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On equivalence classes of Butson Hadamard matrices $BH(4, 2k)$

Research Article

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Abstract: A Butson Hadamard matrix of order n over the k^{th} root of unity is a square matrix H which entries are some complex k^{th} root of unity such that $HH^* = nI_n$, where H^* is the complex conjugate of H. A set of Butson Hadamard matrices of order n over the k^{th} root of unity is denoted by $BH(n, k)$. It is well-known that a Butson Hadamard matrices is a generalization of a Hadamard matrix. In this paper, we give some properties of Butson Hadamard matrices of order 4 which implies to the upper and the lower bounds of the number of its equivalence classes. We also showed that the entries of Butson Hadamard matrices of order 4 is $2k$-th root of unity for some integer k. Furthermore, we describe the equivalence classes of Butson Hadamard matrices of order 4 by constructing the representative of the class.

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Keywords: Butson Hadamard matrices, Equivalence class, Monomial equivalence, Root of unity

1. Introduction

A Hadamard matrix of order n is a square $\{\pm 1\}$ matrix H such that $HH^T = nI_n$, where I_n is the identity matrix of order n and H^T is the transpose of H. We write $H(n)$ to denote the set of all Hadamard matrices of order n. There is a well known unsolved problem that is related to Hadamard matrices which states that there exists a Hadamard matrix of order n, for $n = 1, 2$, or a multiple of 4. The problem is known as the conjecture of Hadamard [3]. There is a possible method to solve the conjecture and it involves a Butson Hadamard matrix.

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Let n and k be positive integers. A Butson Hadamard matrix of order n over the k^{th} root of unity is a square matrix H which entries are some complex k^{th} root of unity such that $HH^* = nI_n$, where H^* is the complex conjugate of H. We use $BH(n,k)$ to denote the set of all Butson Hadamard matrices of order n over the k^{th} root of unity. Some results regarding Butson Hadamard matrices can be found in [8]. A complex Hadamard matrix is a special case of a Butson Hadamard matrix, and it was firstly introduced by Turyn [13]. Turyn also find a method to obtain Hadamard matrices of order $2n$ from a complex Hadamard matrix of order n, which conclude that the existence of complex Hadamard matrix is crucial for Hadamard's conjecture. We remark that the properties of complex Hadamard matrices are given in [12, 14]. Furthermore, Compton, et al. found a method to obtain Hadamard matrices of order $4n$ from Butson Hadamard matrices $BH(n,6)$ with no real entries [2]. This leads us to predict the relation between the construction of Butson Hadamard matrices and Hadamard matrices. Therefore, we give some properties of Butson Hadamard matrices and classify them by using an equivalence relation called a monomial equivalence.

In this paper, we give some properties of Butson Hadamard matrices and give equivalence classes of Butson Hadamard matrices of order 4. We also prove that if H is a Butson Hadamard matrix of order 4, then H has entries of $2k^{th}$ complex root of unity. Furthermore, the result of this paper is based on a parameterization of $BH(4,2k)$ and this implies to the upper and lower bound of the number of equivalence classes of $BH(4,2k)$.

2. Preliminaries

In this section, we will introduce some previous and preliminary results for preparation into the main results. Throughout this paper, we write -1 as $-$ for simplification.

Definition 2.1. *A* ***Hadamard matrix*** *of order n is a square matrix H with entries in $\{\pm 1\}$ that satisfies*

$$HH^T = nI_n,$$

where H^T denotes the transpose of H and I_n is the identity matrix of order n. We denote the set of all Hadamard matrices of order n by $H(n)$.

Example 2.2. *These are some examples of Hadamard matrices of order $n = 1, 2, 4$.*

$$\begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}1 & 1\\ 1 & -\end{bmatrix}, \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & - & 1 & -\\ 1 & 1 & - & -\\ 1 & - & - & 1\end{bmatrix}.$$

It is well known that if $H \in H(n)$, then we can easily obtain another $H' \in H(n)$ by the following steps: swapping rows or columns of H, multiplying a row or column of H with -1, transposing H, or by using finitely many compositions of the steps.

Now, suppose H is a Hadamard matrix. Then, we can obtain another Hadamard matrix H' from H with entries in the first column and the first row are all ones by using the finitely steps from operations above, and the matrix H' is called a **normalized Hadamard matrix**. Some notes related to the normalized Hadamard matrices can be found in [4]. Generally, we say a matrix is **normalized** if the first column and the first row of the matrix are all ones. Note that a normalized Hadamard matrix is not necessarily unique.

Currently, the unknown smallest orders of a Hadamard matrix is 668. In 2005, a Hadamard matrix of order 428, which was the unknown smallest order, was discovered by Kharaghani and Tayfeh-Rezaie [5] by using some classes of complementary sequences. The properties of complementary sequences can be found in [6, 9]. There are many known methods to construct Hadamard matrices, and we refer the reader to see [2, 4, 9–11, 13, 15] for further information.

A Butson Hadamard matrix was introduced by Butson in 1962 as a generalization of Hadamard matrix [1]. A complex number z is called a k^{th} root of unity, if $z^k = 1$.

Definition 2.3. *A **Butson Hadamard matrix** of order n over the k^{th} root of unity is a square matrix H with entries are k^{th} root of unity and satisfies*

$$HH^* = nI_n,$$

where H^* denotes the conjugate transpose of H.

Denote $\mathbb{T}_k$ as the set of all k^{th} root of unity and $\mathbb{T}$ as all complex numbers in the unit circle, that is $\mathbb{T} = \{z \in \mathbb{C}, |z| = 1\}$.

Example 2.4. *The Fourier matrix of order n,*

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & & \alpha^{n-1} \\ 1 & \alpha^2 & \alpha^4 & & \alpha^{2(n-1)} \\ \vdots & & & & \vdots \\ 1 & \alpha^{n-1} & \alpha^{2(n-1)} & \cdots & \alpha^{(n-1)^2} \end{bmatrix}$$

where α is an nth root of unity, is a $BH(n, n)$ matrix.

Note that $H(n) = BH(n, 2)$, since in this case, $H^* = H^T$ for every $H \in BH(n, 2)$. Similar to Hadamard matrices, if we have a matrix $H \in BH(n, k)$, we can obtain another matrix $H' \in BH(n, k)$ by swapping rows or columns of H, multiplying a row or column of H with any k^{th} root of unity, transposing H, or conjugate transposing H. We note that it is also possible to obtain a normalized Butson Hadamard matrix from a given Butson Hadamard matrix H by using some steps of the operations above.

3. Main results

We begin to find the method to classify Butson Hadamard matrices of order 4 by using the monomial equivalence. In this section, we will give the definition and properties of the monomial equivalence. As our main results, the upper and the lower bound of the number of equivalence classes in $BH(4, 2k)$ will be presented.

Denote the set of square matrices of order n with entries in $\mathbb{T}_k$ as T_k^n.

Definition 3.1. *A **monomial matrix** of order n is a square matrix of order n with exactly one nonzero entry in each row and each column. Denote the set of all monomial matrices of order n with nonzero entries in $\mathbb{T}_k$ as $\boldsymbol{S}(n, k)$.*

Definition 3.2. *Let $M_1, M_2 \in T_k^n$. We write $M_1 \sim M_2$ if there exist $S_r, S_c \in \boldsymbol{S}(n, k)$ such that $M_2 = S_r M_1 S_c$.*

It is obvious that the relation $\sim$ defined above is an equivalence relation, and the equivalence is usually known as **monomial equivalence**. More on monomial equivalence can be seen on [8] and [16].

The equivalence classes of Butson Hadamard matrices of order 1, 2, and 3 are trivial as there is just one equivalence class. Therefore, we proceed to the order 4. In this case, we see that it has one parameterization equivalence class.

Now, before we proceed to the main results, we prove the following lemma. Note that there is a more general result due to Lam and Leung [7], but the proof used here is elementary.

Lemma 3.3. *Let $x, y, z, t \in \mathbb{T}$. Then the solutions of*

$$x + y + z + t = 0 \tag{1}$$

are all x, y, z, t such that $\{x, y, z, t\} = \{p, -p, q, -q\}$, for some $p, q \in \mathbb{T}$. Furthermore, if $x, y, z, t \in \mathbb{T}_l$, then Eq. 1 have solutions if and only if l is even.

Proof. Let $x, y, z, t \in \mathbb{T}$. We will prove that $x + y + z + t = 0$ only have $\{x, y, z, t\} = \{p, q, -p, -q\}$ as solution for some $p, q \in \mathbb{T}$. Observe that $x + y = -(z + t)$. So $|x + y| = |z + t|$. Let θ_{xy} be the angle between x and y, and θ_{zt} be the angle between z and t. Then,

$$\begin{aligned}
\sqrt{|x|^2 + |y|^2 + 2|x||y|\cos\theta_{xy}} &= \sqrt{|z|^2 + |t|^2 + 2|z||t|\cos\theta_{zt}} \\
\sqrt{2 + 2\cos\theta_{xy}} &= \sqrt{2 + 2\cos\theta_{zt}} \\
\cos\theta_{xy} &= \cos\theta_{zt}.
\end{aligned}$$

As a result, $\theta_{xy} = \pm\theta_{zt}$. Notice that

$$0 = x + y + z + t = x + x(e^{i\theta_{xy}}) + z + z(e^{i\theta_{zt}}) = x(1 + e^{i\theta_{xy}}) + z(1 + e^{i\theta_{zt}}).$$

Consider the following cases:

- If $\theta_{xy} = \theta_{zt}$, then $0 = x(1 + e^{i\theta_{xy}}) + z(1 + e^{i\theta_{xy}}) = (x + z)(1 + e^{i\theta_{xy}})$, and hence we get $x = -z$ or $e^{i\theta_{xy}} = -1$.
- If $\theta_{xy} = -\theta_{zt}$, write $z = te^{i\theta_{tz}} = te^{-i\theta_{zt}} = te^{i\theta_{xy}}$. This implies

  $$0 = x(1 + e^{i\theta_{xy}}) + (te^{i\theta_{xy}})(1 + e^{-i\theta_{xy}}) = x(1 + e^{i\theta_{xy}}) + t(1 + e^{i\theta_{xy}}),$$

  and therefore, we have $x = -t$ or $e^{i\theta_{xy}} = -1$.

The solutions are of the formc x, y, z, t such that $\{x, y, z, t\} = \{p, q, -p, -q\}$ for $p, q \in \mathbb{T}$.

Now, assume that $x, y, z, t \in \mathbb{T}_l \subseteq \mathbb{T}$. We cannot just have the solution of Eq. 1 is $\{p, q, -p, -q\}$, for some $p, q \in \mathbb{T}_l$ as $-p$ and $-q$ are not guaranteed to be in $\mathbb{T}_l$.

We will prove the last statement in the lemma. If l is even, then $-1 \in \mathbb{T}_l$, and therefore $-p = (-1)(p) \in \mathbb{T}_l$. The same argument applies to show that $-q \in \mathbb{T}_l$. As a result, the solution of Eq. 1 is $\{p, q, -p, -q\}$, for some $p, q \in \mathbb{T}_q$.

On the other hand, assume that $p, q \in \mathbb{T}_l$ and $\{p, q, -p, -q\}$ is a solution of Eq. 1. Then $-p \in \{x, y, z, t\} \subseteq \mathbb{T}_l$. Note that $-1 = (-p)(\overline{p}) \in \mathbb{T}_l$, so we can conclude that l is even. □

Corollary 3.4. *If $H \in BH(4, l)$, then l is even.*

Proof. It is sufficient to show the statement holds for any normalized Butson Hadamard matrices. Let H be normalized and the first two rows of H be

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & h_{22} & h_{23} & h_{24} \end{bmatrix}.$$

Then by orthogonality of those two rows,

$$1 + h_{22} + h_{23} + h_{24} = 0.$$

Since $1, h_{22}, h_{23}, h_{24} \in \mathbb{T}_l$, we conclude that l must be even by lemma 3.3. □

Theorem 3.5. *Let $H \in BH(4, 2k)$, Then H is equivalent to*

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & s & -1 & -s \\ 1 & -s & -1 & s \end{bmatrix}$$

for some $s \in \mathbb{T}_{2k}$.

Proof. Let H be a normalized Butson Hadamard matrix of order 4. Then

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & h_{22} & h_{23} & h_{24} \\ 1 & h_{32} & h_{33} & h_{34} \\ 1 & h_{42} & h_{43} & h_{44} \end{bmatrix}$$

with $h_{ij} \in \mathbb{T}_{2k}$ for some $k \in \mathbb{N}$. Observe that column 2, 3, and 4 are orthogonal to column 1, so

$$1 + \sum_{i=2}^{4} h_{ij} = 0 \text{ for all } j \in \{2, 3, 4\}.$$

Using the same approach for the rows, we get

$$1 + \sum_{i=2}^{4} h_{ji} = 0 \text{ for all } j \in \{2, 3, 4\}.$$

Notice that the second column needs to be orthogonal to the first column. By considering permutations of row 2, 3, and 4, we obtain

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & h_{23} & h_{24} \\ 1 & s & h_{33} & h_{34} \\ 1 & -s & h_{43} & h_{44} \end{bmatrix}$$

for an $s \in \mathbb{T}_{2k}$.

If $s = \pm 1$, we obtain a Hadamard matrix of order 4. It can be easily seen that $H \in H(4)$ is equivalent to

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & s & -1 & -s \\ 1 & -s & -1 & s \end{bmatrix}$$

for $s = \pm 1$.

Assume $s \neq \pm 1$. Because row 3 is orthogonal to row 1, and by considering permutations of row 3 and 4, we obtain

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & h_{23} & h_{24} \\ 1 & s & -1 & -s \\ 1 & -s & h_{43} & h_{44} \end{bmatrix}$$

for some $s \in \mathbb{T}_{2k}$. As row 4 is orthogonal to row 1, we have $\{h_{43}, h_{44}\} = \{-1, s\}$.

If $h_{43} = s$, then

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -s & s \\ 1 & s & -1 & -s \\ 1 & -s & s & -1 \end{bmatrix}$$

As row 2 is orthogonal to row 3, then $0 = 1 - \bar{s} + s - 1 = s - \bar{s}$. As a result, $s \in \mathbb{R}$, and $s = \pm 1$. This is a contradiction to the assumption of $s \neq \pm 1$.

If $h_{43} = -1$, then

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & s & -1 & -s \\ 1 & -s & -1 & s \end{bmatrix}$$

for some $s \in \mathbb{T}_{2k}$. It can be easily checked that $HH^* = 4I_4$ and every entry of H is an element in $\mathbb{T}_{2k}$. Thus, the result holds. $\square$

3.1. Upper bound of the number of equivalence classes in $BH(4, 2k)$

We begin with a particular case of $BH(4, 6)$, which may be easier to understand. By Theorem 3.5, every matrix H in $BH(4, 6)$ is equivalent to

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & s & -1 & -s \\ 1 & -s & -1 & s \end{bmatrix}$$

with $s \in \mathbb{T}_6 = \{\pm 1, \pm\gamma, \pm\gamma^2\}$, which γ takes the value of $e^{2\pi i/3}$. Notice that for $s = -t$, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & t & -1 & -t \\ 1 & -t & -1 & t \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & s & -1 & -s \\ 1 & -s & -1 & s \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -t & -1 & t \\ 1 & t & -1 & -t \end{bmatrix} \tag{2}$$

are equivalent (note that swapping the third and fourth row of the former matrix can give you the latter). Hence, there are at most three classes of equivalence classes in $BH(4, 6)$ matrix, namely

$$H_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, H_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & \gamma & -1 & -\gamma \\ 1 & -\gamma & -1 & \gamma \end{bmatrix}, H_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & \gamma^2 & -1 & -\gamma^2 \\ 1 & -\gamma^2 & -1 & \gamma^2 \end{bmatrix}. \tag{3}$$

Notice that

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} H_1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\gamma^2 \end{bmatrix} = H_2,$$

and therefore we can see that H_1 and H_2 are actually equivalent, so there are at most two equivalence classes in $BH(4,6)$, which can be represented by H_0 and H_1. Thus we have the following lemma.

Lemma 3.6. *There are at most 2 equivalence classes in $BH(4,6)$.*

Now, we will use a similar approach to determine the upper bound of the number of equivalence classes in $BH(4,2k)$. Throughout this section, we write $\omega_{2k}^j = e^{2\pi ij/2k}$ to be a $2k^{th}$ root of unity and let

$$H_{j,2k} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & s & -1 & -s \\ 1 & -s & -1 & s \end{bmatrix},$$

where $s = \omega_{2k}^j$. Also, we define $X_{j,2k}$ to be the set of all 2×2 submatrices of $H_{j,2k}$.

We present some illustration for better understanding by using graphs. The graphs are drawn at a complex plane, with the circle as the unit circle. There are $2k$ points on the circle (which are $2k^{th}$ root of unity), with each value t corresponds to the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & t & -1 & -t \\ 1 & -t & -1 & t \end{bmatrix}.$$

If some of those matrices are equivalent, we would simply mark the corresponding point on the unit circle with the same number.

By using a similar analogy on the last part (Eq. 2), we can see for $s = -t$, with $t \in \mathbb{T}_{2k}$,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & t & -1 & -t \\ 1 & -t & -1 & t \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & s & -1 & -s \\ 1 & -s & -1 & s \end{bmatrix}$$

are equivalent. In other words, $H_{j,2k} \sim H_{j+k,2k}$. As the negation of a $2k^{th}$ root of unity is also a $2k^{th}$ root of unity, we have at most k equivalence classes on $BH(4,2k)$. We give the illustrations for two values of $2k = 8$ and $2k = 10$ shown in Fig. 1.

The label of points shows that the corresponding normalized matrices for $t = 1$ and $t = -1$ are equivalent (those two points are both labeled by the same number). For other points in the graph, the same analogy applies. Based on the observation, we can see that there are at most four and five equivalence classes in $BH(4,8)$ and $BH(4,10)$, respectively. As there are some $H_{j,2k}$ matrix that are equivalent with each other, we can just consider the Butson Hadamard matrices $H_{j,2k}$ with $\omega_{2k}^j \in \{e^{i\theta} \mid -\pi/2 < \theta \leq \pi/2\}$ (that corresponds to the points in the right of real axis in the graph, including the complex number i, if i is also in $\mathbb{T}_{2k}$).

With some further calculations, that is

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} H_{j,2k} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega_{2k}^{-j} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\omega_{2k}^{-j} \end{bmatrix} = H_{-j,2k},$$

we can see that $H_{j,2k}$ is equivalent with $H_{-j,2k}$. In other words, two Butson Hadamard matrices that correspond to t and $\bar{t}$ are equivalent. In the graph, it would be shown as in Fig. 2 (for two values of $2k = 8$ and $2k = 10$).

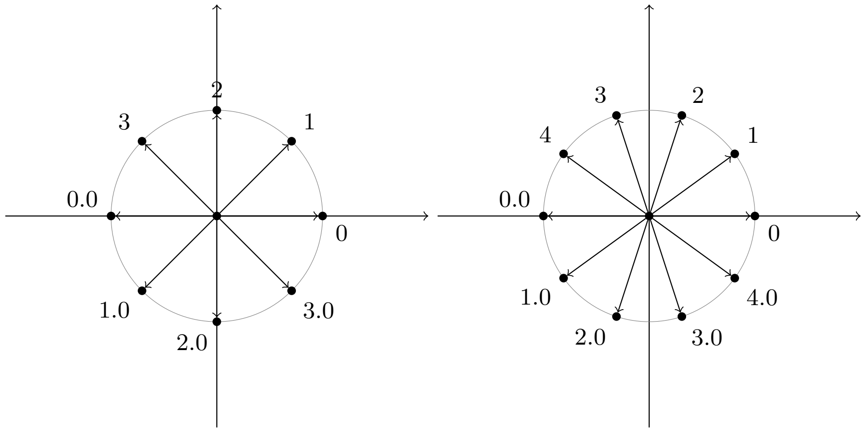

Figure 1. Inequivalent Butson Hadamard matrices on $BH(4, 8)$ and $BH(4, 10)$ on unit circle

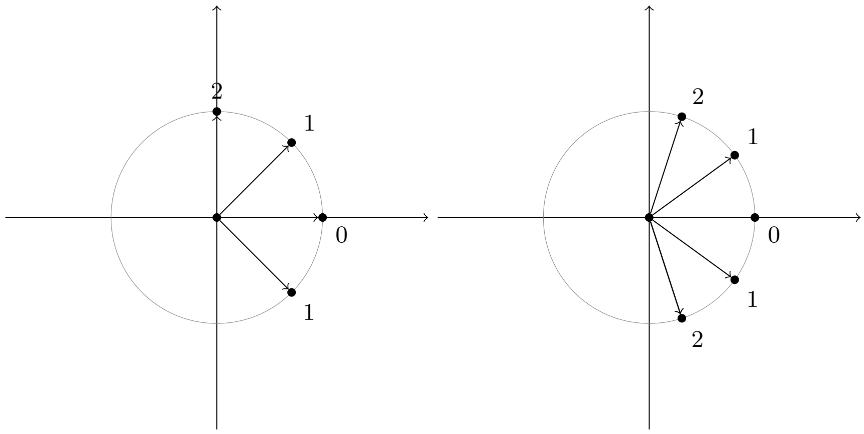

Figure 2. Inequivalent Butson Hadamard matrices on $BH(4, 8)$ and $BH(4, 10)$ after further analysis

There is a difference depending on the parity of k (for even k, the imaginary number i is a $2k^{th}$ root of unity, while for odd k, i is not a $2k^{th}$ root of unity). Here is some explanations:

- for odd k, there exist $k-1$ points that are not on the real and imaginary axis. Furthermore, those $k-1$ points have their conjugates in the set too. Thus, there are at most $k - \frac{k-1}{2} = \frac{k+1}{2}$ equivalence classes in $BH(4, 2k)$.
- for even k, there exist $k-2$ points that are not on the real and imaginary axis. Furthermore, those $k-2$ points have their conjugates in the set too. As a result, there are at most $k - \frac{k-2}{2} = \frac{k+2}{2}$ equivalence classes in $BH(4, 2k)$.

Therefore, we have the following theorem.

Theorem 3.7. *Let p_{2k} be the upper bound of the number of equivalence classes on $BH(4, 2k)$. Then $p_{2k} = \frac{k+1}{2}$ for odd k, and $p_{2k} = \frac{k+2}{2}$ for even k.*

3.2. Lower bound of the number of equivalence classes in $BH(4, 2k)$

After determining the upper bound of the number of equivalence classes in $BH(4, 2k)$, which is p_{2k} (in Theorem 3.7), we proceed to prove that there are at least p_{2k} matrices that are not pairwise equivalent in $BH(4, 2k)$. First, we will present the proof for a certain case of $BH(4, 6)$. Before we proceed to the next part, we will introduce an important lemma that contributes to the next part.

Lemma 3.8. *Let $G = [g_{ij}]\,, H = [h_{ij}] \in BH(n, 2k)$ and $G \sim H$. Then for every 2×2 submatrix of G, there exists a 2×2 submatrix of H such that those two submatrix are equivalent.*

Proof. Let $G, H \in BH(4, 2k)$ and $G = S_r H S_c = P_r D_r H D_c P_c$ with $D_r = diag\{r_i\}, D_c = diag\{c_i\}$, and P_r, P_c are permutation matrix corresponding to σ and τ^{-1} permutation respectively. Consider that the entries of G can be written as

$$g_{ij} = r_{\sigma(i)} h_{\sigma(i)\tau(j)} c_{\tau(j)}$$

By choosing any 2×2 submatrix of G, we obtain

$$\begin{aligned} \begin{bmatrix} g_{ab} & g_{ab'} \\ g_{a'b} & g_{a'b'} \end{bmatrix} &= \begin{bmatrix} r_{\sigma(a)} h_{\sigma(a)\tau(b)} c_{\tau(b)} & r_{\sigma(a)} h_{\sigma(a)\tau(b')} c_{\tau(b')} \\ r_{\sigma(a')} h_{\sigma(a')\tau(b)} c_{\tau(b)} & r_{\sigma(a')} h_{\sigma(a')\tau(b')} c_{\tau(b')} \end{bmatrix} \\ &= \begin{bmatrix} r_{\sigma(a)} & 0 \\ 0 & r_{\sigma(a')} \end{bmatrix} \begin{bmatrix} h_{\sigma(a)\tau(b)} & h_{\sigma(a)\tau(b')} \\ h_{\sigma(a')\tau(b)} & h_{\sigma(a')\tau(b')} \end{bmatrix} \begin{bmatrix} c_{\tau(b)} & 0 \\ 0 & c_{\tau(b')} \end{bmatrix}. \end{aligned}$$

Observe that

$$\begin{bmatrix} h_{\sigma(a)\tau(b)} & h_{\sigma(a)\tau(b')} \\ h_{\sigma(a')\tau(b)} & h_{\sigma(a')\tau(b')} \end{bmatrix}$$

is a 2×2 submatrix of H and $r_{\sigma(a)}, r_{\sigma(a')}, c_{\tau(b)}, c_{\tau(b')} \in \mathbb{T}_{2k}$. As a result,

$$\begin{bmatrix} g_{ab} & g_{ab'} \\ g_{a'b} & g_{a'b'} \end{bmatrix} \text{ and } \begin{bmatrix} h_{\sigma(a)\tau(b)} & h_{\sigma(a)\tau(b')} \\ h_{\sigma(a')\tau(b)} & h_{\sigma(a')\tau(b')} \end{bmatrix}$$

are equivalent. □

In the last section, we have proven that there are at most two equivalence classes in $BH(4, 6)$. In this part, we will prove that $H_0 \not\sim H_1$ where H_0 and H_1 are written in Eq. 3, and prove that there are at least two equivalence classes in $BH(4, 6)$.

Proposition 3.9. *The matrices H_0 and H_1 are not equivalent.*

Proof. Consider the submatrices of H_0. There are two equivalence classes, which are represented by

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}.$$

Let G be a normalized $BH(4, 6)$ matrix that is equivalent to H_0. As G is normalized, we can write

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & * & * & * \\ 1 & * & * & * \\ 1 & * & * & * \end{bmatrix}.$$

Observe the submatrices of G that includes the entry on $(1, 1)$ position, that is

$$\begin{bmatrix} 1 & 1 \\ 1 & * \end{bmatrix}.$$

Because $G \sim H_0$, then by lemma 3.8, there are two cases to be considered:

1. If
$$\begin{bmatrix}1 & 1\\ 1 & *\end{bmatrix} \sim \begin{bmatrix}1 & 1\\ 1 & -\end{bmatrix},$$
then $* = -1$ since Butson Hadamard matrix are only equivalent to another Butson Hadamard matrix.

2. If
$$\begin{bmatrix}1 & 1\\ 1 & *\end{bmatrix} \sim \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},$$
then there exist permutation matrices P_r, P_c and diagonal matrices $D_r = diag\{r_i\}$, $D_c = diag\{c_i\}$ with $r_i, c_i \in \mathbb{T}_6$ for $i \in \{1,2\}$, such that
$$\begin{bmatrix}1 & 1\\ 1 & *\end{bmatrix} = D_r P_r \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} P_c D_c = \begin{bmatrix}r_1 & 0\\ 0 & r_2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}c_1 & 0\\ 0 & c_2\end{bmatrix} = \begin{bmatrix}r_1c_1 & r_1c_2\\ r_2c_1 & r_2c_2\end{bmatrix}.$$
By simple algebraic calculation, we get $* = 1$.

This concludes that every normalized $BH(4,6)$ matrix G that is equivalent to H_0 is a ± 1 matrix. Since H_1 is not a ± 1 matrix, then $H_1 \not\sim H_0$. Thus, the result holds. □

This implies that there are at least two equivalence classes in $BH(4,6)$. Next, we aim to prove that there are p_{2k} matrices in $BH(4,2k)$ are pairwise inequivalent. The approach is similar to the proof of Proposition 3.9, that is, we consider any normalized $BH(4,2k)$ matrix G that is equivalent to $H_{j,2k}$ to eliminate matrix that are not equivalent to $H_{j,2k}$.

Firstly, we determine the number of equivalence classes in $X_{j,2k}$, that is, the set of all 2×2 submatrices of $H_{j,2k}$. Using exhaustive search, there are four equivalence classes in $X_{j,2k}$, which is represented by
$$\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, \begin{bmatrix}1 & 1\\ 1 & \omega_{2k}^j\end{bmatrix}, \text{ and } \begin{bmatrix}1 & 1\\ 1 & -\omega_{2k}^j\end{bmatrix}.$$
Note that for $\omega_{2k}^j = \pm 1$, there are two equivalence classes (omitting the last two) and for $\omega_{2k}^j = \pm i$, there are three equivalence classes (omitting the last). Consider the submatrices of G that includes the entry on $(1,1)$ position, that is
$$\begin{bmatrix}1 & 1\\ 1 & *\end{bmatrix}.$$
Because $G \sim H_{j,2k}$, there are some cases to consider by lemma 3.8:

1. If
$$\begin{bmatrix}1 & 1\\ 1 & *\end{bmatrix} \sim \begin{bmatrix}1 & 1\\ 1 & -\end{bmatrix},$$
then $* = -1$.

2. If
$$\begin{bmatrix}1 & 1\\ 1 & *\end{bmatrix} \sim \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},$$
then there exist permutation matrices P_r, P_c and diagonal matrices $D_r = diag\{r_i\}$, $D_c = diag\{c_i\}$ with $r_i, c_i \in \mathbb{T}_{2k}$, such that
$$\begin{bmatrix}1 & 1\\ 1 & *\end{bmatrix} = D_r P_r \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} P_c D_c = \begin{bmatrix}r_1 & 0\\ 0 & r_2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}c_1 & 0\\ 0 & c_2\end{bmatrix} = \begin{bmatrix}r_1c_1 & r_1c_2\\ r_2c_1 & r_2c_2\end{bmatrix}.$$
By simple algebraic calculation, we get $* = 1$.

3. If

$$\begin{bmatrix} 1 & 1 \\ 1 & * \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 1 & \omega_{2k}^j \end{bmatrix} \text{ with } j \in \{1, 2, \ldots, p_{2k} - 1\},$$

then there exist permutation matrices P_r, P_c and diagonal matrices $D_r = diag\{r_i\}$, $D_c = diag\{c_i\}$ with $r_i, c_i \in \mathbb{T}_{2k}$, such that

$$\begin{bmatrix} 1 & 1 \\ 1 & * \end{bmatrix} = P_r D_r \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} D_c P_c.$$

Since P_r and P_c are permutation matrices, there exist the inverse of the matrices, which are also permutation matrices, namely Q_r and Q_c, respectively. Thus, we have

$$Q_r \begin{bmatrix} 1 & 1 \\ 1 & * \end{bmatrix} Q_c = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & \omega_{2k}^j \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} r_1c_1 & r_1c_2 \\ r_2c_1 & \omega_{2k}^j r_2c_2 \end{bmatrix}.$$

From here, it is sufficient to check four possibilities of permutations using exhaustive search. By direct computation, there are two possible values for $*$, which is $* = \omega_{2k}^j$ or $* = \omega_{2k}^{-j}$.

4. If

$$\begin{bmatrix} 1 & 1 \\ 1 & * \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 1 & -\omega_{2k}^j \end{bmatrix} \text{ with } j \in \{1, 2, \ldots, p_{2k} - 1\},$$

then by using a similar argument, we get $* = -\omega_{2k}^j$ or $* = -\omega_{2k}^{-j}$.

As a result, every normalized $BH(4, 2k)$ matrix $G = [g_{pq}]$ that is equivalent to $H_{j,2k}$ must satisfy

$$g_{pq} \in \{\pm 1, \pm\omega_{2k}^j, \pm\omega_{2k}^{-j}\}$$

for all $p, q \in \{2, 3, 4\}$. Using this result, we can deduce that

1. $H_{0,2k} \not\sim H_{j,2k}$ for all $j \in \{1, 2, \ldots, p_{2k} - 1\}$ as there is an entry ω_{2k}^j of $H_{j,2k}$, which is not in the set $\{\pm 1\}$.
2. $H_{j,2k} \not\sim H_{l,2k}$ for $j, l \in \{1, 2, \ldots, p_{2k} - 1\}$ with $l \neq j$ because there is an entry ω_{2k}^l of $H_{l,2k}$, which is not in the set $\{\pm 1, \pm\omega_{2k}^j, \pm\omega_{2k}^{-j}\}$.

Therefore, every two matrices in $\{H_{0,2k}, \cdots, H_{p_{2k}-1,2k}\}$ are not equivalent. Thus, we proved that there are at least p_{2k} equivalence classes in $BH(4, 2k)$. We restate this statement in the following theorem.

Theorem 3.10. *There are at least p_{2k} equivalence classes in $BH(4, 2k)$.*

By combining Theorem 3.7 and Theorem 3.10, we have the following theorem

Theorem 3.11. *Let p_{2k} be the number of equivalence classes on $BH(4, 2k)$. Then $p_{2k} = \frac{k+1}{2}$ for odd k, and $p_{2k} = \frac{k+2}{2}$ for even k.*

As seen above, we can choose $\{H_{0,2k}, \cdots, H_{p_{2k}-1,2k}\}$ as the representative of each equivalence class of $BH(4, 2k)$. In the matrix form, every equivalence class of $BH(4, 2k)$ can be represented by one of the matrix in the set

$$\left\{ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & e^{\pi ij/k} & -1 & -e^{\pi ij/k} \\ 1 & -e^{\pi ij/k} & -1 & e^{\pi ij/k} \end{bmatrix} \right\}_{j=0}^{j=p_{2k}-1}.$$

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