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Walled Klein-4 Brauer algebras

Research Article

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Abstract: The new class of diagram algebras known as walled Klein-4 Brauer algebras, denoted by $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$, where $r, s \in \mathbb{N}$ and $l \in K$, is an indeterminate, is studied in this paper. The walled klein-4 Brauer algebras are explained in terms of generators and relations. The indexing set of the simple modules of the walled Klein-4 Brauer algebras was described. We established that walled Klein-4 Brauer algebras are iterated inflations of the group algebra of the group $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_r \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_s$, and we concluded that $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ is cellular as a consequence.

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Keywords: Cellularity, Walled Brauer algebras, Walled signed Brauer algebras

1. Introduction

Richard Brauer introduced the Brauer algebra $B_n(\delta)$ in the representation theory of orthogonal groups. Undirected diagrams are the basis elements of Brauer algebra. The centralizers of the orthogonal groups are these algebras. G-Brauer algebras were introduced by Parvathi and Savithri, with signed diagram as basis and studied in [7] its structure over $k(x)$, where x is an indeterminate.

The symmetric group $\mathbf{S}_r$ and the general linear group $GL_n(\mathbb{C})$ representation theories over $\mathbb{C}$ are linked through Schur-Weyl duality. The walled Brauer algebras $B_{r,s}(\delta)$ is the result of a third version of Schur-Weyl duality. This algebra was examined by Turaev, Koike, and Benkart et al.

Kethesan introduced a new class of diagram algebras called walled signed Brauer algebras denoted by $\overrightarrow{D}_{r,s}(x)$, where x is an indeterminate. The $f_{r,s}$ map gives a vector space isomorphism between the group algebra of the hyperoctahedral group $\mathbb{Z}_2 \wr \mathbf{S}_{r+s}$ and the walled singed Brauer algebras $\overrightarrow{D}_{r,s}(x)$. The structure of walled signed Brauer algebras has been studied in [4]. In [11], Tamilselvi et al., studied

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the Robinson-Schensted correspondence for the walled Brauer algebras and the walled signed Brauer algebras. Cellularity and representations of walled cyclic G-Brauer algebras were studied by Tamilselvi [10].

In [8], Parvathi and Sivakumar studied about a new class of diagram algebras called Klein-4 diagarm algebras denoted by $R_k(n)$. These algebras are the centralizer algebras of the group $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_n$ acting on $V^{\otimes k}$, where V is the signed permutation module for $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_n$. In that paper [8], they give a combinatorial rule for the decomposition of the tensor powers of the signed permutation representation of $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_n$, by explicitly constructing the basis for the irreducible modules.

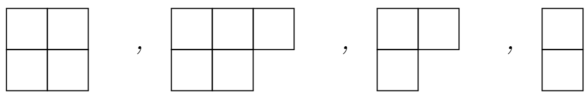
We were motivated by this work to create a new class of algebras over K, known as walled Klein-4 Brauer algebras and denoted by $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$, where $l \in K$ is an indeterminate. In section 3, we define walled Klein-4 Brauer algebras and show how they are represented in terms of generators and relations. We define the map $\mathbf{Flip}_{r,s}$, which is a vector space isomorphism between the group algebra of the group $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r+s}$ and the walled Klein-4 Brauer algebras, in that section. Using the representations of the group algebra of the group [8, 9], we described the indexing set of the simple modules of the walled Klein-4 Brauer algebras in section 4. In section 5, we proved that the walled Klein-4 Brauer algebras are the iterated inflations of the group algebra of the group $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_r \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_s$, and we concluded that walled Klein-4 diagram algebras are cellular.

2. Preliminaries

Now, we will present some definitions that we will use to proceed on.

Definition 2.1. *[3] A partition $\beta = (\beta_1, \beta_2, \dots, \beta_t)$ of $(r+s)$ is said to be p-regular if either $p > 0$ and there is no $1 \leq i \leq t$ such that $\beta_i = \beta_{i+1} = \dots = \beta_{i+p}$ or $p = 0$.*

Definition 2.2. *([9]) A 4-partition $((\beta_1), (\beta_2), (\beta_3), (\beta_4))$ of size $(r+s)$, is an ordered 4-tuple of partitions (β_1), (β_2), (β_3) and (β_4) such that $|(\beta_1), (\beta_2), (\beta_3), (\beta_4)| = |(\beta_1)| + |(\beta_2)| + |(\beta_3)| + |(\beta_4)| = (r+s)$.*



The 4-partition $((\beta_1), (\beta_2), (\beta_3), (\beta_4)) = ([2,2][3,2][2,1][1^2])$ is represented by the above 4-tableau and the sum $|(\beta_1)| + |(\beta_2)| + |(\beta_3)| + |(\beta_4)| = 14$. The set of all 4-partitions of $(r+s)$ is denoted by $B_{4(r+s)}$.

Definition 2.3. *If the partitions (β_1), (β_2), (β_3) and (β_4) are p-regular then the 4-partition $((\beta_1), (\beta_2), (\beta_3), (\beta_4))$ of $(r+s)$ is said to be p-regular.*

Theorem 2.4. *([2], theorem (10.33)) Let K be a field which is a splitting field for the finite groups G_1 and G_2. Then:*

- *For each simple KG_1 module M and each simple KG_2 module N, the outer tensor product $M \boxtimes N$ is a simple $K(G_1 \times G_2)$-module.*
- *Every simple $K(G_1 \times G_2)$-module is of the above form $M \boxtimes N$, with M and N are uniquely determined (up to isomorphism).*

Remark 2.5. *[8] A finite dimensional associative algebra $\mathcal{A}$ with unit over $\mathbb{C}$, the field of complex numbers is said to be semi-simple if $\mathcal{A}$ is isomorphic to a direct sum of full matrix algebras*

$$\mathcal{A} \cong \bigoplus_{\lambda \in \hat{\mathcal{A}}} \mathcal{M}_{d_\lambda}(\mathbb{C})$$

for $\hat{\mathcal{A}}$ a finite index set, and d_λ positive integers. Corresponding to each $\lambda \in \hat{\mathcal{A}}$, there is a single irreducible $\mathcal{A}$ module call it V^λ which has dimension d_λ. If $\hat{\mathcal{A}}$ is a singleton set then $\mathcal{A}$ is said to be simple. Maschke's theorem says that for G finite, $C[G]$ is semi-simple.

Remark 2.6. *[5] An algebra is cellular if an only if it can be written as the iterated inflations of copies of full matrix algebras. Moreover iterated inflation of cellular algebras is always cellular again ([5], proposition 3.4).*

3. Walled Klein-4 Brauer algebras

In this section, we define the walled Klein-4 Brauer algebras which are subalgebras of the G-Brauer algebras [7], where G is the Klein-4 group and give a presentation of walled Klein-4 Brauer algebras in terms of generators and relations. Throughout this paper we are considering 1 as the identity.

Fix an algebraically closed field K of characteristic $p \geq 0$ and l an indeterminate. For $r, s \in \mathbb{N}$ the walled Klein-4 Brauer algebras $\overrightrightarrows{D}_{r,s}(l)$ can be defined in the following manner.

A diagram is said to be signed diagram if every edge is labeled by a sign. In the walled Klein-4 Brauer algebras we are considering four types of signs (e, e), (e, b), (a, e) and (a, b).

The Klein-4 Brauer algebra $\overrightrightarrows{D}_{r+s}$ (diagrams without wall) which is the G-Brauer algebra defined in [7], is the set of all signed diagrams $\overrightrightarrows{d}$ with n signed edges and $2n$ vertices arranged in two rows of n vertices each. Each signed edge links to exactly two vertices in these signed diagrams, and each vertex links to exactly one signed edge. Edges connecting the top row vertex and the bottom row vertex are called vertical edges, and the remaining edges are called upper (connecting two vertices on the top row) or lower (connecting two veritices on the bottom row) horizontal curves (edges connecting two vertices in the same row).

3.1. Multiplication of walled Klein-4 Brauer diagrams

To find the product of two signed diagrams $\overrightrightarrows{d}_1$ and $\overrightrightarrows{d}_2$, draw $\overrightrightarrows{d}_2$ below $\overrightrightarrows{d}_1$ and connect the i^{th} upper vertex of $\overrightrightarrows{d}_2$ with the i^{th} lower vertex of $\overrightrightarrows{d}_1$.

A new edge obtained in the product $\overrightrightarrows{d}_1.\overrightrightarrows{d}_2$ is labeled by a sign (e, e) or (e, b) or (a, e) or (a, b) according as the product of the signs (coordinate wise) of the edges obtained from $\overrightrightarrows{d}_1$ and $\overrightrightarrows{d}_2$ to form this edge, where the product of the coordinates of the signs defined to be $e = e.e = a.a = b.b$; $a = e.a = a.e$; $b = e.b = b.e$. The product of the diagrams produces a new diagram with closed curves called loops.

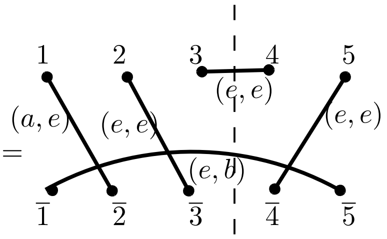
A loop α in $\overrightrightarrows{d}_1.\overrightrightarrows{d}_2$ is replaced by the variable l^2, if the product of signs of the edges obtained from $\overrightrightarrows{d}_1$ and $\overrightrightarrows{d}_2$ to form this loop is (e, e) or (a, b). Similary the loop α in $\overrightrightarrows{d}_1.\overrightrightarrows{d}_2$ is replaced by the variable l, if the product of signs of the edges obtained from $\overrightrightarrows{d}_1$ and $\overrightrightarrows{d}_2$ to form this loop is (a, e) or (e, b). Now we have $\overrightrightarrows{c}$, which is a signed diagram with each edge is marked as above and $\overrightrightarrows{d}_1.\overrightrightarrows{d}_2 = l^{2e_1+e_2}.\overrightrightarrows{c}$, where e_1 is the number of loops with the product of signs of the edges obtained from $\overrightrightarrows{d}_1$ and $\overrightrightarrows{d}_2$ to form those loops is (e, e) or (a, b), e_2 is the number of loops with the product of signs of the edges obtained from $\overrightrightarrows{d}_1$ and $\overrightrightarrows{d}_2$ to form those loops is (e, b) or (a, e).

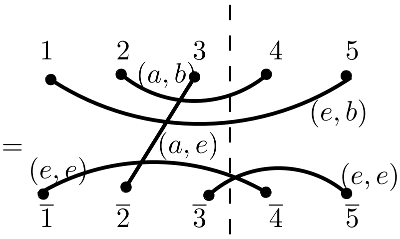
It is usual to represent basis elements by diagrams with $(r + s)$ vertices on the top row numbered 1 to $(r + s)$ from left to right and $(r + s)$ vertices on the bottom row numbered $\overline{1}$ to $\overline{(r + s)}$ from left to right, where each vertex is connected to precisely one other by a signed edge.

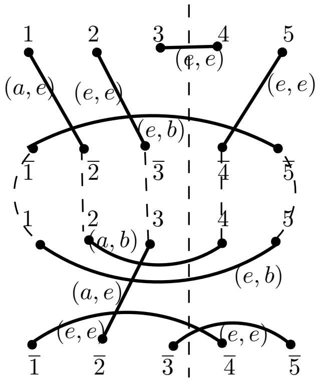
We now define the walled Klein-4 Brauer algebra when $n = r + s$. Partition the basis diagrams of $\overrightrightarrows{D}_{r+s}$ with the wall separating the first r vertices in the top row and the first r vertices in the bottom row from the remaining s vertices in the top row and s vertices in the bottom row, obtained the walled

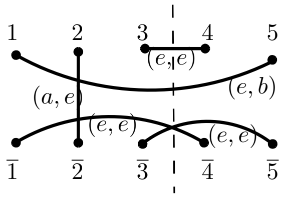
Klein-4 Brauer algebras $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ which is the diagram algebra with basis of those signed diagrams with no vertical edges crosses the wall and every upper or lower horizontal curves does cross the wall.

The example of the basis elements $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$ in $\overrightarrow{\overrightarrow{D}}_{3,2}(l)$, and the multiplication of $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$ are given below,

$\overrightarrow{d}_1 =$ 

$\overrightarrow{d}_2 =$ 

$\overrightarrow{d}_1 . \overrightarrow{d}_2 =$ 

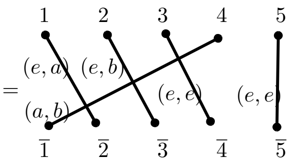
$= l^2$ 

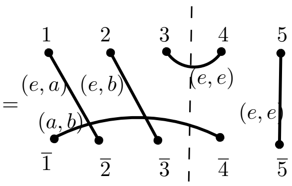
The edge $(1, 5)$ in the resulting diagram is labeled by the sign (e, b), since the coordinate wise product of the signs of the edges obtained from $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$ to form this edge is $(a, e).(a, b).(e, e) = (a.a.e, e.b.e)$ $= (e, b)$. Similarly the other edges are labeled by the corresponding signs in the resulting diagram. Also the resulting diagram is multiplied with l^2 since the product of the signs of the edges from $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$ to form the loop in the product $\overrightarrow{d}_1 . \overrightarrow{d}_2$ is $(e, b).(e, b) = (e.e, b.b) = (e, e)$.

It is useful to compare $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ with the group algebra $K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r+s})$ of the group $\mathbf{G} = (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r+s}$; where $\mathbf{S}_{r+s}$ is a symmetric group of $(r + s)$ symbols and $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is the Klein-4 group consisting of four elements. The group algebra can be viewed diagrammatically if we identify the element g of $\mathbf{G}$ with its signed diagram with no horizontal edge.

We define a map, $\mathbf{Flip}_{r,s} : K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r+s}) \to \overrightarrow{\overrightarrow{D}}_{r,s}(l)$, mapping a Klein-4 signed diagram with no horizontal curves to the walled Klein-4 signed diagram obtained by adding a wall between the $r^{th}$and $(r + 1)^{th}$ vertices, then without disconnecting any edges or changing the sign flip the portion of the diagram that is already on the right side of the wall.

For example,

$\overrightarrow{f} =$ 

$\overrightarrow{h} =$ 

The walled Klein-4 Brauer diagram $\overrightarrow{h}$ above arise by applying $\mathbf{Flip}_{3,2}$ to the Klein-4 signed diagram

$\overset{\rightrightarrows}{f}$. Once add the wall in $\overset{\rightrightarrows}{f}$ between the vertices 3 and 4 flip it on the right side of the wall, so that the edges $(3,\overline{4})$ and $(\overline{1},4)$ in $\overset{\rightrightarrows}{f}$ changes to $(3,4)$ and $(\overline{1},\overline{4})$ in $\overset{\rightrightarrows}{h}$ respectively. It follows in particular that the map $\mathbf{Flip}_{r,s}$ is a vector space isomorphism. since

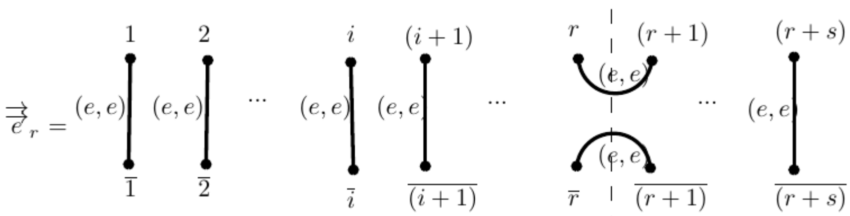
$$\dim \overset{\rightrightarrows}{D}_{r,s}(l) = \dim K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r+s}) = 4^{r+s}.(r+s)!$$

3.2. Generators and relations of the group algebra $K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r+s})$ [9]

The algebra $\mathbf{G}_{(r+s)} = K\mathbf{G}$ of the group $\mathbf{G} = (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r+s}$ is generated by t_1, t_2 and the transpositions $s_i := (i, i+1)$ for $i = 1, 2, ..., (r+s-1)$ subject to the following relations

- $t_1^2 = 1$
- $t_2^2 = 1$
- ${s_i}^2 = 1$ for $1 \leq i \leq r+s-1$
- $t_1t_2 = t_2t_1$
- $t_1s_1t_1s_1 = s_1t_1s_1t_1$
- $t_2s_1t_2s_1 = s_1t_2s_1t_2$
- $t_2s_1t_1s_1 = s_1t_1s_1t_2$
- $s_is_j = s_js_i$ if $|i-j| \geq 2$
- $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ for $1 \leq i \leq r+s-1$
- $s_it_1 = t_1s_i$ for $i \geq 2$
- $s_it_2 = t_2s_i$ for $i \geq 2$

Let $\mathbf{Flip}_{r,s}(t_1) = q_1$, $\mathbf{Flip}_{r,s}(t_2) = q_2$, and for $i = 1, 2, \ldots, (r-1), (r+1), \ldots, (r+s-1)$, $\mathbf{Flip}_{r,s}(s_i) = s_i$(with the addition of the wall). When $i = r$ then $\mathbf{Flip}_{r,s}(s_r) = e_r$ is the diagram,

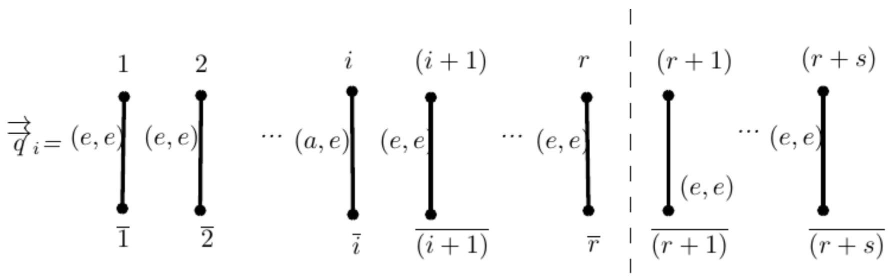


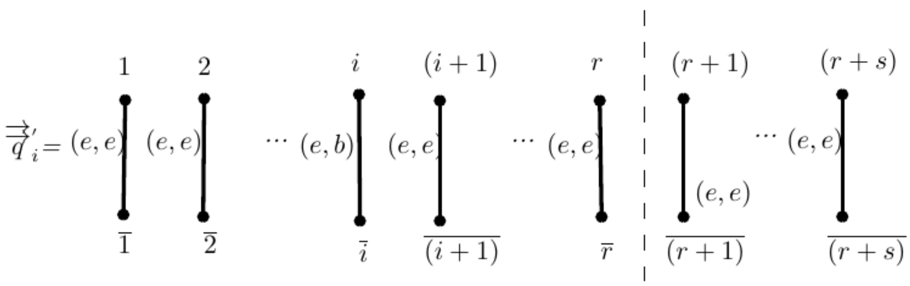
3.3. Generators and relations of the walled Klein-4 Brauer algebras

Theorem 3.1. *The walled Klein-4 Brauer algebras $\overset{\rightrightarrows}{D}_{r,s}(l)$ is generated by the elements $\overset{\rightrightarrows}{q}_1$, $\overset{\rightrightarrows}{q}_{1'}$, $\overset{\rightrightarrows}{q}_{r+1}$, $\overset{\rightrightarrows}{q}_{(r+1)'}$, s_1, s_2, ..., s_{r-1}, e_r, s_{r+1}, ..., s_{r+s-1} and satisfying the following relations:*

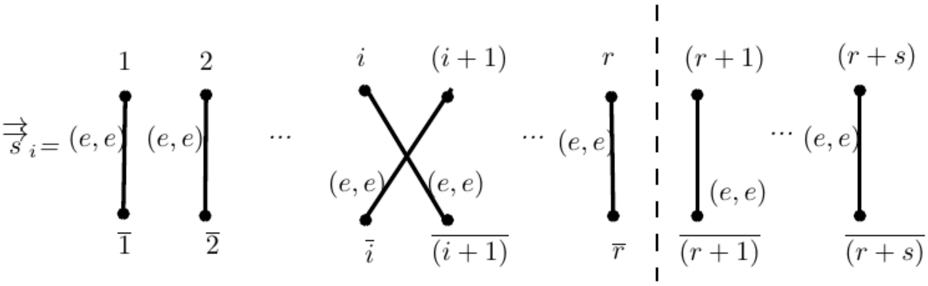
1. $s_i^2 = 1$ *for* $i = 1, 2, ...(r-1), (r+1)...(r+s-1)$.
2. $s_is_j = s_js_i$ *if* $|i-j| > 1$.
3. $s_{i+1}s_is_{i+1} = s_is_{i+1}s_i$ *for* $i \neq r, r-1$.

4. $e_r s_i = s_i e_r$ *for* $1 \leq i \leq r-2$ *or* $(r+2) \leq i \leq (r+s-1)$.
5. $(e_r)^2 = l^2 e_r$.
6. $e_r s_{r-1} e_r = e_r$.
7. $e_r s_{r+1} e_r = e_r$.
8. $s_{r-1} s_{r+1} e_r s_{r-1} s_{r+1} e_r = e_r s_{r-1} s_{r+1} e_r$.
9. $e_r s_{r-1} s_{r+1} e_r s_{r-1} s_{r+1} = e_r s_{r-1} s_{r+1} e_r$.
10. $\overset{\rightrightarrows}{q}_i^2 = 1$ *for* $i = 1, 1^{'}, 2, 2^{'}, ...(r+s), (r+s)^{'}$.
11. $\overset{\rightrightarrows}{q}_i\ s_j = s_j \overset{\rightrightarrows}{q}_i$ *for* $i = 1, 1^{'}$ *and* $j \neq i$.
12. $\overset{\rightrightarrows}{q}_i\ s_1\ \overset{\rightrightarrows}{q}_i\ s_1 = s_1\ \overset{\rightrightarrows}{q}_i\ s_1 \overset{\rightrightarrows}{q}_i$ *for* $i = 1, 1'$.
13. $\overset{\rightrightarrows}{q}_i s_j = s_j \overset{\rightrightarrows}{q}_i$ *for* $i = (r+1), (r+1)^{'}$ *and* $j \neq r, (r+1)$.
14. $\overset{\rightrightarrows}{q}_i s_{r+1} \overset{\rightrightarrows}{q}_i s_{r+1} = s_{r+1} \overset{\rightrightarrows}{q}_i s_{r+1} \overset{\rightrightarrows}{q}_i$ *for* $i = (r+1), (r+1)^{'}$.
15. $e_r \overset{\rightrightarrows}{q}_i e_r = l e_r$ *for* $i = r, r^{'}, (r+1), (r+1)^{'}$
16. $s_i \overset{\rightrightarrows}{q}_{i+1} = \overset{\rightrightarrows}{q}_i s_i$ *for* $i \neq r, r^{'}$.
17. $e_r \overset{\rightrightarrows}{q}_j = e_r \overset{\rightrightarrows}{q}_{j-1}$ *for* $j = (r+1), (r+1)^{'}$.
18. $\overset{\rightrightarrows}{q}_j e_r = \overset{\rightrightarrows}{q}_{j-1} e_r$ *for* $j = (r+1), (r+1)^{'}$.
19. $e_r \overset{\rightrightarrows}{q}_i = \overset{\rightrightarrows}{q}_i e_r$ *for* $i \neq r, r^{'}, (r+1), (r+1)'$.
20. $e_r \overset{\rightrightarrows}{q}_i s_{r+1} e_r = e_r \overset{\rightrightarrows}{q}_{i+2}$ *for* $i = r, r'$.
21. $e_r \overset{\rightrightarrows}{q}_i s_{r+1} e_r = e_r \overset{\rightrightarrows}{q}_{i+1}$ *for* $i = r+1, (r+1)'$.
where $r' + i = (r+i)'$

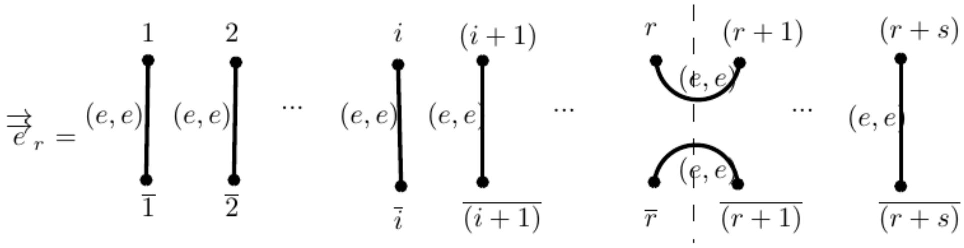




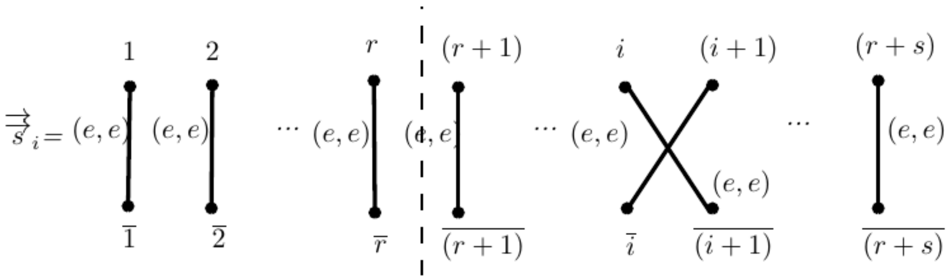
For $1 \leq i \leq r-1$,



For $i = r$,



For $r+1 \leq i \leq r+s-1$



Proof. The walled Brauer algebra $D_{r,s}(l^2)$ for the symmetric group $\mathbf{S}_{r+s}$ is generated by $s_1,\ s_2,\ \ldots,\ s_{r-1},\ e_r,\ s_{r+1},\ \ldots,\ s_{r+s-1}$ with the relations from (1) to (9).

By the universal property of free algebra any homomorphism from $\{s_1,\ s_2,\ \ldots,\ s_{r-1},\ e_r,\ s_{r+1},\ \ldots,\ s_{r+s-1}\}$ to B, where B is the associative algebra over K and is generated by $p_1,\ p_{1'},\ p_{r+1},\ p_{(r+1)'},\ t_1,\ t_2,\ \ldots,\ t_{r-1},\ x_r,\ t_{r+1},\ \ldots,\ t_{r+s-1}$ satisfies the relations from (1) to (21) can be extended to a homomorphism from $D_{r,s}(l^2)$ to B. So there exist a unique algebra homomorphism,

$\Psi' : D_{r,s}(l^2) \to B$ such that $\Psi'(s_1),\ \Psi'(s_2),\ \ldots,\ \Psi'(s_{r-1}),\ \Psi'(e_r),\ \Psi'(s_{r+1}),\ \ldots,\ \Psi' s_{r+s-1}$ satisfies the relation from (1) to (9), where $\Psi'(s_i) = t_i$, for $i = 1,\ 2,\ \ldots,\ r-1,\ r+1,\ \ldots,\ r+s-1$ and $\Psi'(e_r) = x_r$.

Similarly the generators $\overrightarrow{q}_1,\ \overrightarrow{q}_{1'},\ \overrightarrow{q}_{r+1},\ \overrightarrow{q}_{(r+1)'},\ s_1,\ s_2,\ \ldots,\ s_{r-1},\ s_{r+1},\ \ldots,\ s_{r+s-1}$ satisfies the relations of the groups $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_r$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_s$ that is the relations (1), (2), (3), (10), (11), (12), (13), (14) and (16).

Again by the universal property of free algebra any homomorphism from $\{\ \overrightarrow{q}_1,\ \overrightarrow{q}_{1'},\ \overrightarrow{q}_{r+1},\ \overrightarrow{q}_{(r+1)'},\ s_1,\ s_2,\ \ldots,\ s_{r-1},\ s_{r+1},\ \ldots,\ s_{r+s-1}\ \}$ to B can be extended to a homomorphism from $K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_r \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_s)$ to B. So there exist a unique algebra homomorphism,

$\Psi'' : K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_r \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_s) \to B$ such that $\Psi''(\overrightarrow{q}_1),\ \Psi''(\overrightarrow{q}_{1'}),\ \Psi''(\overrightarrow{q}_{r+1}),\ \Psi''(\overrightarrow{q}_{(r+1)'}),\ \Psi''(s_1),\ \Psi''(s_2), \ldots,\ \Psi''(s_{r-1}),\ \Psi''(s_{r+1}), \ldots,\ \Psi''(s_{r+s-1})$ satisfies the relations of the groups $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_r$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_s$ that is the relations (1), (2), (3), (10), (11), (12), (13), (14) and (16). Also $\Psi'|_{K(\mathbf{S}_r \times \mathbf{S}_s)} = \Psi''|_{K(\mathbf{S}_r \times \mathbf{S}_s)}$ and $\Psi''(\overrightarrow{q}_1) = p_1$, $\Psi''(\overrightarrow{q}_{1'}) = p_{1'}$, $\Psi''(\overrightarrow{q}_{r+1}) = p_{r+1}$, $\Psi''(\overrightarrow{q}_{(r+1)'}) = p_{(r+1)'}$ and $\Psi''(s_i) = t_i$.

First, for the walled Brauer diagram $w \in D_{r,s}(l^2)$, we prove the following

(i) If $w\overrightarrow{\overrightarrow{q}}_i\overrightarrow{\overrightarrow{q}}_j = w$, then $\Psi'(w)\Psi''(\overrightarrow{\overrightarrow{q}}_i)\Psi''(\overrightarrow{\overrightarrow{q}}_j) = \Psi'(w)$ for $1, 1' \leq i \leq r, r'$ and $(r+1), (r+1)' \leq j \leq (r+s), (r+s)'$.

(ii) If $\overrightarrow{\overrightarrow{q}}_i\overrightarrow{\overrightarrow{q}}_j w = w$, then $\Psi''(\overrightarrow{\overrightarrow{q}}_i)\Psi''(\overrightarrow{\overrightarrow{q}}_j)\Psi'(w) = \Psi'(w)$ for $1, 1' \leq i \leq r, r'$ and $(r+1), (r+1)' \leq j \leq (r+s), (r+s)'$.

(iii) If $\overrightarrow{\overrightarrow{q}}_s w\overrightarrow{\overrightarrow{q}}_t = w$, then $\Psi''(\overrightarrow{\overrightarrow{q}}_s)\Psi'(w)\Psi''(\overrightarrow{\overrightarrow{q}}_t) = \Psi'(w)$ for $1, 1' \leq s, t \leq r, r'$ and $(r+1), (r+1)' \leq s, t \leq (r+s), (r+s)'$.

By extending the calculations for $1' \leq i \leq r'$, $(r+1)' \leq j \leq (r+s)'$ and considering Ψ' is an algebra homomorphism on $D_{r,s}(l^2)$, Ψ'' is an algebra homomorphism on $K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_r \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_s)$ and $\Psi'|_{K(\mathbf{S}_r \times \mathbf{S}_s)} = \Psi''|_{K(\mathbf{S}_r \times \mathbf{S}_s)})$, the proofs of (i), (ii) and (iii) are very similar to the proofs (a), (b) and (c) of the theorem 3.2, [4]. Next, using the results (i), (ii) and (iii), we define the homomorphism from $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ to B by the following way.

If $\overrightarrow{\overrightarrow{w}} \in \overrightarrow{\overrightarrow{D}}_{r,s}(l)$, then there exist $\overrightarrow{\overrightarrow{q}}, \overrightarrow{\overrightarrow{q'}} \in H$ so that $\overrightarrow{\overrightarrow{w}} = \overrightarrow{\overrightarrow{q}}\ w\ \overrightarrow{\overrightarrow{q'}}$, with w being the underlying walled Klein-4 Brauer diagram of $\overrightarrow{\overrightarrow{w}}$ and H being the subgroup of $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_r \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_s$ generated by $\overrightarrow{\overrightarrow{q}}_i$, where $1, 1' \leq i \leq (r+s), (r+s)'$.

If there are other elements $\overrightarrow{\overrightarrow{q''}}, \overrightarrow{\overrightarrow{q'''}} \in H$ and make $\overrightarrow{\overrightarrow{w}} = \overrightarrow{\overrightarrow{q''}}w\overrightarrow{\overrightarrow{q'''}}$ then $\overrightarrow{\overrightarrow{q}}w\overrightarrow{\overrightarrow{q'}} = \overrightarrow{\overrightarrow{q''}}w\overrightarrow{\overrightarrow{q'''}}$ implies that $w = \overrightarrow{\overrightarrow{q}}\overrightarrow{\overrightarrow{q''}}w\overrightarrow{\overrightarrow{q'''}}\overrightarrow{\overrightarrow{q'}}$. So we get,

$$\begin{aligned} \Psi''(\overrightarrow{\overrightarrow{q}}\overrightarrow{\overrightarrow{q''}})\Psi'(w)\Psi''(\overrightarrow{\overrightarrow{q'''}}\overrightarrow{\overrightarrow{q'}}) &= \Psi'(w) \\ \Psi''(\overrightarrow{\overrightarrow{q}})\Psi''(\overrightarrow{\overrightarrow{q''}})\Psi'(w)\Psi''(\overrightarrow{\overrightarrow{q'''}})\Psi''(\overrightarrow{\overrightarrow{q'}}) &= \Psi'(w) \\ \Psi''(\overrightarrow{\overrightarrow{q''}})\Psi'(w)(\Psi''(\overrightarrow{\overrightarrow{q'''}}) &= \Psi''(\overrightarrow{\overrightarrow{q}})\Psi'(w)\Psi''(\overrightarrow{\overrightarrow{q'}}) \end{aligned}$$

Now define $\Psi : \overrightarrow{\overrightarrow{D}}_{r,s}(l) \to B$ by $\Psi(\overrightarrow{\overrightarrow{w}}) = \Psi''(\overrightarrow{\overrightarrow{q}})\ \Psi'(w)\ \Psi''(\overrightarrow{\overrightarrow{q'}})$. $\Psi = \Psi'$ on $D_{r,s}(l^2)$, $\Psi = \Psi''$ on $K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_r \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_s)$ are obvious. We can use the linearity condition to extend it to the entire space. Again by extending the calculations for $1' \leq i \leq r'$, $(r+1)' \leq j \leq (r+s)'$, the proof of the following identity,

$$\Psi(e_{\check{i},\check{j}}\overrightarrow{\overrightarrow{q}}e_{\breve{u},\breve{v}}) = \Psi'(e_{\check{i},\check{j}})\Psi''(\overrightarrow{\overrightarrow{q}})\Psi'(e_{\breve{u},\breve{v}}) \tag{1}$$

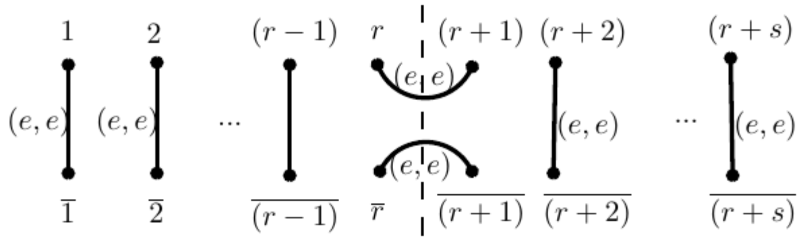
where, $e_{\check{i},\check{j}} = \prod\limits_p e_{i_p,j_p}$, $\quad e_{\breve{u},\breve{v}} = \prod\limits_q e_{u_q,v_q}$, $\quad \overrightarrow{\overrightarrow{q}} = \prod\limits_n \overrightarrow{\overrightarrow{q}}_{k_n}$, where $1 \leq i_p, u_q \leq r$, $(r+1) \leq j_p, v_q \leq (r+s)$, $1, 1' \leq k_n \leq (r+s), (r+s)'$, is similar to the proof of the identity (3.1.1) in theorem 3.2, [4]. Using the identity (1) and the proof of theorem 3.2, [4], we get that Ψ is an isomorphism. □

4. Indexing set for the simple modules of the walled Klein-4 Brauer algebras

This section will cover the indexing set for the simple modules of the walled Klein-4 Brauer algebra $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$, using the idempotent of that algebra and the simple modules of the group algebra of the groups $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_r$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_s$. The walled Klein-4 Brauer algebra $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ is also used to build the recollment towers discussed in [1].

For K an algebraically closed field, $r, s > 0$ and $l \neq 0$ the following is a description of an idempotent element of the walled Klein-4 Brauer algebra. Let $\overrightarrow{\overrightarrow{e}}_{r,s}$ be the walled Klein-4 Brauer diagram multiplied by l^{-2}, with one horizontal curve linking r and $(r+1)$ at the top marked by the sign (e,e), one horizontal curve linking $\overline{r}$ and $\overline{(r+1)}$ at the bottom marked by the sign (e,e). The remaining edges for $i = 1,2,\dots r-1, r+2,\dots,s$, are vertical edges that link i and $\overline{i}$, and they are marked by the sign (e,e). The diagram below demonstrates this.

$\overrightarrow{\overrightarrow{e}}_{r,s} = \frac{1}{l^2}$



$\overrightarrow{\overrightarrow{e}}_{r,s}$ is certainly an idempotent in $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$. Because $(\overrightarrow{\overrightarrow{e}}_{r,s})^2$ equals $\overrightarrow{\overrightarrow{e}}_{r,s}$.

Lemma 4.1. *The algebra $\overrightarrow{\overrightarrow{D}}_{r-1,s-1}(l)$ is isomorphic to the algebra $\overrightarrow{\overrightarrow{e}}_{r,s}\ \overrightarrow{\overrightarrow{D}}_{r,s}(l)\ \overrightarrow{\overrightarrow{e}}_{r,s}$, for $r,s>0$.*

Proof. For every diagram $\overrightarrow{\overrightarrow{d}}$ that belongs to $\overrightarrow{\overrightarrow{e}}_{r,s}\ \overrightarrow{\overrightarrow{D}}_{r,s}(l)\ \overrightarrow{\overrightarrow{e}}_{r,s}$, $\overrightarrow{\overrightarrow{d}}$ can be constructed from $\overrightarrow{\overrightarrow{D}}_{r-1,s-1}(l)$ by inserting two signed vertical edges (vertical edges marked by the sign (e,e) or (a,e) or (e,b) or (a,b)) before and after the wall in some $\overrightarrow{\overrightarrow{f}}$ that belongs to $\overrightarrow{\overrightarrow{D}}_{r-1,s-1}(l)$, so that r is linked with $\overline{r}$ and $(r+1)$ is linked with $\overline{(r+1)}$. By this way we can define an injective homomorphism between the algebras $\overrightarrow{\overrightarrow{D}}_{r-1,s-1}(l)$ and $\overrightarrow{\overrightarrow{e}}_{r,s}\ \overrightarrow{\overrightarrow{D}}_{r,s}(l)\ \overrightarrow{\overrightarrow{e}}_{r,s}$.

For any image $\overrightarrow{\overrightarrow{d}}$ belongs to $\overrightarrow{\overrightarrow{e}}_{r,s}\ \overrightarrow{\overrightarrow{D}}_{r,s}(l)\ \overrightarrow{\overrightarrow{e}}_{r,s}$, we can get a pre-image $\overrightarrow{\overrightarrow{f}}$ belongs to $\overrightarrow{\overrightarrow{D}}_{r-1,s-1}(l)$ by removing the vertical edges with the sign that we inserted to get $\overrightarrow{\overrightarrow{d}}$. As a result, the algebras $\overrightarrow{\overrightarrow{D}}_{r-1,s-1}(l)$ and $\overrightarrow{\overrightarrow{e}}_{r,s}\ \overrightarrow{\overrightarrow{D}}_{r,s}(l)\ \overrightarrow{\overrightarrow{e}}_{r,s}$ are isomorphic. □

Between the algebras, the preceding lemma aids in defining the necessary exact localization and the right exact globalisation functor [3].

Next we will collect some details about the representations and the simple modules of the group $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_n$ from [9].

The collection of all 4-partitions of $(r+s)$, that is $B_{4(r+s)}$ is the indexing set for the inequivalent irreducible representations of the group $(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{r+s}$. For each $\lambda \in B_{4(r+s)}$ the set, $\{\pi_\lambda\colon (\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{r+s} \to \mathrm{End}(V_\lambda)\}$ is a complete set of inequivalent irreducible representation of the group $(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{r+s}$. The set $\{V_\lambda\colon \lambda \in B_{4(r+s)}\}$ is a complete set of simple modules of $(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{r+s}$.

By extending the multiplication turn the representations of the group $(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{r+s}$ into the group algebra $K((\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{r+s})$ the simple modules of $K((\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_r\times(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_s)$ are the outer tensor product of the simple modules of the form $V_{\lambda^L}\boxtimes V_{\lambda^R}$, where V_{λ^L} is the simple module of the group algebra $K((\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_r)$, V_{λ^R} is the simple module of the group algebra $K(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_s)$ and $\lambda^L \in B_{4r}$, $\lambda^R \in B_{4s}$, λ^L and λ^R are the p-regular 4-partition of r and p-regular 4-partition of s respectively [[2], theorem (10.33)].

Consider the indexing set of the simple modules of the group algebra $K((\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_r\times(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_s)$ as $\overrightarrow{\overrightarrow{\mathbf{In}}}{}^{r,s}_{4-reg}$. So we get, $\overrightarrow{\overrightarrow{\mathbf{In}}}{}^{r,s}_{4-reg} = \{(\lambda^L,\ \lambda^R) : \lambda^L$ is a p-regular 4-partition of r and λ^R is a p-regular 4-partition of $s\}$.

If the characteristic p of the field K is zero or $p > max(r,s)$ then the group algebra $K((\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_r\times(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_s)$ is semisimple [2].

Next, we will give the definition and lemma that we will need to construct the indexing set simple modules of the algebra $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$.

Definition 4.2. *The vertical pair of a diagram $\overrightarrow{\overrightarrow{d}}$ that belongs to $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ is an ordered pair (α, β), where α represents the number of signed vertical edges on the left side of the wall and β represents the number of singed vertical edges on the right side of the wall, and the remaining top and bottom vertices are linked in pairs with signed horizontal curves. The edges marked by the signs (e, e) or (a, e) or (e, b) or (a, b) are referred to as signed edges.*

Remark 4.3. *Whenever we multiply the two walled Klein-4 Brauer diagrams $\overrightarrow{\overrightarrow{d}}$ and $\overrightarrow{\overrightarrow{e}}$ with the vertical pairs (α, β) and (γ, δ), we get a walled Klein-4 Brauer diagram with the vertical pair (σ, τ), with $\sigma \leq min(\alpha, \gamma)$ and $\tau \leq min(\beta, \delta)$.*

Lemma 4.4. *(a) The set $\overrightarrow{\overrightarrow{D}}_{r,s}(l)\overrightarrow{\overrightarrow{e}}_{r,s}\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ has a basis of all diagrams with vertical pair (a, b) for some $a \leq (r-1)$ and $b \leq (s-1)$.*

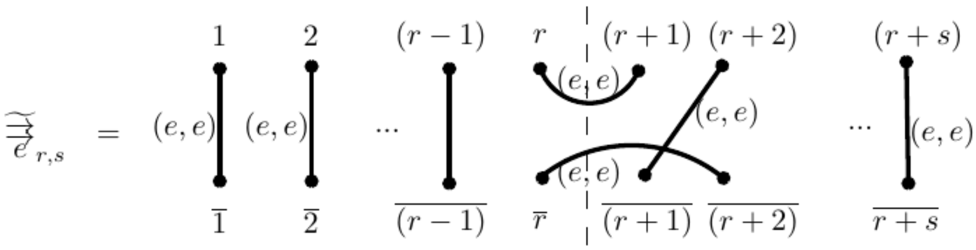
(b) $\overrightarrow{\overrightarrow{D}}_{r,s}(l)\overrightarrow{\overrightarrow{e}}_{r,s}\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ is an ideal of $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$.

(c) The quotient $\overrightarrow{\overrightarrow{D}}_{r,s}(l) \,/\, \overrightarrow{\overrightarrow{D}}_{r,s}(l)\overrightarrow{\overrightarrow{e}}_{r,s}\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ has a basis of all diagrams with vertical pair (r, s).

(d) The quotient $\overrightarrow{\overrightarrow{D}}_{r,s}(l) \,/\, \overrightarrow{\overrightarrow{D}}_{r,s}(l)\overrightarrow{\overrightarrow{e}}_{r,s}\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ is isomorphic to the algebra $K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_r \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_s)$.

Proof. The proofs of *(a)*, *(b)* and *(c)*, follows from the multiplication of the walled Klein-4 Brauer diagrams that we defined in section(3.1), and the definition(4.2), of the vertical pair of the diagram. Again the proof *(d)*, follows from *(c)*. □

Subsequently, for $l = 0$, We define an alternate idempotent $\widetilde{\overrightarrow{\overrightarrow{e}}}_{r,s}$ in a different way as we cannot define the idempotent element of the walled klein-4 algebra as we did before. If r or s is greater than or equal to 2, then $\widetilde{\overrightarrow{\overrightarrow{e}}}_{r,s}$ is the walled Klein-4 Brauer diagram with one upper horizontal curve marked by the sign (e, e) linking r and $(r+1)$, one lower horizontal curve marked by the sign (e, e) connecting $\overline{r}$ and $\overline{(r+2)}$, nodes $(r+2)$ and $\overline{(r+1)}$ are linked by a vertical edge marked by the sign (e, e), and all other edges being vertical lines from i to $\overline{i}$ marked by the sign (e, e). This is depicted in the diagram below,



$\widetilde{\overrightarrow{\overrightarrow{e}}}_{r,s}$ is definitely an idempotent. The lemma(4.1) and lemma(4.4) also applicable to $\widetilde{\overrightarrow{\overrightarrow{e}}}_{r,s}$.

Theorem 4.5. *The indexing set $\overrightarrow{\overrightarrow{\mathbf{In}}}_{r,s}$ of the simple modules of the walled Klein-4 Brauer algebra $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ equals to the disjoint union of $\overrightarrow{\overrightarrow{\mathbf{In}}}^{r-i,s-i}_{4-reg}$, where $0 \leq i \leq min(r, s)$, for $l \neq 0$ or $r \neq s$.*

Proof. The Isomorphism defined in lemma(4.1) allows us to define an exact localization functor [3],

$F : \overrightarrow{\overrightarrow{D}}_{r,s}$- mod $\to \overrightarrow{\overrightarrow{D}}_{r-1,s-1}$- mod by

$L_{r,s}(M) = \overrightarrow{\overrightarrow{e}}_{r,s}(M)$; $M \in \overrightarrow{\overrightarrow{D}}_{r,s}$- mod, and a corresponding right exact globalization functor, $G_{r-1,s-1}$: $\overrightarrow{\overrightarrow{D}}_{r-1,s-1}$- mod $\to \overrightarrow{\overrightarrow{D}}_{r,s}$ - mod by,

$G_{r-1,s-1}(N) = \overrightarrow{D}_{r,s}\ \overrightarrow{e}_{r,s} \otimes_{\overrightarrow{e}_{r,s}\overrightarrow{D}_{r,s}\overrightarrow{e}_{r,s}} (N);\ N \in \overrightarrow{D}_{r-1,s-1}$ - mod.

By theorem 1 in [1] and the isomorphism (d) in lemma(4.4), we have that for $r, s > 0$,

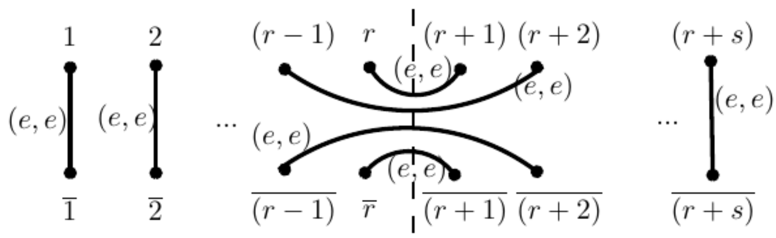
$$\begin{aligned}\overrightarrow{\mathbf{In}}_{r,s} &= \overrightarrow{\mathbf{In}}_{r-1,s-1} \sqcup \overrightarrow{\mathbf{In}}^{r,s}_{4-reg}\\ &= \overrightarrow{\mathbf{In}}_{r-2,s-2} \sqcup \overrightarrow{\mathbf{In}}_{r-1,s-1} \sqcup \overrightarrow{\mathbf{In}}^{r,s}_{4-reg} \text{ and so on, we have}\\ \overrightarrow{\mathbf{In}}_{r,s} &= \bigsqcup_{i=0}^{min(r,s)} \overrightarrow{\mathbf{In}}^{r-i,s-i}_{4-reg} \text{ as } \overrightarrow{D}_{r,0} \cong \overrightarrow{D}_{0,r} \cong K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_r)\end{aligned}$$

□

Next we define the sequence of idempotents $\overrightarrow{e}_{r,s,i}$ in the walled Klein-4 Brauer algebra $\overrightarrow{D}_{r,s}$ (l) to produce the chain of two sided ideals in that algebra.

Consider, $\overrightarrow{e}_{r,s,0} = 1$ and $\overrightarrow{e}_{r,s,i}$ is the isomorphic image of $(\overrightarrow{e}_{r-1,s-1,i-1})$, which we defined in the lemma(4.1), for $1 \leq i \leq min(r,s)$. The element $\overrightarrow{e}_{r,s,i}$ is undefined when $l = 0$, $i = r$, and $r = s$. To these sequence of idempotents we define associate quotients, $\overrightarrow{D}_{r,s,i} = \overrightarrow{D}_{r,s} / \overrightarrow{D}_{r,s}\ \overrightarrow{e}_{r,s,i}\ \overrightarrow{D}_{r,s}$.

When $l \neq 0$, we can use our explicit explanation of the isomorphism defined in the lemma 4.1 to provide an equivalent description of $\overrightarrow{e}_{r,s,i}$ as l^{-2i} times the walled Klein-4 Brauer diagram with i upper horizontal curves marked by the sign (e,e) linking $r-t$ to $r+1+t$, j lower horizontal curves marked by the sign (e,e) linking $\overline{r-t}$ to $\overline{r+1+t}$ for $0 \leq t \leq i-1$, and the remaining edges are all vertical marked by the sign (e,e) and linking i to $\overline{i}$. In the case of $l = 0$, a similar justification can be provided. When $i = 2$ and $l \neq 0$, the idempotent $\overrightarrow{e}_{r,s,2}$ is shown in the diagram below.

$\overrightarrow{e}_{r,s,2} = \frac{1}{l^4}$ 

Let $\overrightarrow{A}_i = \overrightarrow{D}_{r,s}\ \overrightarrow{e}_{r,s,i}\ \overrightarrow{D}_{r,s}$ then, $\overrightarrow{A}_0 = \overrightarrow{D}_{r,s}\ \overrightarrow{e}_{r,s,0}\ \overrightarrow{D}_{r,s} = \overrightarrow{D}_{r,s}$, $\overrightarrow{A}_1 = \overrightarrow{D}_{r,s}\ \overrightarrow{e}_{r,s,1}\ \overrightarrow{D}_{r,s} \subset \overrightarrow{D}_{r,s}$ and so on. So we get the sequence of ideals $\overrightarrow{A}_i$ such that,

$$\ldots \subset \overrightarrow{A}_{i+1} \subset \overrightarrow{A}_i \subset \overrightarrow{A}_{i-1} \subset \ldots \subset \overrightarrow{A}_1 \subset \overrightarrow{A}_0 = \overrightarrow{D}_{r,s} \tag{2}$$

Remark 4.6. *The ideal* $\overrightarrow{A}_j$, *for* $j = 1, 2, 3, \ldots, i-1, i, i+1, \ldots,$ *has a basis of all diagrams with veritcal pair* $\overrightarrow{(\alpha, \beta)}$ *for some* $\alpha \leq r-j$ *and* $\beta \leq s-j$.

In particular the quotient $\overrightarrow{A}_j / \overrightarrow{A}_{j+1}$ *for* $j = 1, 2, 3, \ldots, i-1, i, i+1, \ldots,$ *has a basis of all diagrams with vertical pair* $\overrightarrow{(r-j, s-j)}$.

5. Walled Klein-4 Brauer algebras are iterated inflations

We show that walled Klein-4 Brauer algebras are iterated inflations [5], of group algebra of the group $((\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_r\times(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_s)$ in this section. As a result, we will see that walled Klein-4 Brauer algebras are cellular, and we can also see that the simple modules of the walled Klein-4 Brauer algebra over a field K are described. We will start by proving some results for constructing inflated algebras.

5.1. One row representation of Walled Klein-4 Brauer diagram

Each of the Walled Klein-4 Brauer diagram $\overset{\rightrightarrows}{d}\in\overset{\rightrightarrows}{D}_{r,s}$ with f top signed horizontal curves, f bottom signed horizontal curves and the remaining edges are being vertical can be represented uniquely by a partial one row diagram such that $\overset{\rightrightarrows}{d}=Y_{\overset{\rightrightarrows}{\mu},\overset{\rightrightarrows}{\nu},\rho_{(\mu\nu)}}$. Here $\overset{\rightrightarrows}{\mu}$ stands for the top signed horizontal curves configuration, $\overset{\rightrightarrows}{\nu}$ for the bottom signed horizontal curves configuration and $\rho_{(\mu\nu)}$ for the remaining signed vertical edge configuration obtained by renumbering the top vertices of the signed vertical edges from left to right as $1,2,\ldots,r-f,r-f+1,r-f+2,\ldots,r+s-2f$ and their bottom vertices from left to right as $\bar{1},\bar{2},\ldots,\overline{r-f},\overline{r-f+1},\overline{r-f+2},\ldots,\overline{r+s-2f}$. Also $\rho_{(\mu\nu)}=(\rho,\zeta)\in(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{r-f}\times(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{s-f}$ such that $\rho(i)=\bar{j}$ if the i^{th} top vertex of the the vertical edge is linked to the bottom vertex $\bar{j}$, where $\rho\in\mathbf{S}_{r-f}\times\mathbf{S}_{s-f}$ and ζ is a map from $\{1,2,\ldots,r-f,r-f+1,r-f+2,\ldots,r+s-2f\}\longrightarrow\mathbb{Z}_2\times\mathbb{Z}_2$ with,

$$\zeta(i)=\begin{cases}(e,e), & \text{if the sign } (e,e) \text{ is marked on the vertical edge from } i \text{ to some } \bar{j};\\ (a,e), & \text{if the sign } (a,e) \text{ is marked on the vertical edge from } i \text{ to some } \bar{j};\\ (e,b), & \text{if the sign } (e,b) \text{ is marked on the vertical edge from } i \text{ to some } \bar{j};\\ (a,b), & \text{if the sign } (a,b) \text{ is marked on the vertical edge from } i \text{ to some } \bar{j}.\end{cases}$$

The set of elements $\overset{\rightrightarrows}{\mu}$ that arise in this way is denoted by $v^f_{r,s}$, and the same notation is used to represent the set of elements $\overset{\rightrightarrows}{\nu}$ that arise. This is considered to as the set of partial one-row (r,s,f) diagrams.

Lemma 5.1. *An involution i from $\overset{\rightrightarrows}{D}_{r,s}(l)$ to $\overset{\rightrightarrows}{D}_{r,s}(l)$ is defined by transferring a diagram $\overset{\rightrightarrows}{d}\in\overset{\rightrightarrows}{D}_{r,s}(l)$ to a diagram $i(\overset{\rightrightarrows}{d})\in\overset{\rightrightarrows}{D}_{r,s}(l)$, which is the reflection in the horizontal axis of that diagram $\overset{\rightrightarrows}{d}$.*

Proof. It is obvious that $i^2=i$ since the diagram $\overset{\rightrightarrows}{d}\in\overset{\rightrightarrows}{D}_{r,s}(l)$ will be the same if we reflect it two times around its horizontal axis. □

We now prove the algebra $V_m\otimes V_m\otimes K((\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{r-m}\times(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{s-m})$ is an inflation of the group algebra of the group $(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{r-m}\times(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{s-m}$ along a free K-module V_m of rank $|v^m_{r,s}|$ with respect to some bilinear form ϕ_m from $V_m\otimes V_m\to K((\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{r-m}\times(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{s-m})$ which we will define in the following lemma. Also the multiplication is defined on the basis elements of the algebra $V_m\otimes V_m\otimes K((\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{r-m}\times(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{s-m})$ as follows. For $\overset{\rightrightarrows}{\mu}_1\otimes\overset{\rightrightarrows}{\nu}_1\otimes\rho_{(\mu\nu)_1}$, $\overset{\rightrightarrows}{\mu}_2\otimes\overset{\rightrightarrows}{\nu}_2\otimes\rho_{(\mu\nu)_2}\in V_m\otimes V_m\otimes K((\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{r-m}\times(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{s-m})$ we have,

$$(\overset{\rightrightarrows}{\mu}_1\otimes\overset{\rightrightarrows}{\nu}_1\otimes\rho_{(\mu\nu)_1}).(\overset{\rightrightarrows}{\mu}_2\otimes\overset{\rightrightarrows}{\nu}_2\otimes\rho_{(\mu\nu)_2})=\overset{\rightrightarrows}{\mu}_1\otimes\overset{\rightrightarrows}{\nu}_2\otimes\rho_{(\mu\nu)_1}\phi_m(\overset{\rightrightarrows}{\nu}_1,\overset{\rightrightarrows}{\mu}_2)\rho_{(\mu\nu)_2}. \tag{3}$$

Lemma 5.2. *Fix an index m, then the K-algebra $\overset{\rightrightarrows}{A}_m/\overset{\rightrightarrows}{A}_{m+1}$ is isomorphic to the algebra $V_m\otimes V_m\otimes K((\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{r-m}\times(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{s-m})$ with respect to some bilinear form which we will describe later in the proof.*

Proof. By applying the partial one row diagram representation of a walled Klein-4 Brauer diagram that we mentioned previously we can define a K-isomorphism $\Psi:V_m\otimes V_m\otimes K((\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{r-m}\times(\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbf{S}_{s-m})\to\overset{\rightrightarrows}{A}_m/\overset{\rightrightarrows}{A}_{m+1}$ as follows,

$$\Psi(\overrightarrow{\overrightarrow{\mu}} \otimes \overrightarrow{\overrightarrow{\nu}} \otimes \rho_{(\mu\nu)}) = Y_{\overrightarrow{\overrightarrow{\mu}},\overrightarrow{\overrightarrow{\nu}},\rho_{(\mu\nu)}}$$

We now define the bilinear form ϕ_m, which is required for multiplication of elements from the algebra $V_m \otimes V_m \otimes K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r-m} \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{s-m})$. For $\overrightarrow{\overrightarrow{\mu}}_1 \otimes \overrightarrow{\overrightarrow{\nu}}_1 \otimes \rho_{(\mu\nu)_1}$, $\overrightarrow{\overrightarrow{\mu}}_2 \otimes \overrightarrow{\overrightarrow{\nu}}_2 \otimes \rho_{(\mu\nu)_2} \in V_m \otimes V_m \otimes K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r-m} \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{s-m})$ we have,

$$\begin{aligned}\Psi(\overrightarrow{\overrightarrow{\mu}}_1 \otimes \overrightarrow{\overrightarrow{\nu}}_1 \otimes \rho_{(\mu\nu)_1}) &= Y_{\overrightarrow{\overrightarrow{\mu}}_1,\overrightarrow{\overrightarrow{\nu}}_1,\rho_{(\mu\nu)_1}} \in \overrightarrow{\overrightarrow{D}}_{r,s}(l) \text{ and}\\ \Psi(\overrightarrow{\overrightarrow{\mu}}_2 \otimes \overrightarrow{\overrightarrow{\nu}}_2 \otimes \rho_{(\mu\nu)_2}) &= Y_{\overrightarrow{\overrightarrow{\mu}}_2,\overrightarrow{\overrightarrow{\nu}}_2,\rho_{(\mu\nu)_2}} \in \overrightarrow{\overrightarrow{D}}_{r,s}(l)\end{aligned}$$

Using the multiplication of the elements of the walled klein-4 diagram algebra $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$, that we described in the section 3.1 we have,

$$Y_{\overrightarrow{\overrightarrow{\mu}}_1,\overrightarrow{\overrightarrow{\nu}}_1,\rho_{(\mu\nu)_1}} . Y_{\overrightarrow{\overrightarrow{\mu}}_2,\overrightarrow{\overrightarrow{\nu}}_2,\rho_{(\mu\nu)_2}} = l^t . \overrightarrow{\overrightarrow{(\mu\nu)}} \tag{4}$$

where t is the number of closed loops in the product (4), and $\overrightarrow{\overrightarrow{(\mu\nu)}} \in \overrightarrow{\overrightarrow{D}}_{r,s}(l)$ having $2m$ or more signed horizontal edges.

If the product (4), does not have vertical pair $\overrightarrow{\overrightarrow{(r-m, s-m)}}$ then, set $\phi_m(\overrightarrow{\overrightarrow{\nu}}_1, \overrightarrow{\overrightarrow{\mu}}_2) = 0$, otherwise $\phi_m(\overrightarrow{\overrightarrow{\nu}}_1, \overrightarrow{\overrightarrow{\mu}}_2) = l^t.\rho_{\mu\nu}$; where $\rho_{(\mu\nu)} \in K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r-m} \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{s-m})$ such that,

$$\begin{aligned}Y_{\overrightarrow{\overrightarrow{\mu}}_1,\overrightarrow{\overrightarrow{\nu}}_1,\rho_{(\mu\nu)_1}} \cdot Y_{\overrightarrow{\overrightarrow{\mu}}_2,\overrightarrow{\overrightarrow{\nu}}_2,\rho_{(\mu\nu)_2}} &= l^t \cdot \overrightarrow{\overrightarrow{(\mu\nu)}}\\ &= l^t \cdot Y_{\overrightarrow{\overrightarrow{\mu}}_1,\overrightarrow{\overrightarrow{\nu}}_2,\rho_{(\mu\nu)_1}\rho_{(\mu\nu)}\rho_{(\mu\nu)_2}}\end{aligned}$$

So we have,

$$\begin{aligned}\Psi(\overrightarrow{\overrightarrow{\mu}}_1 \otimes \overrightarrow{\overrightarrow{\nu}}_1 \otimes \rho_{(\mu\nu)_1} \cdot \overrightarrow{\overrightarrow{\mu}}_2 \otimes \overrightarrow{\overrightarrow{\nu}}_2 \otimes \rho_{(\mu\nu)_2}) &= \Psi(\overrightarrow{\overrightarrow{\mu}}_1 \otimes \overrightarrow{\overrightarrow{\nu}}_2 \otimes \rho_{(\mu\nu)_1}\phi_m(\overrightarrow{\overrightarrow{\mu}}_1, \overrightarrow{\overrightarrow{\nu}}_2)\rho_{(\mu\nu)_2})\\ &= \Psi(\overrightarrow{\overrightarrow{\mu}}_1 \otimes \overrightarrow{\overrightarrow{\nu}}_2 \otimes \rho_{(\mu\nu)_1}(l^t\rho_{(\mu\nu)})\rho_{(\mu\nu)_2})\\ &= l^t \cdot Y_{\overrightarrow{\overrightarrow{\mu}}_1^+ \otimes \overrightarrow{\overrightarrow{\nu}}_1^- \otimes \rho_{(\mu\nu)_1}\rho_{(\mu\nu)}\rho_{(\mu\nu)_2}}\\ &= Y_{\overrightarrow{\overrightarrow{\mu}}_1,\overrightarrow{\overrightarrow{\nu}}_1,\rho_{(\mu\nu)_1}} \cdot Y_{\overrightarrow{\overrightarrow{\mu}}_2,\overrightarrow{\overrightarrow{\nu}}_2^-,\rho_{(\mu\nu)_2}}\\ &= \Psi(\overrightarrow{\overrightarrow{\mu}}_1 \otimes \overrightarrow{\overrightarrow{\nu}}_1^- \otimes \rho_{(\mu\nu)_1}) \cdot \Psi(\overrightarrow{\overrightarrow{\mu}}_2 \otimes \overrightarrow{\overrightarrow{\nu}}_2 \otimes \rho_{(\mu\nu)_2})\end{aligned}$$

Ψ is an algebra homomorphism. Suppose that,

$$\begin{aligned}\Psi(\overrightarrow{\overrightarrow{\mu}}_1 \otimes \overrightarrow{\overrightarrow{\nu}}_1 \otimes \rho_{(\mu\nu)_1}) &= \Psi(\overrightarrow{\overrightarrow{\mu}}_2 \otimes \overrightarrow{\overrightarrow{\nu}}_2 \otimes \rho_{(\mu\nu)_2}) \text{ then,}\\ Y_{\overrightarrow{\overrightarrow{\mu}}_1,\overrightarrow{\overrightarrow{\nu}}_1,\rho_{(\mu\nu)_1}} &= Y_{\overrightarrow{\overrightarrow{\mu}}_2^+,\overrightarrow{\overrightarrow{\nu}}_2,\rho_{(\mu\nu)_2}}\end{aligned}$$

So we have $\overrightarrow{\overrightarrow{\mu}}_1 = \overrightarrow{\overrightarrow{\mu}}_2$, $\overrightarrow{\overrightarrow{\nu}}_1 = \overrightarrow{\overrightarrow{\nu}}_2$, $\rho_{(\mu\nu)_1} = \rho_{(\mu\nu)_2}$.

Now let $\overrightarrow{\overrightarrow{(\mu\nu)}} \in A_m/A_{m+1}$ then $\overrightarrow{\overrightarrow{(\mu\nu)}} = Y_{\overrightarrow{\overrightarrow{\mu}}^+,\overrightarrow{\overrightarrow{\nu}}^-,\rho_{(\mu\nu)}}$ and $\Psi(\overrightarrow{\overrightarrow{\mu}} \otimes \overrightarrow{\overrightarrow{\nu}} \otimes \rho_{(\mu\nu)}) = Y_{\overrightarrow{\overrightarrow{\mu}},\overrightarrow{\overrightarrow{\nu}},\rho_{(\mu\nu)}} = \overrightarrow{\overrightarrow{(\mu\nu)}}$, here $\overrightarrow{\overrightarrow{\mu}} \otimes \overrightarrow{\overrightarrow{\nu}} \otimes \rho_{(\mu\nu)} \in V_m \otimes V_m \otimes K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r-m} \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{s-m})$. Hence Ψ is an isomorphism. □

Lemma 5.3. *The involution i from $\overrightarrow{\overrightarrow{A}}_m/\overrightarrow{\overrightarrow{A}}_{m+1}$ to $\overrightarrow{\overrightarrow{A}}_m/\overrightarrow{\overrightarrow{A}}_{m+1}$ under the map Ψ corresponds to the standard involution on $V_m \otimes V_m \otimes K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r-m} \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{s-m})$, which sends $\overrightarrow{\overrightarrow{\mu}} \otimes \overrightarrow{\overrightarrow{\nu}} \otimes \rho_{(\mu\nu)}$ to $\overrightarrow{\overrightarrow{\nu}} \otimes \overrightarrow{\overrightarrow{\mu}} \otimes \rho_{(\mu\nu)^-}$.*

Proof. The involution on $\overrightarrow{\overrightarrow{A}}_m/\overrightarrow{\overrightarrow{A}}_{m+1}$ is derived from the involution on $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ stated in lemma 5.1. As a result, the proof follows from the previous lemma 5.2. $\square$

Lemma 5.4. *Let $\overrightarrow{\overrightarrow{d}}_1 \in \overrightarrow{\overrightarrow{A}}_m/\overrightarrow{\overrightarrow{A}}_{m+1}$ and $\overrightarrow{\overrightarrow{d}}_2 \in \overrightarrow{\overrightarrow{A}}_n/\overrightarrow{\overrightarrow{A}}_{n+1}$ be two walled Klein-4 Brauer diagrams from $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ and let their respective Ψ-pre images be $\overrightarrow{\overrightarrow{\mu}}_1 \otimes \overrightarrow{\overrightarrow{\nu}}_1 \otimes \rho_{(\mu\nu)_1}$ and $\overrightarrow{\overrightarrow{\mu}}_2 \otimes \overrightarrow{\overrightarrow{\nu}}_2 \otimes \rho_{(\mu\nu)_2}$. We assume that $n \geq m$. Then the product $\overrightarrow{\overrightarrow{d}}_1.\overrightarrow{\overrightarrow{d}}_2$ is either an element of $\overrightarrow{\overrightarrow{A}}_n/\overrightarrow{\overrightarrow{A}}_{n+1}$, or is an element of $\overrightarrow{\overrightarrow{A}}_{n+1}$. Under Ψ it corresponds to the scalar multiple of an element $\overrightarrow{\overrightarrow{\mu'}} \otimes \overrightarrow{\overrightarrow{\mu}}_2 \otimes \eta\rho_{(\mu\nu)_2}$; where $\overrightarrow{\overrightarrow{\mu'}} \in V_n$ and $\eta \in K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r-n} \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{s-n})$.*

In the case of $n \leq m$, a similar assumption holds true.

Proof. Both statements have a proof that is extremely similar to lemma 5.2. $\square$

As a result of the lemma 5.4, we can now define the bilinear mappings α, β, γ described in [5], which are required for iterated inflation. We have proven the following theorem in total.

Theorem 5.5. *The walled Klein-4 Brauer algebra $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ is an iterated inflation of group algebras of groups $((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r-m} \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{s-m})$, where $0 \leq m \leq min(r,s)$. Specifically as a free K-module $\overrightarrow{\overrightarrow{D}}_{r,s}$ is equal to $K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_r \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_s) \oplus (V_1 \otimes V_1 \otimes K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r-1} \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{s-1})) \oplus (V_2 \otimes V_2 \otimes K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r-2} \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{s-2})) \oplus \cdots$, and inflation begins with $K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_r \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_s)$ inflates it along $V_1 \otimes V_1 \otimes K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r-1} \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{s-1})$ and so on, concluding with an inflation of $K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_1)$ or $K((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_0)$ as bottom layer depending on whether $(r+s)$ is even or odd.*

Proof. The proof follows from the above lemma 5.2, lemma 5.3 and lemma 5.4. $\square$

Corollary 5.6. *The walled Klein-4 Brauer algebra $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ over any field of characteristic $p = 0$ or $p > max(r,s)$ is cellular.*

Proof. From the theorem 5.5, the walled Klein-4 Brauer algebra $\overrightarrow{\overrightarrow{D}}_{r,s}$ is an iterated inflation of group algebras of the group $((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{r-m} \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbf{S}_{s-m})$ for $0 \leq m \leq min(r,s)$ along V_m. So the proof follows from the remarks 2.5 and 2.6. $\square$

Corollary 5.7. *If $l \neq 0$ or $r \neq s$ then the simple modules of the walled Klein-4 Brauer algebras $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ are indexed by all pair $(m, \lambda^L, \lambda^R)$, where $0 \leq m \leq min(r,s)$ and $(\lambda^L, \lambda^R) \in \wedge_{4-reg}^{r-m,s-m}$.*

If $l = 0$ and $r = s$ we get the same indexing set for the simple modules as above but with the single simple corresponding to $l = min(r,s)$.

Proof. By theorem 4.5, for $l \neq 0$ or $r \neq s$, the simple modules of the walled Klein-4 Brauer algebras $\overrightarrow{\overrightarrow{D}}_{r,s}(l)$ are indexed by all pair $(m, \lambda^L, \lambda^R)$, where $0 \leq m \leq min(r,s)$ and $(\lambda^L, \lambda^R) \in \wedge_{4-reg}^{r-m,s-m}$.

In the case of $l = 0$, the above assertion is also valid except that the case $m = 0$ (which occurs only for r even) does not contribute a simple module [6].

The exception in this situation arises from the fact that in the iterated inflation, there is a piece with ϕ_m equal to zero [6]. Here ϕ_m is the bilinear form defined in the lemma 5.2.

$\square$

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