2=\sum_{v\in V(G)}[n-1-d_G(v)]\,d_G(v)^2\\ &\le \sum_{v\in V(G)}(n-1-\delta)\Delta d(v)=2m(n-1-\delta)\Delta. \qquad (2)\end{aligned}$$

and, analogously

$$\mathrm{Lz}(G) \geq \sum_{v\in V(G)} (n-1-\Delta)\delta^2 \geq n(n-1-\Delta)\delta^2 \,. \tag{3}$$

By inequalities (2) and (3),

$$n\delta^2(n-1-\Delta) \leq \mathrm{Lz}(G) \leq 2m(n-1-\delta)\Delta \leq n\Delta^2(n-1-\delta)\,.$$

Equalities hold if and only if $d_G(v_i) = \delta = \Delta$ for any $v_i \in V(G)$. □

Theorem 2.5. *Let G be a graph of order n with maximum degree Δ, minimum degree δ and $(n-1) = 2\Delta$. Then*

$$\mathrm{Lz}(G) \leq \begin{cases} \frac{n\Delta}{2}\left[\delta^2 + (n-1-\delta)^2\right], & \text{if } \delta+\Delta \leq n-1, \\ \frac{n\Delta}{2}\left[\Delta^2 + (n-1-\Delta)^2\right], & \text{if } \delta+\Delta \geq n-1, \end{cases} \tag{4}$$

with equality holding if and only if G is $(n-1)/2$-regular.

Proof. Consider the function $f(x) = x^2 + (n-1-x)^2$ for $\delta \leq x \leq \Delta$. Then $f'(x) = 2-2n+4x = 4(x-\frac{n-1}{2})$, and so $f(x)$ is increasing on $\frac{n-1}{2} \leq x \leq \Delta$ and is decreasing on $\delta \leq x < \frac{n-1}{2}$. For each vertex $u \in V(G)$,

$$d_G(u)^2 + \big[n-1-d_G(u)\big]^2 \leq \begin{cases} \delta^2 + (n-1-\delta)^2\,, & \text{if } \delta+\Delta \leq n-1, \\ \Delta^2 + (n-1-\Delta)^2\,, & \text{if } \delta+\Delta \geq n-1\,, \end{cases}$$

which combined with the definition of the Lanzhou index

$$\mathrm{Lz}(G) = \sum_{u\in V(G)} \big[n-1-d_G(v)\big]d_G(v)^2 \leq \sum_{u\in V(G)} \frac{\Delta}{2}\left[(n-1-d_G(v))^2 + d_G(v)^2\right]$$

yields Eq. (4). □

Theorem 2.6. *Let G be a graph of order n and size m with maximum degree Δ and minimum degree δ. Then*

$$2m\big[\delta(n-1)-\Delta^2\big] \leq \mathrm{Lz}(G) \leq \frac{m(n-1)^2}{2},$$

with equality if and only if n is even and $d_{\overline{G}}(u) = d(u)$ holds for all $u \in V(G)$.

Proof. Use the arithmetic–geometric mean inequality:

$$\prod_{i=1}^{k} x_i \leq \left(\frac{1}{k}\sum_{i=1}^{k} x_i\right)^k,$$

with equality if and only if $x_1 = \cdots = x_k$.

For $k = 2$, let $x_1 = d_{\overline{G}}(v)$ and $x_2 = d_G(v)$. Then

$$d_G(v)\, d_{\overline{G}}(v) \leq \left(\frac{d_G(v) + d_{\overline{G}}(v)}{2}\right)^2.$$

For any $v \in V(G)$, $d_G(v) + d_{\overline{G}}(v) \leq n-1$, and thus

$$d_G(v)\, d_{\overline{G}}(v) \leq \left(\frac{d_G(v) + d_{\overline{G}}(v)}{2}\right)^2 \leq \frac{(n-1)^2}{4}$$

and

$$d_G(v)\,d_{\overline{G}}(v) = d_G(v)\big[n-1-d_G(v)\big] = d_G(v)(n-1) - d_G(v)^2 \geq (n-1)\delta - \Delta^2\,.$$

Then we obtain

$$(n-1)\delta - \Delta^2 \leq d_G(v)\,d_{\overline{G}}(v) \leq \frac{(n-1)^2}{4},$$

with equality if and only if $d_G(v) = \delta = \Delta = d_{\overline{G}}(v)$ for any $v \in V(G)$.

From the definition of the Lanzhou index, we have

$$\mathrm{Lz}(G) = \sum_{v\in V(G)} \big[d_G(v)\,d_{\overline{G}(v)}\big] d_G(v) \leq \frac{(n-1)^2}{4} \sum_{v\in V(G)} d_G(v) \leq \frac{(n-1)^2\,m}{2}$$

and

$$\mathrm{Lz}(G) = \sum_{v\in V(G)} \big[d_G(v)\,d_{\overline{G}}(v)\big] d_G(v) \geq \big[(n-1)\delta - \Delta^2\big] \sum_{v\in V(G)} d_G(v) \geq 2m\big[(n-1)\delta - \Delta^2\big].$$

□

Theorem 2.7. *Let G be a graph such that the degree each vertex of G is either Δ or δ. Then*

$$n(n-1) - n\Delta^3 \leq \mathrm{Lz}(G) \leq n(n-1) - n\delta^3,$$

with equality if and only if G is a regular graph.

Proof. For the sake of description, let $\mathfrak{S}_\delta = \{v_i \,|\, d(v_i) = \delta,\ v_i \in V(G)\}$, $\mathfrak{S}_\Delta = \{v_i \,|\, d(v_i) = \Delta,\ v_i \in V(G)\}$, $m_1 = |\mathfrak{S}_\delta|$ and $m_2 = |\mathfrak{S}_\Delta|$.

Clearly, $m_1 + m_2 = n$, we have

$$\begin{aligned}
\mathrm{Lz}(G) &= \sum_{i=1}^{n} (n-1-d_i)d_i^2 = \sum_{v_i\in\mathfrak{S}_\delta} (n-1-\delta)\delta^2 + \sum_{v_i\in\mathfrak{S}_\Delta} (n-1-\Delta)\Delta^2 \\
&= m_1(n-1-\delta)\delta^2 + m_2(n-1-\Delta)\Delta^2 \\
&= (m_1+m_2)(n-1) - m_1\delta^3 - m_2\Delta^3 = n(n-1) - m_1\delta^3 - m_2\Delta^3 \\
&\leq n(n-1) - m_1\delta^3 - m_2\delta^3 = n(n-1) - (m_1+m_2)\delta^3 = n(n-1) - n\delta^3
\end{aligned}$$

Similarly, we can get the following result,

$$\begin{aligned}
\mathrm{Lz}(G) &= \sum_{i=1}^{n} (n-1-d_i)d_i^2 = \sum_{v_i\in\mathfrak{S}_\delta} (n-1-\delta)\delta^2 + \sum_{v_i\in\mathfrak{S}_\Delta} (n-1-\Delta)\Delta^2 \\
&= n(n-1) - m_1\delta^3 - m_2\Delta^3 \geq n(n-1) - m_1\Delta^3 - m_2\Delta^3 = n(n-1) - n\Delta^3
\end{aligned}$$

with equality if and only if $\delta = \Delta$. □

3. Nordhaus–Gaddum type results

Let $f(G)$ be a graph invariant and n a positive integer. The *Nordhaus–Gaddum Problem* is to determine sharp bounds for $f(G)+f(\overline{G})$ and $f(G)\cdot f(\overline{G})$, as G ranges over the class of all graphs of order n, and to characterize the extremal graphs, i.e., graphs that achieve the bounds. Nordhaus–Gaddum type relations have received wide attention; see the recent survey by Aouchiche and Hansen [2] and the book chapter by Mao [13].

Theorem 3.1. *Let G be a simple graph of order n and size m. Then*

$$0 \leq \mathrm{Lz}(G) + \mathrm{Lz}(\overline{G}) \leq 2m(n-1)^2 - \frac{4m^2(n-1)}{n}.$$

The upper bounds are attained in case of regular graphs and the lower bounds are attained in case of $G \cong K_n$, where K_n is a complete graph of order n.

Proof. From the definition of Lanzhou index, we have

$$\begin{aligned}\mathrm{Lz}(G) + \mathrm{Lz}(\overline{G}) &= \sum_{v \in V(G)} d_{\overline{G}}(v) d_G^2(v) + \sum_{v \in V(\overline{G})} d_G(v) d_{\overline{G}}^2(v) \\ &= \sum_{v \in V(G)} (n-1-d_G(v)) d_G^2(v) + \sum_{v \in V(G)} d_G(v)(n-1-d_G(v))^2 \\ &= \sum_{v \in V(G)} \big((n-1)^2 d_G(v) - (n-1) d_G(v)^2\big) \\ &= \sum_{v \in V(G)} (n-1)^2 d_G(v) - \sum_{v \in V(G)} (n-1) d_G(v)^2 \\ &= 2m(n-1)^2 - (n-1) \sum_{v \in V(G)} d_G(v)^2\end{aligned}$$

and

$$\sum_{v_i \in V(G)} d_G(v_i)^2 \geq n \left(\frac{\sum_{v_i \in V(G)} d_G(v_i)}{n} \right)^2 = n \frac{(2m)^2}{n^2} = \frac{4m^2}{n}.$$

$$\mathrm{Lz}(G) + \mathrm{Lz}(\overline{G}) = 2m(n-1)^2 - (n-1) \sum_{v \in V(G)} d_G(v)^2 \leq 2m(n-1)^2 - \frac{4m^2(n-1)}{n}.$$

The above inequality obtains an equal if and only if $d_G(v_i) = d_G(v_j)$, which means G is a regular graph.

On the other hand,

$$\mathrm{Lz}(G) + \mathrm{Lz}(\overline{G}) = \sum_{v \in V(G)} d_{\overline{G}}(v) d_G^2(v) + \sum_{v \in V(\overline{G})} d_G(v) d_{\overline{G}}^2(v) \geq 0$$

we know $\mathrm{Lz}(K_n) = \mathrm{Lz}(\overline{K_n}) = 0$, so, the lower bonds are attained in case of $G \cong K_n$, where K_n is a complete graph of order n. □

The following corollary is immediate.

Corollary 3.2. *Let G be a simple graph of order n and size m. Then*

$$0 \leq \mathrm{Lz}(G)\,\mathrm{Lz}(\overline{G}) \leq \frac{m^2(n-1)^2(n-1-2m)^2}{n^2}.$$

4. Relations between Lanzhou and other indices

At this point we recall that the first Zagreb index and the forgotten index satisfy the relations [6]

$$\mathrm{M}_1(G) = \sum_{v_i v_j \in E(G)} \left[d_G(v_i) + d_G(v_j)\right] \tag{5}$$

and

$$\mathrm{F}(G) = \sum_{v_i v_j \in E(G)} \left[d_G(v_i)^2 + d_G(v_j)^2\right]. \tag{6}$$

4.1. First hyper-Zagreb/forgotten topological index

As an extension of Eq. (5), the *hyper-Zagreb index* of a graph G is defined as

$$\mathrm{HM}(G) = \sum_{v_i v_j \in E(G)} \left[d_G(v_i) + d_G(v_j)\right]^2.$$

See [3, 4, 11] for details.

Theorem 4.1. *Let G be a simple graph of order n and size m. Then*

$$\frac{4(n-1)m^2}{n} - \mathrm{HM}(G) \leq \mathrm{Lz}(G) \leq \frac{(n-1)m}{2} + \frac{(n-3)}{2}\,\mathrm{HM}(G).$$

Equality holds if and only if $G \cong K_n$.

Proof. By Jensen's inequality,

$$\sum_{v_i v_j \in E(G)} \big(d_G(v_i) + d_G(v_i)\big) = \sum_{v_i \in V(G)} d_G(v_i)^2 \geq n \left(\frac{1}{n} \sum_{v_i \in V(G)} d_G(v_i)\right)^2 = n\frac{(2m)^2}{n^2} = \frac{4m^2}{n}.$$

For any two non-negative real numbers x and y,

$$x^2 + y^2 \leq (x+y)^2,$$

with equality if and only if $xy = 0$. For $x = d(v_i)$ and $x = d(v_j)$, this yields

$$\sum_{v_i v_j \in E(G)} \left[d_G(v_i)^2 + d_G(v_j)^2\right] \leq \sum_{v_i v_j \in E(G)} \left[d_G(v_i) + d_G(v_j)\right]^2,$$

and

$$\sum_{v_i v_j \in E(G)} \left[d_G(v_i) + d_G(v_j)\right] \geq \frac{4m^2}{n}.$$

Combining this with the above results, we obtain

$$\mathrm{Lz}(G) = \sum_{v \in V(G)} d_{\overline{G}}(v)\, d_G(v)^2 = \sum_{v \in V(G)} [n - 1 - d_G(v)]\, d_G(v)^2$$

$$= (n-1) \sum_{v_i v_j \in E(G)} \left[d_G(v_i) + d_G(v_j)\right] - \sum_{v_i v_j \in E(G)} \left[d_G(v_i)^2 + d_G(v_j)^2\right]$$

$$\geq \frac{4m^2(n-1)}{n} - \sum_{v_i v_j \in E(G)} \left[d_G(v_i)^2 + d_G(v_j)^2\right] \geq \frac{4(n-1)m^2}{n} - HM(G).$$

Since

$$\sum_{v_iv_j\in E(G)} \big[d_G(v_i)+d_G(v_j)\big]\times 1 \le \sum_{v_iv_j\in E(G)} \frac{1}{2}\Big[\big[d_G(v_i)+d_G(v_j)\big]^2+1^2\Big]$$

$$\le \frac{1}{2}\sum_{v_iv_j\in E(G)} \big[d_G(v_i)+d_G(v_j)\big]^2+\frac{1}{2}\sum_{v_iv_j\in E(G)} 1$$

$$\le \frac{1}{2}\sum_{v_iv_j\in E(G)} \big[d_G(v_i)+d_G(v_j)\big]^2+\frac{1}{2}\,m.$$

it follows that

$$(n-1)\sum_{v_iv_j\in E(G)} \big[d_G(v_i)+d_G(v_j)\big] \le \frac{(n-1)}{2}\sum_{v_iv_j\in E(G)} \big[d_G(v_i)+d_G(v_j)\big]^2+\frac{(n-1)m}{2},$$

and hence

$$\begin{aligned}\mathrm{Lz}(G) &= (n-1)\sum_{v_iv_j\in E(G)} \big[d_G(v_i)+d_G(v_j)\big]-\sum_{v_iv_j\in E(G)} \big[d_G(v_i)^2+d_G(v_j)^2\big]\\ &\le \frac{(n-1)}{2}\sum_{v_iv_j\in E(G)} \big[d_G(v_i)+d_G(v_j)\big]^2+\frac{(n-1)m}{2}-\sum_{v_iv_j\in E(G)} \big[d_G(v_i)+d_G(v_j)\big]^2\\ &\le \frac{(n-1)m}{2}+\frac{(n-3)}{2}\,\mathrm{HM}(G).\end{aligned}$$

□

Corollary 4.2. *Let G be a connected graph with n vertices and m edges. Let $\mathrm{F}(G)$ be the forgotten index, Eq. (1). Then*

$$\mathrm{Lz}(G) \ge \frac{(n-m-2)}{m+1}\,\mathrm{F}(G)\,.$$

Equality holds if and only if $d_G(v_i)+d_G(v_j)=m+1$ for all $v_iv_j\in E(G)$.

Proof. Bearing in mind Eq. (6), we have

$$\mathrm{Lz}(G) = (n-1)\sum_{v_iv_j\in E(G)} \big[d_G(v_i)+d_G(v_j)\big]-\sum_{v_iv_j\in E(G)} \big[d_G(v_i)^2+d_G(v_j)^2\big]$$

$$= (n-1)\sum_{v_iv_j\in E(G)} \frac{(d_G(v_i)+d_G(v_j))^2}{d_G(v_i)+d_G(v_j)}-\sum_{v_iv_j\in E(G)} \big[d_G(v_i)^2+d_G(v_j)\big]^2$$

$$\ge (n-1)\sum_{v_iv_j\in E(G)} \frac{d_G(v_i)^2+d_G(v_j)^2}{m+1}-\sum_{v_iv_j\in E(G)} \big[d_G(v_i)^2+d_G(v_j)^2\big]$$

$$\ge \frac{n-m-2}{m+1}\,\mathrm{F}(G)\,.$$

□

4.2. Sombor index

One of the present authors proposed recently a new vertex–degree–based topological index, called Sombor index [8]. It is defined via the term $\sqrt{\deg(u)^2+\deg(v)^2}$; see [8, 12, 14, 19] for more details.

Theorem 4.3. *Let G be a connected graph of order n and size m having minimum vertex degree δ and maximum vertex degree Δ. Then*

$$\mathrm{Lz}(G) \geq (n-1)m\left(\frac{\delta}{2}+\frac{\delta^2}{2\sqrt{8\Delta^2+\delta^2}+4\sqrt{2}\Delta}\right)+(n-1)(1-\sqrt{2})\,\mathrm{SO(G)}\,.$$

The bound is attained in case of regular graphs.

Proof. Since

$$\begin{aligned}\sqrt{d_G(v_i)^2+d_G(v_j)^2} &\leq \sqrt{d_G(v_i)^2+d_G(v_j)^2+\frac{\delta^2}{4}}-\frac{\delta^2}{2\sqrt{8\Delta^2+\delta^2}+4\Delta\sqrt{2}} \\ &\leq d_G(v_i)+d_G(v_j)-\frac{\delta}{2}-\frac{\delta^2}{2\sqrt{8\Delta^2+\delta^2}+4\Delta\sqrt{2}}\,,\end{aligned}$$

it follows that

$$\sqrt{d_G(v_i)^2+d_G(v_j)^2}+\frac{\delta}{2}+\frac{\delta^2}{2\sqrt{8\Delta^2+\delta^2}+4\Delta\sqrt{2}} \leq d_G(v_i)+d_G(v_j)\,.$$

Summing both sides of this inequality, we get

$$\begin{aligned}&(n-1)\sum_{v_iv_j\in E(G)}\left(\sqrt{d_G(v_i)^2+d_G(v_j)^2}+\frac{\delta}{2}+\frac{\delta^2}{2\sqrt{8\Delta^2+\delta^2}+4\Delta\sqrt{2}}\right) \\ &\leq (n-1)\sum_{v_iv_j\in E(G)}\left[d_G(v_i)+d_G(v_j)\right].\end{aligned}$$

Thus

$$\begin{aligned}\sum_{v_iv_j\in E(G)}\left[d_G(v_i)^2+d_G(v_j)^2\right] &= \sum_{v_iv_j\in E(G)}\sqrt{d_G(v_i)^2+d_G(v_j)^2}\sqrt{d_G(v_i)^2+d_G(v_j)^2} \\ &\geq \sqrt{2}\delta\sum_{v_iv_j\in E(G)}\sqrt{d_G(v_i)^2+d_G(v_j)^2}=\sqrt{2}\delta\,\mathrm{SO}(G),\end{aligned}$$

and

$$\begin{aligned}\sum_{v_iv_j\in E(G)}\left[d_G(v_i)^2+d_G(v_j)^2\right] &= \sum_{v_iv_j\in E(G)}\sqrt{d_G(v_i)^2+d_G(v_j)^2}\sqrt{d_G(v_i)^2+d_G(v_j)^2} \\ &\leq (n-1)\sqrt{2}\sum_{v_iv_j\in E(G)}\sqrt{d_G(v_i)^2+d_G(v_j)^2} \\ &\leq \sqrt{2}(n-1)\,\mathrm{SO(G)}\,.\end{aligned}$$

Taking all this into account, we get

$$\begin{aligned}
\mathrm{Lz}(G) &= (n-1)\sum_{v_iv_j\in E(G)}\big[d_G(v_i)+d_G(v_j)\big] - \sum_{v_iv_j\in E(G)}\big[d_G(v_i)^2+d_G(v_j)^2\big] \\
&\geq (n-1)\sum_{v_iv_j\in E(G)}\left(\sqrt{d_G(v_i)^2+d_G(v_j)^2}+\frac{\delta}{2}+\frac{\delta^2}{2\sqrt{8\Delta^2+\delta^2}+4\Delta\sqrt{2}}\right) \\
&- (n-1)\sqrt{2}\sum_{v_iv_j\in E(G)}\sqrt{d_G(v_i)^2+d_G(v_j)^2} \\
&\geq (n-1)m\left(\frac{\delta}{2}+\frac{\delta^2}{2\sqrt{8\Delta^2+\delta^2}+4\sqrt{2}\Delta}\right)+(n-1)(1-\sqrt{2})\,\mathrm{SO(G)}\ .
\end{aligned}$$

□

Theorem 4.4. *Let G be the same as in Theorem 4.3. Then*

$$\mathrm{Lz}(G) \leq (n-1-\sqrt{\delta})\sqrt{2}\,\mathrm{SO(G)}\,.$$

Equality holds if and only if G is an empty (edgeless) graph.

Proof. For any two non-negative real numbers x and y,

$$x+y \leq \sqrt{2(x^2+y^2)}$$

with equality if and only if $x=y$. Then for $x=d_G(v_i)$ and $y=d_G(v_j)$,

$$\sum_{v_iv_j\in E(G)}\big[d_G(v_i)+d_G(v_j)\big] \leq \sum_{v_iv_j\in E(G)}\sqrt{2\big[d_G(v_i)^2+d_G(v_j)^2)\big]}$$

and thus

$$\begin{aligned}
\mathrm{Lz}(G) &= (n-1)\sum_{v_iv_j\in E(G)}\big[d_G(v_i)+d_G(v_j)\big] - \sum_{v_iv_j\in E(G)}\big[d_G(v_i)^2+d_G(v_j)^2\big] \\
&\leq (n-1)\sqrt{2}\sum_{v_iv_j\in E(G)}\sqrt{(d_G(v_i)^2+d_G(v_j)^2} \\
&- \sum_{v_iv_j\in E(G)}\sqrt{d_G(v_i)^2+d_G(v_j)^2}\sqrt{d_G(v_i)^2+d_G(v_j)^2} \\
&\leq (n-1)\sqrt{2}\sum_{v_iv_j\in E(G)}\sqrt{d_G(v_i)^2+d_G(v_j)^2} - \sqrt{2\delta}\sum_{v_iv_j\in E(G)}\sqrt{d_G(v_i)^2+d_G(v_j)^2} \\
&\leq \sqrt{2}\big[n-1-\sqrt{\delta}\big]\,\mathrm{SO(G)}\,.
\end{aligned}$$

□

4.3. Randić index

The *Randić index* is defined as [20]

$$\mathrm{R}(G) = \sum_{uv\in E(G)}\frac{1}{\sqrt{d_G(u)\,d_G(u)}} \tag{7}$$

whereas the *reduced reciprocal Randić index* is

$$\mathrm{RR}(G) = \sum_{uv\in E(G)} \sqrt{d_G(u)\, d_G(u)}\,. \tag{8}$$

This latter index has the second-best correlating ability among many vertex–degree–based molecular structure descriptors; see [1] for more details.

Theorem 4.5. *Let G be a simple graph of order n and size m having minimum degree δ and maximum degree Δ. Then*

$$2\big[\delta^2(n-1)-\Delta^3\big]\,\mathrm{R}(G) \le \mathrm{Lz}(G) \le \big[\Delta(m-1)(n-1))-\delta^3\big]\,\mathrm{R}(G)\,.$$

Equalities hold if and only if G is regular and satisfies $d_G(v_i)+d_G(v_j)=m+1$ for $v_iv_j\in E(G)$.

Proof. Note that

$$\begin{aligned}
\mathrm{Lz}(G) &= (n-1)\sum_{v_iv_j\in E(G)}\big[d_G(v_i)+d_G(v_j)\big] - \sum_{v_iv_j\in E(G)}\big[d_G(v_i)^2+d_G(v_j)^2\big]\\
&= (n-1)\sum_{v_iv_j\in E(G)}\frac{1}{\sqrt{d_G(v_i)\,d_G(v_j)}}\sqrt{d_G(v_i)d_G(v_j)}\,\big[d_G(v_i)+d_G(v_j)\big]\\
&- \sum_{v_iv_j\in E(G)}\frac{1}{\sqrt{d_G(v_i)d_G(v_j)}}\sqrt{d_G(v_i)d_G(v_j)}\,\big[d_G(v_i)^2+d_G(v_j)^2\big]\\
&= (n-1)\sum_{v_iv_j\in E(G)}\frac{1}{\sqrt{d_G(v_i)d_G(v_j)}}\Delta(m+1) - \sum_{v_iv_j\in E(G)}\frac{1}{\sqrt{d_G(v_i)d_G(v_j)}}\,\delta^3\\
&\le \big[\Delta(m-1)(n-1))-\delta^3\big]\,\mathrm{R(G)}\ .
\end{aligned}$$

The lower bound can be verified in an analogous manner. □

Theorem 4.6. *Let G be a connected graph of order n with m edges having minimum vertex degree δ and maximum vertex degree Δ. Then*

$$\left(2(n-1)-\frac{2\Delta^2}{\delta}\right)\mathrm{RR}(G) \le \mathrm{Lz}(G) \le \left(\sqrt{2}(n-1)(\frac{\Delta}{\delta}+\frac{\delta}{\Delta})-2\right)\mathrm{RR}(G),$$

with equality if and only if G is regular.

Proof. Using an analogous reasoning as in the proof of Theorem 4.4, we have

$$\begin{aligned}
\mathrm{Lz}(G) &= (n-1)\sum_{v_iv_j\in E(G)}\big[d_G(v_i)+d_G(v_j)\big] - \sum_{v_iv_j\in E(G)}\big[d_G(v_i)^2+d_G(v_j)^2\big]\\
&\le (n-1)\sqrt{2}\sum_{v_iv_j\in E(G)}\sqrt{d_G(v_i)\,d_G(v_j)}\sqrt{\frac{d_G(v_i)}{d_G(v_j)}+\frac{d_G(v_j)}{d_G(v_i)}} - \sum_{v_iv_j\in E(G)} 2\sqrt{d_G(v_i)\,d_G(v_j)}\\
&\le (n-1)\sqrt{2}\sum_{v_iv_j\in E(G)}\sqrt{d_G(v_i)\,d_G(v_j)}\sqrt{\frac{\Delta}{\delta}+\frac{\delta}{\Delta}} - \sum_{v_iv_j\in E(G)} 2\sqrt{d_G(v_i)\,d_G(v_j)}\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{Lz}(G) &= (n-1)\sum_{v_iv_j\in E(G)}\left[d_G(v_i)+d_G(v_j)\right]-\sum_{v_iv_j\in E(G)}\left[d_G(v_i)^2+d_G(v_j)^2\right]\\ &\geq (n-1)\sum_{v_iv_j\in E(G)}2\sqrt{d_G(v_i)\,d_G(v_j)}-\frac{2\Delta^2}{\delta}\sum_{v_iv_j\in E(G)}\sqrt{d_G(v_i)\,d_G(v_j)}\\ &\geq \left(2(n-1)-\frac{2\Delta^2}{\delta}\right)\mathrm{RR}(G)\,.\end{aligned}$$

□

4.4. Symmetric division deg index

The *symmetric division deg index* is defined as

$$\mathrm{SDD}(G)=\sum_{v_iv_j\in E(G)}\frac{d_G(v_i)^2+d_G(v_j)^2}{d_G(v_i)\,d_G(v_j)}.$$

This index, proposed several years ago by Vukičevć et al. [16], was found to be useful in predicting physico-chemical properties of molecules.

Theorem 4.7. *Let G be same as in Theorem 4.6. Then*

$$\left(\frac{2(n-1)\Delta\,\delta^2}{\Delta^2+\delta^2}-\Delta^2\right)\mathrm{SDD}(G)\leq \mathrm{Lz}(G)\leq\left(\frac{(n-1)(m+1)\Delta}{2\delta}-\delta^2\right)\mathrm{SDD}(G),$$

with equality if and only if G is regular and satisfies $d_G(v_i)+d_G(v_j)=m+1$ for all $v_iv_j\in E(G)$.

Proof.

$$\begin{aligned}\mathrm{Lz}(G) &= (n-1)\sum_{v_iv_j\in E(G)}\left[d_G(v_i)+d_G(v_j)\right]-\sum_{v_iv_j\in E(G)}\left[d_G(v_i)^2+d_G(v_j)^2\right]\\ &= (n-1)\sum_{v_iv_j\in E(G)}\left[d_G(v_i)+d_G(v_j)\right]\left(\frac{d_G(v_i)^2+d_G(v_j)^2}{d_G(v_i)\,d_G(v_j)}\right)\left(\frac{d_G(v_i)\,d_G(v_j)}{d_G(v_i)^2+d_G(v_j)^2}\right)\\ &-\sum_{v_iv_j\in E(G)}\left(\frac{d_G(v_i)^2+d_G(v_j)^2}{d_G(v_i)\,d_G(v_j)}\right)d_G(v_i)\,d_G(v_j)\\ &= (n-1)\sum_{v_iv_j\in E(G)}\left(\frac{d_G(v_i)^2+d_G(v_j)^2}{d_G(v_i)\,d_G(v_j)}\right)\left(\frac{d_G(v_i)\,d_G(v_j)(d_G(v_i)+d_G(v_j))}{d_G(v_i)^2+d_G(v_j)^2}\right)\\ &-\sum_{v_iv_j\in E(G)}\left(\frac{d_G(v_i)^2+d_G(v_j)^2}{d_G(v_i)\,d_G(v_j)}\right)d_G(v_i)\,d_G(v_j)\\ &= (n-1)\sum_{v_iv_j\in E(G)}\left(\frac{d_G(v_i)^2+d_G(v_j)^2}{d_G(v_i)\,d_G(v_j)}\right)\left(\frac{(d_G(v_i)+d_G(v_j))}{\frac{d_G(v_i)}{d_G(v_j)}+\frac{d_G(v_j)}{d_G(v_i)}}\right)\end{aligned}$$

$$
\begin{aligned}
&- \sum_{v_iv_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i)\, d_G(v_j)} \right) d_G(v_i)\, d_G(v_j) \\
&\leq (n-1) \sum_{v_iv_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i)\, d_G(v_j)} \right) \left(\frac{(m+1)}{2\frac{\delta}{\Delta}} \right) - \sum_{v_iv_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i)\, d_G(v_j)} \right) \delta^2 \\
&\leq \left(\frac{(n-1)(m+1)\Delta}{2\delta} - \delta^2 \right) \mathrm{SDD(G)}\ .
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
\mathrm{Lz}(G) &= (n-1) \sum_{v_iv_j \in E(G)} \left[d_G(v_i) + d_G(v_j) \right] - \sum_{v_iv_j \in E(G)} \left[d_G(v_i)^2 + d_G(v_j)^2 \right] \\
&= (n-1) \sum_{v_iv_j \in E(G)} \left[d_G(v_i) + d_G(v_j) \right] \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i)\, d_G(v_j)} \right) \left(\frac{d_G(v_i)\, d_G(v_j)}{d_G(v_i)^2 + d_G(v_j)^2} \right) \\
&- \sum_{v_iv_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i)\, d_G(v_j)} \right) d_G(v_i)\, d_G(v_j) \\
&= (n-1) \sum_{v_iv_j) \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i)\, d_G(v_j)} \right) \left(\frac{d_G(v_i)\, d_G(v_j)(d_G(v_i) + d_G(v_j))}{d_G(v_i)^2 + d_G(v_j)^2} \right) \\
&- \sum_{v_iv_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i)\, d_G(v_j)} \right) d_G(v_i)\, d_G(v_j) \\
&= (n-1) \sum_{v_iv_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i)\, d_G(v_j)} \right) \left(\frac{(d_G(v_i) + d_G(v_j))}{\frac{d_G(v_i)}{d_G(v_j)} + \frac{d_G(v_j)}{d_G(v_i)}} \right) \\
&- \sum_{v_iv_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i)\, d_G(v_j)} \right) d_G(v_i)\, d_G(v_j) \\
&\geq (n-1) \sum_{v_iv_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i)\, d_G(v_j)} \right) \left(\frac{2\delta}{\frac{\Delta}{\delta} + \frac{\delta}{\Delta}} \right) - \sum_{v_iv_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i)\, d_G(v_j)} \right) \Delta^2 \\
&\geq (n-1) \left(\frac{2\delta}{\frac{\Delta}{\delta} + \frac{\delta}{\Delta}} \right) \sum_{v_iv_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i)\, d_G(v_j)} \right) - \Delta^2 \sum_{v_iv_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i)\, d_G(v_j)} \right) \\
&\geq \left((n-1)\left(\frac{(2\delta))}{\frac{\Delta}{\delta} + \frac{\delta}{\Delta}} \right) - \Delta^2 \right) - \sum_{v_iv_j \in E(G)} \left(\frac{d_G(v_i)^2 + d_G(v_j)^2}{d_G(v_i)\, d_G(v_j)} \right) \\
&= \left(\frac{2(n-1)\Delta\, \delta^2}{\Delta^2 + \delta^2} - \Delta^2 \right) \mathrm{SDD}(G)\, .
\end{aligned}
$$

□

4.5. Reciprocal sum-connectivity index

Motivated by the definition of the Randić index, Eq. (7), its variant, called *sum-connectivity index* was recently proposed [21], defined as

$$\mathrm{SC}(G) = \sum_{v_i v_j \in E(G)} \frac{1}{\sqrt{d_G(v_i) + d_G(v_j)}} .$$

In parallel to Eq. (8), the *reciprocal sum-connectivity index* of a graph G is

$$\mathrm{RSC}(G) = \sum_{v_i v_j \in E(G)} \sqrt{d_G(v_i) + d_G(v_j)}.$$

Theorem 4.8. *Let G be a connected graph of order n with m edges, having minimum vertex degree δ and maximum vertex degree Δ. Then*

$$\left(\sqrt{2}(n-1)\delta - \frac{\sqrt{2}\Delta^2}{\delta}\right) \mathrm{RSC}(G) \leq \mathrm{Lz}(G) \leq \left(\sqrt{2}(n-1)\Delta - \frac{\sqrt{2}\delta^2}{\Delta}\right) \mathrm{RSC}(G),$$

with equalities if and only if G is regular.

Proof. Starting with

$$d_G(v_i) + d_G(v_i) = \sqrt{d_G(v_i) + d_G(v_i)}\sqrt{d_G(v_i) + d_G(v_i)}$$

we obtain

$$\sqrt{2\delta}\sqrt{d_G(v_i) + d_G(v_i)} \leq \sqrt{d_G(v_i) + d_G(v_i)}\sqrt{d_G(v_i) + d_G(v_i)} \leq \sqrt{2\Delta}\sqrt{d_G(v_i) + d_G(v_i)},$$

and thus

$$\sqrt{2\delta}(n-1) \sum_{v_i v_j \in E(G)} \sqrt{d_G(v_i) + d_G(v_i)} \leq (n-1) \sum_{v_i v_j \in E(G)} \left[d_G(v_i) + d_G(v_i)\right]$$

$$\leq \sqrt{2\Delta}(n-1) \sum_{v_i v_j \in E(G)} \sqrt{d_G(v_i) + d_G(v_i)},$$

which means that

$$\sqrt{2\delta}(n-1)\,\mathrm{RSC}(G) \leq (n-1) \sum_{v_i v_j \in E(G)} \left[d_G(v_i) + d_G(v_i)\right] \leq \sqrt{2\Delta}(n-1)\ \mathrm{RSC}(G) .$$

Using the relation

$$d_G(v_i)^2 + d_G(v_i)^2 = \frac{d_G(v_i)^2 + d_G(v_i)^2}{\sqrt{d_G(v_i) + d_G(v_j)}}\sqrt{d_G(v_i) + d_G(v_j)}$$

we have

$$\frac{2\delta^2}{\sqrt{2}\Delta}\sqrt{d_G(v_i) + d_G(v_j)} \leq \frac{d_G(v_i)^2 + d_G(v_i)^2}{\sqrt{d_G(v_i) + d_G(v_j)}}\sqrt{d_G(v_i) + d_G(v_j)} \leq \frac{2\Delta^2}{\sqrt{2}\delta}\sqrt{d_G(v_i) + d_G(v_j)},$$

from which it follows

$$\frac{2\delta^2}{\sqrt{2}\Delta} \sum_{v_i v_j \in E(G)} \sqrt{d_G(v_i) + d_G(v_j)} \leq \sum_{v_i v_j \in E(G)} \left[d_G(v_i)^2 + d_G(v_i)^2\right]$$

$$\leq \frac{2\Delta^2}{\sqrt{2}\delta} \sum_{v_i v_j \in E(G)} \sqrt{d_G(v_i) + d_G(v_j)}$$

and

$$\begin{aligned}\mathrm{Lz}(G) &= (n-1)\sum_{v_iv_j\in E(G)}\left[d_G(v_i)+d_G(v_j)\right]-\sum_{v_iv_j\in E(G)}\left[d_G(v_i)^2+d_G(v_j)^2\right]\\ &\le \sqrt{2\Delta}(n-1)\sum_{v_iv_j\in E(G)}\sqrt{d_G(v_i)+d_G(v_i)}-\frac{2\Delta^2}{\sqrt{2\delta}}\sum_{v_iv_j\in E(G)}\sqrt{d_G(v_i)+d_G(v_j)}\\ &\le \left(\sqrt{2\Delta}(n-1)-\frac{\sqrt{2}\delta^2}{\Delta}\right)\mathrm{RSC}(G)\end{aligned}$$

as well as

$$\begin{aligned}\mathrm{Lz}(G) &= (n-1)\sum_{v_iv_j\in E(G)}\left[d_G(v_i)+d_G(v_j)\right]-\sum_{v_iv_j\in E(G)}\left[d_G(v_i)^2+d_G(v_j)^2\right]\\ &\ge \sqrt{2\delta}(n-1)\sum_{v_iv_j\in E(G)}\sqrt{d_G(v_i)+d_G(v_i)}-\frac{2\Delta^2}{\sqrt{2\delta}}\sum_{v_iv_j\in E(G)}\sqrt{d_G(v_i)+d_G(v_j)}\\ &\ge \left(\sqrt{2}(n-1)\delta-\frac{\sqrt{2}\Delta^2}{\delta}\right)\mathrm{RSC}(G)\,.\end{aligned}$$

□

Theorem 4.9. *Let G be same as in Theorem 4.8. Then*

$$\left[2\sqrt{2}(n-1)\delta^{\frac{5}{2}}-2\Delta^2\sqrt{2(m+1)}\right]\mathrm{SC}(G)\le \mathrm{Lz}(G)\le\left[(n-1)(m+1)^{\frac{3}{2}}-2\sqrt{2}\delta^{\frac{5}{2}}\right]\mathrm{SC}(G)\,.$$

The above equalities hold if and only if G is regular and satisfies $d_G(v_i)+d_G(v_j)=m+1$ for all $v_iv_j\in E(G)$.

Proof. Using the identity

$$d_G(v_i)+d_G(v_j)=\frac{d_G(v_i)+d_G(v_j)}{\sqrt{d_G(v_i)+d_G(v_j)}}\sqrt{d_G(v_i)+d_G(v_j)}$$

we obtain

$$\frac{2\delta\sqrt{2\delta}}{\sqrt{d_G(v_i)+d_G(v_j)}}\le d_G(v_i)+d_G(v_j)\le\frac{(m+1)\sqrt{m+1}}{\sqrt{d_G(v_i)+d_G(v_j)}}\,.$$

which after summation yields

$$\begin{aligned}2\delta\sqrt{2\delta}\sum_{v_iv_j\in E(G)}\frac{1}{\sqrt{d_G(v_i)+d_G(v_j)}} &\le \sum_{v_iv_j\in E(G)}\left[d_G(v_i)+d_G(v_j)\right]\\ &\le (m+1)\sqrt{(m+1)}\sum_{v_iv_j\in E(G)}\frac{1}{\sqrt{d_G(v_i)+d_G(v_j)}}\,.\end{aligned}$$

This means that

$$2\delta\sqrt{(2\delta)}\,\mathrm{SC}(G)\le\sum_{v_iv_j\in E(G)}\left[d_G(v_i)+d_G(v_j)\right]\le(m+1)\sqrt{m+1}\;\mathrm{SC}(G)$$

implying

$$(2\delta)^{\frac{3}{2}}\,\mathrm{SC}(G) \le \sum_{v_iv_j\in E(G)} \left[d_G(v_i)+d_G(v_j)\right] \le 2\Delta^2\sqrt{m+1}\,\mathrm{SC}(G)\,.$$

Similarly,

$$(2\delta^2)\sqrt{2\delta}\,\mathrm{SC}(G) \le \sum_{v_iv_j\in E(G)} \left[d_G(v_i)^2+d_G(v_j)^2\right] \le (2\Delta^2\sqrt{m+1})\,\mathrm{SC}(G)$$

and thus

$$\begin{aligned}\mathrm{Lz}(G) &= (n-1)\sum_{v_iv_j\in E(G)} \left[d_G(v_i)+d_G(v_j)\right] - \sum_{v_iv_j\in E(G)} \left[d_G(v_i)^2+d_G(v_j)^2\right] \\ &\le (m+1)(n-1)\sqrt{(m+1)}\,\mathrm{SC}(G) - 2\sqrt{2}(\delta)^{\frac{5}{2}}\,\mathrm{SC}(G) \\ &= \left[(n-1)(m+1)^{\frac{3}{2}} - 2\sqrt{2}\delta^{\frac{5}{2}}\right]\mathrm{SC}(G)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Lz}(G) &= (n-1)\sum_{v_iv_j\in E(G)} \left[d_G(v_i)+d_G(v_j)\right] - \sum_{v_iv_j\in E(G)} \left[d_G(v_i)^2+d_G(v_j)^2\right] \\ &\ge 2(n-1)\delta\,\sqrt{(2\delta)}\,\mathrm{SC}(G) - 2\Delta^2\sqrt{2(m+1)}\,\mathrm{SC}(G) \\ &= \left[2\sqrt{2}(n-1)\delta^{\frac{5}{2}} - 2\Delta^2\sqrt{2(m+1)}\right]\mathrm{SC}(G)\,.\end{aligned}$$

Combining the above two results, we arrive at Theorem 4.9 □

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References

[1] A. Ali, A. A. Bhatti, A note on the minimum reduced reciprocal Randic index of n-vertex unicyclic graphs, Kuwait J. Sci. 44(2) (2017) 27–33.
[2] M. Aouchiche, P. Hansen, A survey of Nordhaus-Gaddum type relations, Discrete Appl. Math. 161(4-5) (2013) 466–546.
[3] A. T. Balaban, I. Motoc, D. Bondchev, O. Mekenyan, Tonpological indices for structure-activity correlations, In: Steric Effects in Drug Design, Topics in Curr. Chem. 114, Springer, Berlin (1983).
[4] N. De, Hyper Zagreb index of bridge and chain graphs, Open J. Math. Sci. 2(1) (2018) 1–17.
[5] N. Dehgardi, J. Liu, Lanzhou index of trees with fixed maximum degree, MATCH Commun. Math. Comput. Chem. 86 (2021) 3–10.
[6] T. Došlić, T. Réti, D. Vukičević, On the vertex degree indices of connected graphs, Chem. Phys. Lett. 512(4-6) (2011) 283–286.
[7] B. Furtula, I. Gutman, A forgotten topological index, J. Math. Chem. 53(4) (2015) 1184–1190.
[8] I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, MATCH Commun. Math. Comput. Chem. 86 (2021) 11–16.

[9] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total φ-electron energy of alternant hydrocarbons, Chem. Phys. Lett. 17(4) (1972) 535–538.
[10] A. L. Horvath, Molecular design: chemical structure generation from the properties of pure organic compounds, Elsevier, Amsterdam (1992).
[11] V. R. Kulli, Reduced second Hyper-Zagreb index and its polynomial of certain silicate networks, J. Math. Inform. 14 (2018) 11–16.
[12] V. R. Kulli, I. Gutman, Computation of Sombor indices of certain networks, Int. J. Appl. Chem. 8(1) (2021) 1–5.
[13] Y. Mao, Nordhaus-Gaddum type results in chemical graph theory, In: Bounds in chemical graph theory-advances, Eds.: I. Gutman, B. Furtula, K. C. Das, E. Milovanović, I. Milovanović, Univ. Kragujevac, Kragujevac (2017).
[14] I. Redžepović, Chemical applicability of Sombor indices, J. Serb. Chem. Soc. 86(5) (2021) 445–457.
[15] R. Todeschini, R. Consonni, Molecular descriptors for chemoinformatics, Wiley-VCH, Weinheim (2009).
[16] D. Vukičević, B. Furtula, Topological index based on the ratios of geometrical and arithmetical means of end-vertex degrees of edges, J. Math. Chem. 46 (2009) 1369–1376.
[17] D. Vukičević, Q. Li, J. Sedlar, T. Došlić, Lanzhou index, MATCH Commun. Math. Comput. Chem. 80 (2018) 863–876.
[18] X. Wang, Y. Mao, Z. Wang, S. Wang, J. Li, On the Lanzhou index, MATCH Commun. Math. Comput. Chem. 85 (2021) 109–119.
[19] Z. Wang, Y. Mao, Y. Li, B. Furtula, On relations between Sombor and other degree-based indices, J. Appl. Math. Comput. 68 (2022) 1âĂŞ-17.
[20] M. Randić, Characterization of molecular branching, J. Am. Chem. Soc. 97 (1975) 6609–6615.
[21] B. Zhou, N. Trinajstić, On a novel connectivity index, J. Math. Chem. 46 (2009) 1252–1270.