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Left-to-right maxima in Dyck prefixes

Research Article

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Abstract: In a Dyck path, a peak which is strictly (weakly) higher than all the preceding peaks is called a strict (weak) left-to-right maximum. By dropping the restrictions for the path to end on the x-axis, one obtains Dyck prefixes. We obtain explicit generating functions for both weak and strict left-to-right maxima in Dyck prefixes. The proofs of the associated asymptotics make use of analytic techniques such as Mellin transforms, singularity analysis and formal residue calculus.

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1. General introduction

A Dyck prefix is a lattice path in the first quadrant, that starts at the origin (0,0) with an up step ($u = (1, 1)$) and thereafter only up and down ($d = (1, -1)$) steps are allowed under the condition that it may not go below the x-axis. If a Dyck prefix has its end at $(x, y) = (n, i)$ then we say that it has n steps. For a detailed study of properties of Dyck paths (i.e., Dyck prefixes which end on the x-axis) see [7]. For further recent works on Dyck paths and prefixes, see [1, 2, 4–6, 12, 18].

Given an arbitrary Dyck prefix, by a *strict left-to-right maximum*, we mean any peak (successive pair of the form ud) in the path whose height is greater than or equal to that of all peaks to its left, or in the case of the last step, any u step which is above all steps to its left. A *weak left-to-right maximum* is a peak which is greater than or equal to all peaks to its left with the obvious exception of the last step as already detailed in the strict case. For ordinary Dyck paths, left-to-right maxima have been studied in [3]. For left-to-right maxima in other combinatorial structures such as set partitions and compositions of integers see [10, 11].

A standard combinatorial problem is the accounting for the number of left-to-right maxima in combinatorial structures such as permutations and words over a fixed alphabet. In this paper we focus on

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obtaining a generating function for the number of left-to-right maxima in Dyck prefixes. This is a bivariate generating function which tracks the number of steps by z and the number of left-to-right maxima by x. From this we also obtain a generating function for the total number of left-to-right maxima in Dyck prefixes with n steps.

As an introduction to the method for the first generating function above, here follows a sketch (Figure 1) of two Dyck prefixes of height 3. The left-to-right maxima are marked in the first case by A and P and in the second by E and P. P also marks the first maximum height attained by the Dyck prefixes. We begin at the origin with a u step tracked in the generating function by z which brings us to the point E. This single up step is followed by a possibly empty upside-down Dyck path of maximum height 1. In the left example in Figure 1, this part is indeed empty (and therefore not requiring x) but not in the right example where the path between E and B is an upside-down Dyck path of height 1 which gives rise to a left to right maximum, thus requiring an x tracker. Then we have another single u step and we proceed recursively in this way leaving us eventually at the next left to right maximum which is point A in the first example and P in the second. In the first example, right of A is again a possibly empty upside-down Dyck path, this time of maximum height 2 where the non- empty case is tracked again by x. We are referring to the path between A and B which is actually of height 1. Once P is reached, it is followed by the rest of the path which is a prefix of bounded height without any further left-to-right maxima. In the section dealing with this, the generating function for these latter Dyck prefixes ending at height r will be used, as will the generating function for Dyck paths of a fixed height h, which is used as indicated above for the possibly empty upside-down Dyck paths that occur sequentially before the point P is attained.

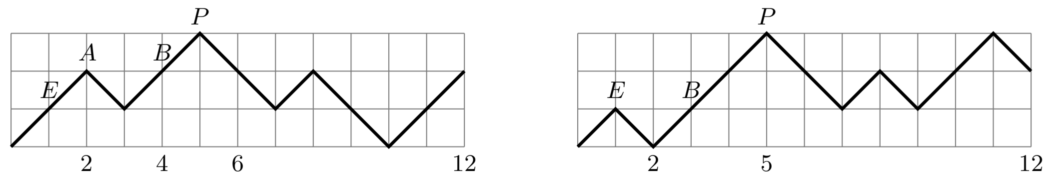

Figure 1. Two Dyck prefixes of length 12 and height 3

2. Strict left-to-right maxima in Dyck prefixes

We start this section by referring to the paper [9] by R. Kemp. We let $C(h)$ be the number of Dyck prefixes of height $\leq h$ whose steps follow all rules of Dyck paths except that they terminate at any height between 0 and h, and we let $A(h)$ be the number of Dyck paths of height $\leq h$ (which by definition end at height zero). It is shown in [9] and in [17] that

$$C(h) := \frac{\left(1-\sqrt{1-4z^2}\right)\left(1-(\frac{\lambda_2}{z})^{h+1}\right)}{z\left(-1+2z+\sqrt{1-4z^2}\right)\left(1+(\frac{\lambda_2}{z})^{h+2}\right)} \tag{2.1}$$

and

$$A(h) := \frac{{\lambda_1}^{h+1}-{\lambda_2}^{h+1}}{{\lambda_1}^{h+2}-{\lambda_2}^{h+2}}, \tag{2.2}$$

where λ_1 and λ_2, are given by

$$\lambda_1 = \frac{1+\sqrt{1-4z^2}}{2}; \lambda_2 = \frac{1-\sqrt{1-4z^2}}{2}. \tag{2.3}$$

From (2.1) we deduce by letting $h \to \infty$ that the generating function for all prefixes of length n is

$$\frac{1-\sqrt{1-4z^2}}{z\left(\sqrt{1-4z^2}+2z-1\right)}$$

and the number of such prefixes is

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}. \tag{2.4}$$

As explained in the introductory section, we consider a sequence of possibly empty Dyck paths of height $\leq h$ for $h \geq 1$. Between each such path in the sequence, we have a single up step that leads to the next left to right maximum and eventually to the first overall maximum of the entire Dyck prefix. Thereafter there is a prefix of bounded height to conclude the path. We let x count the number of left to right maxima attained by the entire Dyck prefix. This leads to our first theorem.

Theorem 2.1. *The generating function for the number of left-to-right maxima tracked by x, for Dyck prefixes of maximum height r and length tracked by z is*

$$F(x,z,r) := z^r x C(r) \prod_{h=1}^{r-1} (1 + x(A(h)-1)). \tag{2.5}$$

So, the total number of left-to-right maxima for Dyck prefixes of fixed height r is found by differentiating the above function with respect to x and setting $x = 1$. The derivative at this point is given by

$$\begin{aligned} \frac{\partial}{\partial x} F(x,z,r)\Big|_{x=1} &= z^r \ C(r) \prod_{h=1}^{r-1} A(h) + z^r C(r) \prod_{h=1}^{r-1} A(h) \sum_{i=1}^{r-1} \frac{A(i)-1}{A(i)} \\ &= z^r \ C(r) \prod_{h=1}^{r-1} A(h) \left(1 + \sum_{i=1}^{r-1} \frac{A(i)-1}{A(i)}\right) \\ &= z^r C(r) \prod_{h=1}^{r-1} A(h) \left(r - \sum_{i=1}^{r-1} \frac{1}{A(i)}\right). \end{aligned} \tag{2.6}$$

Note that $\prod_{h=1}^{r-1} A(h)$ telescopes to become

$$\frac{2^{1+r}\sqrt{1-4z^2}}{-\left(1-\sqrt{1-4z^2}\right)^{1+r} + \left(1+\sqrt{1-4z^2}\right)^{1+r}}$$

but the full generating function becomes very complicated as a function of z.

To simplify this generating function, we substitute

$$z = \frac{v}{1+v^2} \tag{2.7}$$

in (2.6) and the right hand side becomes

$$T(r) := \frac{v^r(1+v)\left(1+v^2\right)}{\left(1+v^{1+r}\right)\left(1+v^{2+r}\right)} \left(r - \sum_{i=1}^{r-1} \frac{-1+v^{4+2i}}{\left(1+v^2\right)\left(-1+v^{2+2i}\right)}\right). \tag{2.8}$$

The full generating function for the total number of left-to-right maxima in all Dyck prefixes of length n is

$$Tot(v) := \sum_{r=1}^{\infty} T(r). \tag{2.9}$$

Consequently, we have the following result.

The generating function $\texttt{Tot}(v)$ for the total number of left-to-right maxima in Dyck prefixes of length n tracked by z is given by

$$Tot(v) = \sum_{r=1}^{\infty} \frac{v^r(1+v)\left(1+v^2\right)}{\left(1+v^{1+r}\right)\left(1+v^{2+r}\right)} \left(r - \sum_{i=1}^{r-1} \frac{1-v^{4+2i}}{\left(1+v^2\right)\left(1-v^{2+2i}\right)} \right), \tag{2.10}$$

where $z = \frac{v}{1+v^2}$.

In order to obtain the series expansion for this, we use the equivalent inverse substitution for v, namely

$$v = \frac{1-\sqrt{1-4z^2}}{2z}, \tag{2.11}$$

and obtain in terms of z,

$$Tot(v) = z + 2z^2 + 3z^3 + 7z^4 + \mathbf{13}z^5 + 28z^6 + 54z^7 + 114z^8 + 222z^9 + O(z^{10}). \tag{2.12}$$

We illustrate the bold term of the series by means of the black dots in Figure 2.

To simplify equation (2.10) we swap the order of the summations in the double sum, and thereafter use partial fractions on the second sum (which then telescopes as in line (2.13)) to obtain

$$\begin{aligned}
&\sum_{r=1}^{\infty} \frac{v^r(1+v)\left(1+v^2\right)}{\left(1+v^{1+r}\right)\left(1+v^{2+r}\right)} \sum_{i=1}^{r-1} \frac{1-v^{4+2i}}{\left(1+v^2\right)\left(1-v^{2+2i}\right)} \\
&= (1+v) \sum_{i=1}^{\infty} \frac{1-v^{4+2i}}{1-v^{2+2i}} \sum_{r=i+1}^{\infty} \frac{v^r}{\left(1+v^{1+r}\right)\left(1+v^{2+r}\right)} \qquad (2.13) \\
&= \frac{1+v}{(1-v)v} \sum_{i=1}^{\infty} \frac{\left(1-v^{4+2i}\right)v^{i+2}}{\left(1-v^{2+2i}\right)\left(1+v^{i+2}\right)}. \qquad (2.14)
\end{aligned}$$

Now changing the index of summation from i to r, $Tot(v)$ simplifies to

$$\begin{aligned}
Tot(v) &= \sum_{r=1}^{\infty} \frac{v^r(1+v)\left(1+v^2\right)r}{\left(1+v^{1+r}\right)\left(1+v^{2+r}\right)} - \frac{1+v}{(1-v)v} \sum_{r=1}^{\infty} \frac{\left(1-v^{4+2r}\right)v^{r+2}}{\left(1-v^{2+2r}\right)\left(1+v^{r+2}\right)} \\
&= \sum_{r=1}^{\infty} \frac{v^r\left(r\left(1-v^4\right)\left(1-v^{1+r}\right) + v(1+v)\left(-1+v^{4+2r}\right)\right)}{(-1+v)\left(-1+v^{1+r}\right)\left(1+v^{1+r}\right)\left(1+v^{2+r}\right)} \\
&= \sum_{r=1}^{\infty} \left(\frac{v(1+v)}{-1+v} + \frac{1+v}{2\left(-1+v^{1+r}\right)} + \frac{2r - v + 2rv - 2v^2 + 2rv^2 - v^3 + 2rv^3}{2(-1+v)v\left(1+v^{1+r}\right)} \right. \\
&\quad \left. - \frac{(1+v)\left(r+rv^2\right)}{(-1+v)v\left(1+v^{2+r}\right)} \right). \qquad (2.15)
\end{aligned}$$

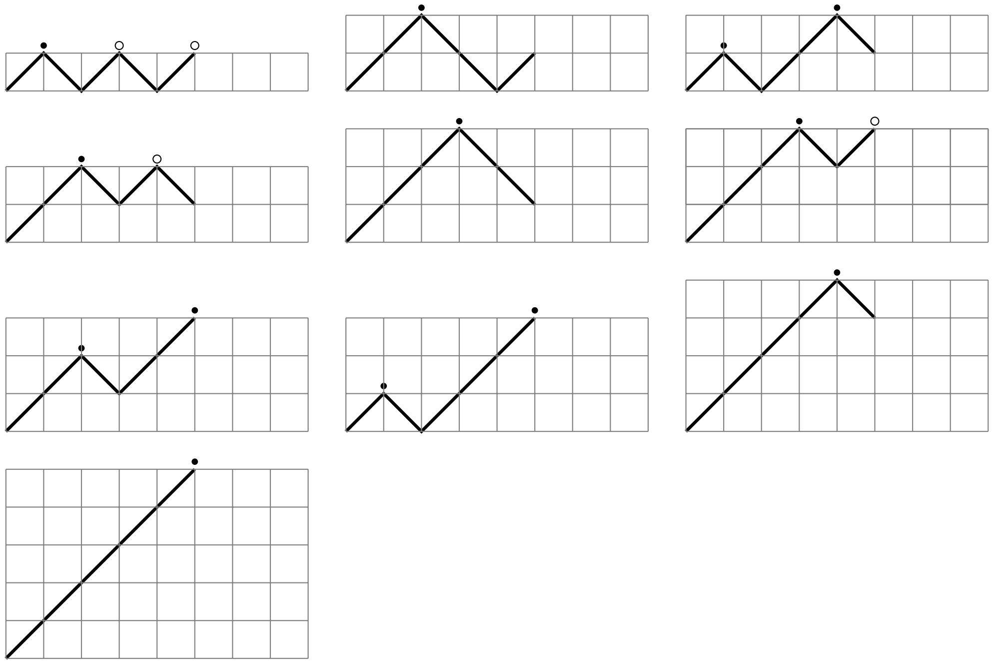

Figure 2. All 10 Dyck prefixes of length 5 with 13 strict left-to-right maxima indicated by black dots and with circles indicating the additional 4 weak left-to-right maxima.

Now ∞ is replaced temporarily by finite M for the last term of the sum above so that

$$\sum_{r=1}^{M} \frac{(1+v)\left(r+rv^2\right)}{(-1+v)v\left(1+v^{2+r}\right)} = \sum_{r=1}^{M} \frac{(1+v)(r-1)\left(1+v^2\right)}{(-1+v)v\left(1+v^{1+r}\right)} + \frac{M(1+v)\left(1+v^2\right)}{(-1+v)v\left(1+v^{2+M}\right)}. \tag{2.16}$$

We now convert the final term above into a sum:

$$\frac{M(1+v)\left(1+v^2\right)}{(-1+v)v\left(1+v^{2+M}\right)} = \sum_{r=1}^{M} \frac{(1+v)\left(1+v^2\right)}{(-1+v)v\left(1+v^{2+M}\right)} = \sum_{r=1}^{M} \frac{(1+v)\left(1+v^2\right)}{(-1+v)v} + O(v^{M+1}). \tag{2.17}$$

Then we again let M tend to ∞, note that then $O(v^{M+1}) \to 0$ and insert (2.16) and (2.17) into (2.15). Simplifying we find that

$$\begin{aligned} Tot(v) &= \sum_{r=1}^{\infty} \left(\frac{1+v}{v-v^2} + \frac{-1-v+v^{1+r}+v^{4+r}}{(-1+v)v\left(-1+v^{2+2r}\right)} \right) \\ &= \frac{1+v}{1-v} \sum_{r=1}^{\infty} \frac{v^r\left(1-v+v^2-v^{1+r}\right)}{1-v^{2+2r}}. \end{aligned} \tag{2.18}$$

This yields the result in the theorem below.

Theorem 2.2. *The simplified generating function for the total number of left-to-right maxima in Dyck prefixes is*

$$\mathtt{Tot}(v) = \frac{1+v^3}{1-v} \sum_{r=1}^{\infty} \frac{v^r}{1-v^{2+2r}} - \frac{1+v}{1-v} \sum_{r=1}^{\infty} \frac{v^{1+2r}}{1-v^{2+2r}}. \tag{2.19}$$

2.1. Formula for total number of left-to-right maxima

In this section, we will obtain an exact formula for the total number of left-to-right maxima in terms of a well-known arithmetic function, namely the divisor function $d(r)$. We define $d^*(i) = d(i) - 1$, $d_o(n)$ is the number of odd divisors of n, and

$$b(n) := \begin{cases} d_o(n) & \text{if } n \text{ is even} \\ d_o(n) - 1 & \text{if } n \text{ is odd} \end{cases}.$$

We shall prove the following theorem.

Theorem 2.3. *The total number of left-to-right maxima in Dyck prefixes of length n is given by*

$$\sum_{r=0}^{n/2} \binom{n-1}{r} \Bigg(b(n-2r-3) + b(n-2r-2) + b(n-2r) + b(n-2r+1) - d^*\left(\frac{1}{2}(n-2r-1)\right) - 2d^*\left(\frac{n-2r}{2}\right) - d^*\left(\frac{1}{2}(n-2r+1)\right)\Bigg). \tag{2.20}$$

We use the convention that $b(j) = d^*(j) = 0$ whenever j is a non-positive integer or non-integer.

Proof. We use the fact that

$$\sum_{r=1}^{\infty} \frac{v^r}{1-v^r} = \sum_{r=1}^{\infty} d(r)\, v^r.$$

To read off the coefficients from equation (2.19), we observe that for any formal power series $f(z)$,

$$[z^n]f(z) = [v^n](1-v^2)(1+v^2)^{n-1} f(z(v)). \tag{2.21}$$

This can be justified by using Cauchy's coefficient formula or by using formal residue calculus, see for example [15]. Let us consider $(1-v^2)Tot[v]$. Firstly

$$\sum_{r=1}^{\infty} \frac{v^{2+2r}}{1-v^{2+2r}} = \sum_{r=1}^{\infty} (d(r)-1) v^{2r}.$$

Because $d^*(r) = d(r) - 1$, it follows that

$$\frac{(1+v)^2}{v} \sum_{r=1}^{\infty} \frac{v^{2+2r}}{1-v^{2+2r}} = \sum_{r=1}^{\infty} d^*(r) v^{2r+1} + 2\sum_{r=1}^{\infty} d^*(r) v^{2r} + \sum_{r=1}^{\infty} d^*(r) v^{2r-1}.$$

Next,

$$\sum_{r=1}^{\infty} \frac{v^{1+r}}{1-v^{2+2r}} = \sum_{i=1}^{\infty} b(i) v^i$$

where $b(n)$ and $d_0(n)$ are defined above.

Thus

$$(1+v)\left(\frac{1}{v} + v^2\right) \sum_{r=1}^{\infty} \frac{v^{1+r}}{1-v^{2+2r}} = \sum_{i=1}^{\infty} (b(i-3) + b(i-2) + b(i) + b(i+1)) v^i$$

with $b(j) = 0$ for $j \leq 0$. Using (2.21), we obtain

$$\begin{aligned}
[z^n]Tot(z) &= [v^n](1-v^2)(1+v^2)^{n-1}Tot(v) \\
&= \sum_{j=0}^{n}[v^j](1+v^2)^{n-1}[v^{n-j}]\sum_{r=1}^{\infty}(1-v^2)Tot(v) \\
&= \sum_{j=0}^{n}\binom{n-1}{j/2}\Bigg(b(n-j-3)+b(n-j-2)+b(n-j)+b(n-j+1) \\
&\quad - d^*\left(\frac{1}{2}(n-j-1)\right) - 2d^*\left(\frac{n-j}{2}\right) - d^*\left(\frac{1}{2}(n-j+1)\right)\Bigg). \qquad (2.22)
\end{aligned}$$

□

3. Asymptotics for strict left-to-right maxima

In this section we find the asymptotic expression for the total number of strict left-to-right maxima in Dyck prefixes. We will follow the approach used to study the height of planted plane trees by Prodinger in [15]. For related asymptotic calculations concerning the height of trees and lattice paths, see [13, 14, 16].

We will prove the following theorem.

Theorem 3.1. *As $n \to \infty$, the average number of strong left-to-right maxima in Dyck prefixes of length n is*

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}^{-1}[z^n]\mathtt{Tot}(v) = \frac{\sqrt{2\pi n}\log(2)}{2} - \frac{\log(n)}{4} + \frac{1}{4}(-1-3\gamma+\log(2)) + O(n^{-1/2}).$$

Proof. We study the asymptotic behaviour of the expression given in Equation 2.19 by analyzing the behaviour of $\frac{1+v^3}{1-v}\sum_{r=1}^{\infty}\frac{v^r}{1-v^{2+2r}}$ and $\frac{1+v}{1-v}\sum_{r=1}^{\infty}\frac{v^{1+2r}}{1-v^{2+2r}}$ separately.

We first consider

$$\frac{1+v^3}{(1-v)v}\sum_{r=1}^{\infty}\frac{v^{1+r}}{1-v^{2+2r}}. \qquad (3.1)$$

When v is in terms of z, by (2.11) the function $Tot(v)$ has its dominant singularity at $z = 1/2$ which is mapped to $v = 1$. To study this further we set $v = e^{-t}$ and let $t \to 0$. Thus

$$\frac{e^t\left(1+e^{-3t}\right)}{1-e^{-t}} = \frac{2}{t} + \frac{13t}{6} + \frac{119t^3}{360} + \cdots . \qquad (3.2)$$

To estimate the harmonic sum $f_1(t) := \sum_{r=2}^{\infty}\frac{e^{-rt}}{1-e^{-2rt}}$ as $t \to 0$, we take the Mellin transform of $f_1(t)$, see [8], which is $f_1^*(s) := \int_0^\infty f_1(t)t^{s-1}\,dt$. Thus

$$f_1^*(s) = 2^{-s}\left(-1+2^s\right)\Gamma(s)\zeta(s)(\zeta(s)-1), \text{ for } \mathrm{Re(s)} > 1.$$

By using the Mellin inversion formula, we have $f_1(t) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} f_1^*(s)\,t^{-s}\,ds$ (again see [8]). By computing residues this yields

$$f_1(t) \sim \frac{-1+\gamma+\log(2)-\log(t)}{2t} + \frac{13t}{144} + \cdots , \qquad (3.3)$$

where γ is Euler's constant. Above $f \sim g$ means $\frac{f}{g} \to 1$ as $t \to 0$. We note that $f_1(t) = \frac{e^t}{1-e^{2t}} + V(t) - V(2t)$ where $V(t)$ is the function studied by Prodinger in [17].

Let

$$g_1(t) := \frac{e^t\left(1+e^{-3t}\right)}{1-e^{-t}}\, f_1(t).$$

From (3.2) and (3.3)

$$g_1(t) \sim \frac{-1+\gamma+\log(2)-\log(t)}{t^2} + \frac{13}{72} + \frac{13}{12}(-1+\gamma+\log(2)-\log(t)) + \cdots . \tag{3.4}$$

By letting $y = 1-2z$ and writing $e^{-t} = v = \frac{1-\sqrt{(2-y)y}}{1-y}$, we find $t = -\log\frac{1-\sqrt{(2-y)y}}{1-y} = \sqrt{2y} + \frac{5y^{3/2}}{6\sqrt{2}} + \cdots$. In terms of the y variable, we obtain

$$g_1(\sqrt{2y} + \frac{5y^{3/2}}{6\sqrt{2}} + \cdots) \sim \frac{-2+2\gamma+\log(2)-\log(y)}{4y} + \frac{1}{36}(-25+24\gamma+12\log(2)-12\log(y)) + \cdots .$$

Replacing y by $1-2z$ gives

$$\frac{-2+2\gamma+\log(2)-\log(1-2z)}{4(1-2z)} + \frac{1}{36}(-25+24\gamma+12\log(2)-12\log(1-2z)) + \cdots .$$

By means of singularity analysis, see [8], we find the coefficient of z^n in the above expression as $n \to \infty$. It is asymptotically equal to

$$2^{n-2}\left(H_n - 2 + 2\gamma + \log(2)\right) + \cdots ., \tag{3.5}$$

where H_n is the harmonic number $\sum_{j=1}^{n} \frac{1}{j}$. To obtain the mean value we must divide by the total number of Dyck prefixes of length n, given by (2.4). As $n \to \infty$ we have

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} = 2^n\left(\sqrt{\frac{2}{\pi n}} - \frac{1}{2n^{3/2}\sqrt{2\pi}} + \frac{1}{16n^{5/2}\sqrt{2\pi}}\right) + \cdots . \tag{3.6}$$

Hence, dividing (3.5) by (3.6) yields the asymptotic contribution to the average from the first sum (3.1) as

$$\frac{1}{4}\sqrt{\frac{\pi}{2}}\sqrt{n}\Big(\log(n) + 3\gamma - 2 + \log(2)\Big) + \cdots . \tag{3.7}$$

Next we consider

$$\frac{1+v}{1-v}\sum_{r=1}^{\infty}\frac{v^{1+2r}}{1-v^{2+2r}}. \tag{3.8}$$

Proceeding as above, we set $v = e^{-t}$ and let $t \to 0$. We have that

$$\frac{e^t\left(1+e^{-t}\right)}{1-e^{-t}} = \frac{2}{t} + 2 + \frac{7t}{6} + \cdots . \tag{3.9}$$

To estimate the harmonic sum $f_2(t) := \sum_{r=2}^{\infty} \frac{e^{-2rt}}{1-e^{-2rt}}$ as $t \to 0$, as before we take the Mellin transform to obtain

$$f_2^*(s) = 2^{-s}\Gamma(s)\zeta(s)(\zeta(s)-1), \text{ for Re(s)} > 1.$$

By using the Mellin inversion formula and computing residues, we find

$$f_2(t) \sim \frac{-1+\gamma-\log(2)-\log(t)}{2t} + \frac{3}{4} - \frac{13t}{72} + \cdots . \tag{3.10}$$

Let

$$g_2(t) := \frac{e^t\,(1+e^{-t})}{1-e^{-t}}\, f_2(t).$$

From (3.9) and (3.10)

$$\begin{aligned} g_2(t) \sim & \frac{-1+\gamma-\log(2)-\log(t)}{t^2} + \frac{\frac{1}{2}+\gamma-\log(2)-\log(t)}{t} \\ & + \left(\frac{41}{36} + \frac{7}{12}(-1+\gamma-\log(2)-\log(t))\right) + \cdots . \end{aligned} \tag{3.11}$$

Let $y = 1-2z$ and as before we find that $t = -\log \frac{1-\sqrt{(2-y)y}}{1-y} = \sqrt{2y} + \frac{5y^{3/2}}{6\sqrt{2}} + \cdots$.
In terms of the y variable, we obtain

$$\begin{aligned} g_2(\sqrt{2y} + \frac{5y^{3/2}}{6\sqrt{2}} + \cdots) \sim & \frac{-2+2\gamma-3\log(2)-\log(y)}{4y} + \frac{1+2\gamma-3\log(2)-\log(y)}{2\sqrt{2}\sqrt{y}} \\ & + \cdots . \end{aligned}$$

Replacing y by $1-2z$ gives

$$\frac{-2+2\gamma-3\log(2)-\log(1-2z)}{4(1-2z)} + \frac{1+2\gamma-3\log(2)-\log(1-2z)}{2\sqrt{2}\sqrt{1-2z}} + \cdots .$$

Similarly, the coefficient of z^n in the expansion above is asymptotically equal to

$$2^{n-2}\left(-2+2\gamma+H_n-\log(8)+\sqrt{\frac{2}{\pi n}}(1+3\gamma-\log 2+\log(n))\right) + \cdots . \tag{3.12}$$

Dividing by the total number of Dyck prefixes of length n yields the asymptotic contribution to the average from the second sum (3.8) as

$$\frac{1}{4}\sqrt{\frac{\pi}{2}}(-2+3\gamma-3\log(2)+\log(n))\sqrt{n} + \frac{1}{4}(1+3\gamma-\log(2)+\log(n)) + \cdots . \tag{3.13}$$

Finally, subtracting (3.13) from (3.7) completes the proof. □

Remark 3.2. *The asymptotic formula of Theorem 3.1 when* $n = 100$ *yields 7.026 for the average number of strong left-to-right maxima. Using the exact formula of Theorem 4.2 divided by the number of prefixes for* $n = 100$ *yields 7.090.*

Remark 3.3. *The number of strong left-to-right maxima is bounded above by the height of the prefix, which is known to be* $\log(2)\sqrt{2\pi n}$ *as* $n \to \infty$*, as shown in [9]. We see that asymptotically the average number of strict left-to-right maxima is half of the height of the prefix.*

4. Weak left-to-right maxima in Dyck prefixes

To study this, we first need a generating function for Dyck paths of height h which have only a single return to the x axis. So using the formula (2.2) above, we obtain the generating function for these where $h \geq 1$ as

$$D(h, z) = z^2 A(h-1). \tag{4.1}$$

Now in order to construct the generating function $E(h, x, z)$ for the number of times a Dyck path of length n tracked by z returns to 0, where the latter is tracked by a variable x in the generating function, we construct a sequence of such Dyck paths where each term in the generating function for this sequence is multiplied by x. Thus we obtain

$$E(h, x, z) = \frac{1}{1 - xD(h, z)}. \tag{4.2}$$

At maximal height r in a Dyck prefix, we will also need to count by x the total number of times that a Dyck path passes through the zero level given by the generating function

$$E_f(r, x, z) = x + \frac{x^2 D(r, z)}{1 - xD(r, z)}. \tag{4.3}$$

The initial x takes care of the case where the final point of the prefix is strictly highest.

We now reiterate the construction in Theorem 2.1. The result is the theorem below in which the bracketed term $(1 + zC(r-1))$ means that the path is either empty after the last maximum or is a down step z followed by a path of height at most $r-1$, ending at any value. The initial down step stops a return to the maximum height.

Theorem 4.1. *The generating function for the number of weak left-to-right maxima, tracked by x, for Dyck prefixes of maximum height r and length tracked by z where $C(h)$ was given in (2.1) is*

$$F(x, z, r) := z^r (1 + zC(r-1)) \prod_{h=1}^{r-1} E(h, x, z) E_f(r, x, z). \tag{4.4}$$

We also obtain the following theorem:

Theorem 4.2. *The simplified generating function for the total number of weak left-to-right maxima for Dyck prefixes of length n tracked by z is*

$$\mathtt{WTot}(v) = \frac{v(1+v)}{1-v} + \frac{1 + v^2 + v^3 + v^5}{(1-v)v^2} \sum_{j=1}^{\infty} \frac{v^j}{1 - v^{2j}} - \frac{(1+v)\left(1+v^2\right)}{(1-v)v^3} \sum_{j=1}^{\infty} \frac{v^{2j}}{1 - v^{2j}} \tag{4.5}$$

where $z = \frac{v}{1+v^2}$.

Proof. To obtain the generating function for the total number of weak left-to-right maxima, we once again differentiate (4.4) with respect to x and evaluate this at $x = 1$.

The derivative of (4.4) is

$$\begin{aligned} \frac{\partial}{\partial x} F(x, z, r)\Big|_{x=1} =& z^r (1 + zC(r-1)) \Big(\frac{\partial}{\partial x} E_f(r, 1, z) \prod_{h=1}^{r-1} E(h, 1, z) \\ &+ E_f(r, 1, z) \prod_{h=1}^{r-1} E(h, 1, z) \sum_{i=1}^{r-1} \frac{\frac{\partial}{\partial x} E(i, 1, z)}{E(i, 1, z)} \Big). \end{aligned} \tag{4.6}$$

Putting $z = \frac{v}{1+v^2}$ in the formula above we obtain

$$z^r(1+zC(r-1))\frac{\partial}{\partial x}E_f(r,1,z)\prod_{h=1}^{r-1}E(h,1,z) = \frac{v^r(1+v)\left(1+v^2\right)^2\left(1-v^{1+r}\right)}{\left(1-v^{2+r}\right)\left(1+v^{2+r}\right)^2}, \tag{4.7}$$

while

$$z^r(1+zC(r-1))E_f(r,1,z)\prod_{h=1}^{r-1}E(h,1,z)\sum_{i=1}^{r-1}\frac{\frac{\partial}{\partial x}E(i,1,z)}{E(i,1,z)} = \frac{v^r(1+v)\left(1+v^2\right)}{\left(1+v^{1+r}\right)\left(1+v^{2+r}\right)}\sum_{i=1}^{r-1}\frac{v^2\left(1-v^{2i}\right)}{1-v^{4+2i}}. \tag{4.8}$$

Thus we have that the generating function for the total number of weak left-to-right maxima for Dyck prefixes of length n tracked by z is

$$WTot(v) := \sum_{r=1}^{\infty}\left(\frac{v^r(1+v)\left(1+v^2\right)^2\left(1-v^{1+r}\right)}{\left(1-v^{2+r}\right)\left(1+v^{2+r}\right)^2} + \frac{v^r(1+v)\left(1+v^2\right)}{\left(1+v^{1+r}\right)\left(1+v^{2+r}\right)}\sum_{i=1}^{r-1}\frac{v^2\left(1-v^{2i}\right)}{1-v^{4+2i}}\right) \tag{4.9}$$

where $z = \frac{v}{1+v^2}$.

Now, we simplify Equation (4.9). The double sum becomes

$$\left(1+v\right)\left(1+v^2\right)v^2\sum_{i=1}^{\infty}\frac{1-v^{2i}}{1-v^{4+2i}}\sum_{r=i+1}^{\infty}\frac{v^r}{\left(1+v^{1+r}\right)\left(1+v^{2+r}\right)}. \tag{4.10}$$

We use partial fractions on the r-sum and then the double sum telescopes to

$$\frac{\left(1+v\right)\left(1+v^2\right)v^2}{1-v}\sum_{i=1}^{\infty}\frac{\left(1-v^{2i}\right)v^{1+i}}{\left(1-v^{4+2i}\right)\left(1+v^{2+i}\right)}. \tag{4.11}$$

This is then combined with the single sum which simplifies to

$$\frac{(1+v)(1+v^2)}{1-v}\sum_{r=1}^{\infty}\frac{v^r\left(1-v+v^2-v^{1+r}\right)}{\left(1-v^{2+r}\right)\left(1+v^{2+r}\right)} = \frac{(1+v)\left(1+v^2\right)\left(1-v+v^2\right)}{(1-v)v^2}\sum_{r=1}^{\infty}\frac{v^{r+2}}{1-v^{4+2r}} - \frac{(1+v)\left(1+v^2\right)}{(1-v)v^3}\sum_{r=1}^{\infty}\frac{v^{4+2r}}{1-v^{4+2r}}. \tag{4.12}$$

□

Equation (4.5) has series expansion

$$z + 2z^2 + 4z^3 + 9z^4 + \mathbf{17}z^5 + 38z^6 + 73z^7 + 159z^8 + 310z^9 + 662z^{10} + O\left(z^{11}\right).$$

This is illustrated in Figure 2, where the dots and circles mark all 17 of the weak left-to-right maxima in Dyck prefixes of length 5.

4.1. Formula for total number of weak left-to-right maxima

In this section, we again obtain an exact formula for the total number of left-to-right maxima in terms of the divisor function $d(r)$. To read off coefficients from Theorem 4.2, as before

$$[z^n]f(z) = [v^n](1-v^2)(1+v^2)^{n-1}f(z(v)). \tag{4.13}$$

Therefore

$$[z^n]WTot(z) = [v^n](1-v^2)(1+v^2)^{n-1}WTot(z(v)). \tag{4.14}$$

From this the next Theorem follows in a similar way as in Section 2.1.

Theorem 4.3. *The total number of weak left-to-right maxima in Dyck paths of semi-length n is given by*

$$\sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \Bigg((d_o(n-2r-4) + d_o(n-2r-3) + d_o(n-2r-2) + 2d_o(n-2r-1) + d_o(n-2r) + d_o(n-2r+1) + d_o(n-2r+2)) - 2d\left(\frac{1}{2}(n-2r)\right) - d\left(\frac{1}{2}(n-2r-1)\right) - 2d\left(\frac{1}{2}(n-2r+1)\right) - 2d\left(\frac{1}{2}(n-2r+2)\right) - d\left(\frac{1}{2}(n-2r+3)\right)\Bigg)\binom{n-1}{r} + \binom{n-1}{\frac{n-1}{2}} + 2\binom{n-1}{\frac{n-2}{2}} + \binom{n-1}{\frac{n-3}{2}}, \tag{4.15}$$

where $d(i)$ and $d_o(i)$ respectively denote the number of divisors and odd divisors of i. Here the binomial coefficients are set to 0 if their lower indices are non-integers. Also $d(i) = 0$ whenever i is non-integer or non-positive.

5. Asymptotics for weak left-to-right maxima

To find an asymptotic expression for $WTot(u)$, we reiterate the approach in Section 3. This yields the following Theorem.

Theorem 5.1. *The average number of weak left-to-right maxima in Dyck prefixes of length n, as $n \to \infty$ is*

$$\sqrt{2\pi n}\log(2) + \frac{1}{2}(1 - 6\gamma + 2\log(2) - 2\log(n)) + O(n^{-1/2}).$$

Remark 5.2. *The asymptotic formula of Theorem 5.1 when $n = 100$ yields 12.231 for the average number of weak left-to-right maxima. Using the exact formula of Theorem 4.3 divided by the number of prefixes for $n = 100$ yields 12.208.*

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