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Rings with annihilator condition and their extensions

Research Article

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Abstract: Let R be an associative unital ring and α an endomorphism of R. In this article, we aim to investigate rings that satisfy Property $(a.c.)$ and explore some related rings. Moreover, we examine many polynomial extensions of R satisfying Property $(a.c.)$. In particular, α-skew quasi-Armendariz rings and α-(sps) quasi-Armendariz rings. We prove that this class of rings has always Property $(a.c.)$ on the left but not necessarily on the right. Furthermore, we investigate the transmission of Property $(a.c)$ from a ring R to $R[x, \alpha]$ and $R[[x, \alpha]]$.

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1. Introduction

In this article, we will denote by R an associative unital ring. The center and the set of all idempotent elements of R will be represented by $C(R)$ and $I(R)$, respectively. Meanwhile, $End(R)$ will denote the endomorphism ring of R. The sets of the left and the right semi-central idempotents of R will be represented by $\mathcal{S}_l(R)$ and $\mathcal{S}_r(R)$, respectively. The right and the left annihilator of a nonempty subset S of R will be denoted, respectively, by $r_R(S) = \{a \in R \mid Sa = 0\}$ and $\ell_R(S) = \{a \in R \mid aS = 0\}$. Let $\alpha \in End(R)$, $R[x; \alpha]$ and $R[[x, \alpha]]$ will represent the skew polynomial ring over R and the skew power series ring over R, respectively. Recall that the addition is defined as usual, and the multiplication is defined as follows: for all $a \in R$, $xa = \alpha(a)x$. When α is the identity map of R, we use the usual notation $R[x, \alpha] = R[x]$ and $R[[x, \alpha]] = R[[x]]$ (where $R[x]$ represents the polynomial ring and $R[[x]]$ represents the skew power series ring over R, respectively).

The concept of the annihilator condition $(a.c.)$ was first introduced for commutative reduced rings by Henriksen and Jerison [12]. In the commutative context, we say that R has the annihilator condition $(a.c.)$ or it has Property $(a.c.)$ if the annihilator of each finitely generated ideal I of R is equal to the annihilator of an element of R. This definition was later extended to any commutative ring by Lucas [18]. Rings having Property $(a.c.)$ form a pretty large class that includes subdirect sums of totally ordered integral

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domains, Bezout rings, complete or weak direct sum of rings having Property ($a.c.$), reduced Noetherian rings, and reduced non trivial graded rings (see [12, 17, 18]). In 2009, Hong et al. [16] extended this notion to noncommutative rings. A unital ring R is said to have Property ($a.c.$) on the right (respectively, on the left), if for any finitely generated right ideal I of R, $r_R(I) = r_R(cR)$ (respectively, $\ell_R(I) = \ell_R(Rc)$) for some $c \in R$. They studied many rings that have Property ($a.c.$) and proved that several types of rings possess this property. These include subdirect products of fully ordered semiprime rings, semiprime rings with many minimal prime ideals, semiprime rings with the ascending chain condition on annihilators, and biregular rings, etc. They also investigated various extensions over semiprime rings that have Property ($a.c.$). Furthermore, they demonstrated that matrix rings over rings with Property ($a.c.$) always possess Property ($a.c.$). Additionally, there is an equivalence when the matrix ring is replaced by the upper triangular matrix ring. They also explored further extensions over semiprime rings and proved that the Ore extension $R[x, \alpha, \delta]$, with δ is a derivation and α is an automorphism, has Property ($a.c.$). Moreover, they established that $R[G]$, $R[[x, \alpha]]$, $R[x,\ x^{-1}, \alpha]$ and $R[[x,\ x^{-1}, \alpha]]$ have Property ($a.c.$) when $G = \{\cdots,\ x^{-2},\ x^{-1},\ 1,\ x^1,\ x^2,\ \cdots\}$ is a unique product monoid with $\mid G \mid \geq 2$. Later in 2019, Bajor and Ziembowsi [2] provided examples to confirm that Property ($a.c.$) is not transferred from R to $R[x, \alpha]$ (respectively, $R[[x, \alpha]]$), nor vice versa.

The concept of Armendariz rings was first proposed by Rege and Chhawchharia [8]. They defined a ring R to be an Armendariz ring if, for any two polynomials in $R[x]$, whenever they have a zero product, the product of any pair of their coefficients is also zero. The name "Armendariz rings" was given to this class of rings in honor of Armendariz [1], who studied this condition for the first time in the context of reduced rings. Later, Hirano [13] introduced the notion of quasi-Armendariz rings, which is a generalization of Armendariz rings. This class of rings satisfies the following condition: for any $p(x) = \sum_{i=0}^{n} p_i x^i,\ q(x) = \sum_{j=0}^{m} q_j x^j \in R[x]$ if, $p(x)R[x]q(x) = 0$ then, $p_i R q_j = 0$ for each i, j. Hong et al. [15] extended this definition to the skew polynomial rings. A ring R is said to be α-skew quasi-Armendariz if, for $p(x) = \sum_{i=0}^{n} p_i x^i,\ q(x) = \sum_{j=0}^{m} q_j x^j \in R[x, \alpha]$, whenever $p(x)R[x, \alpha]q(x) = 0$, then $p_i R \alpha^i(q_j) = 0$ for each i, j.

In this paper, we start by studying Property ($a.c.$) for various rings, including quasi-Baer rings, p.q.-Baer rings, p.p.-rings, chain rings, $\ell.e.a.$ rings and $r.e.a.$ rings. We investigate cases where R/I, for a two-sided ideal I of R, has Property (a.c.). Additionally, we introduce the definition of Property ($a.c.$) for a nonempty subset of R. This definition allows us to minimize the study of this property from the whole ring, to a nonempty subset with a certain condition that we introduce in this paper. Furthermore, we examine Property ($a.c$) over α-skew quasi-Armendariz rings and α-(sps) quasi-Armendariz rings. We prove that this class of rings have Property ($a.c.$) on the left, but not necessarily on the right. We also provide conditions under which this class of rings will have Property ($a.c.$) on the right. Moreover, we show that most of these results still true for $R[[x, \alpha]]$.

2. Preliminary results on rings having Property ($a.c.$)

Throughout this paper, we will use freely these facts [16]:
(1) For all $a, b \in R$, $r_R(aR + bR) = r_R(aR) \cap r_R(bR)$ and $r_R(RaR) = r_R(aR)$.
(2) For all $a, b \in R$, $\ell_R(Ra + Rb) = \ell_R(Ra) \cap \ell_R(Rb)$ and $\ell_R(RaR) = \ell_R(Ra)$.
(3) A ring R has Property ($a.c.$) on the right if and only if for any $a, b \in R$, $r_R(aR + bR) = r_R(cR)$ for some $c \in R$.

Recall that, a quasi-Baer ring is defined as a ring in which the right annihilator of every right ideal is generated by an idempotent of R [9]. Similarly, a right (respectively, a left) principally quasi-Baer ring (for short p.q.-Baer) is a ring in which the right (respectively, the left) annihilator of every principal right (respectively, left) ideal is generated by an idempotent [6]. In [16], the authors mentioned that every quasi-Baer ring has Property ($a.c.$), in the next Proposition we generalize this result.

Proposition 2.1. *Let R be a right (respectively, left) p.q.-Baer ring. Then, R has Property ($a.c.$) on the right (respectively, on the left). In particular, Baer rings and quasi-Baer rings have Property ($a.c.$).*

Proof. Let R be a right p.q.-Baer ring and $a, b \in R$. According to [7, Proposition 3.2.26], in a p.q.-Baer ring, the right annihilator of each finitely generated ideal is generated by a left semicentral idempotent. Therefore, there exists $e \in \mathcal{S}_l(R)$, such that $r_R(aR + bR) = eR$. We will claim that $r_R((1-e)R) = eR$. As e is a left semicentral idempotent, for an arbitrary $r \in R$, we have $re = ere \Leftrightarrow (1-e)re = 0$. Thus $(1-e)Re = 0 \Rightarrow (1-e)ReR = 0$, hence $eR \subseteq r_R((1-e)R)$. Conversely, let $c \in r_R((1-e)R)$, then $(1-e)Rc = 0 \Rightarrow (1-e)c = 0 \Rightarrow c = ec$. Thus $c \in eR$, hence $r_R((1-e)R) \subseteq eR$. Therefore, $r_R((1-e)R) = eR$, equivalently $r_R(aR + bR) = r_R((1-e)R)$ and so R has Property $(a.c.)$ on the right. By the same arguments we prove the left side.

□

Corollary 2.2 ([16, Proposition 1.6])**.** *Biregular rings have Property* $(a.c.)$.

Proposition 2.3. *Let R be a right (respectively, left) p.p.-ring. If $I(R) = \mathcal{S}_l(R)$ (respectively, $I(R) = \mathcal{S}_r(R)$), then R has Property $(a.c.)$ on the right (respectively, left). In particular, abelian p.p.-rings have Property $(a.c.)$.*

Proof. Let R be a right p.p.-ring. First, we will show that for all $a \in R$, $r_R(a) = r_R(aR)$. Let $a \in R$. Since $a \in aR$, then $r_R(aR) \subseteq r_R(a)$. Conversely, as R is a right p.p.-ring then, $r_R(a) = eR$ with $e \in \mathcal{S}_l(R)$. Moreover, as e is a left semi-central idempotent, then for all $r \in R$ we have $re = ere$. Let $d \in r_R(a) = eR$, then for $r \in R$ we have $ard = aered = 0$. In other words, $d \in r_R(aR)$. Therefore, $r_R(aR) = r_R(a)$ and R is a right p.q.-Baer ring. By Proposition 2.1, R has Property $(a.c.)$ on the right. With a similar method, we can prove the left side. □

Following Mazurek et al. [19], a ring R is said to be a right chain ring if every two right principal ideals of R are comparable with respect to inclusion. In other words, for any $a, b \in R$ either $aR \subseteq bR$ or $bR \subseteq aR$. A left chain is defined similarly. A right-left chain ring will be called a chain ring.

Proposition 2.4. *Every right (respectively, left) chain ring has Property $(a.c.)$ on the right (respectively, on the left).*

Proof. Suppose that R is a right chain ring, and let $a, b \in R$. We can suppose that $aR \subseteq bR$. Then, we have $r_R(aR + bR) = r_R(bR)$, thus R has Property $(a.c.)$ on the right. By the same argument, we can prove that if R is a left chain ring, then it has Property $(a.c.)$ on the left. □

Yohe [21], defined a right elemental annihilator ring, for short $r.e.a.$ ring, as a ring in which every right ideal is the right annihilator of an element of that ring. In other words, for each right ideal I of R, there always exists an element a of R such that $I = r_R(a)$. A left elemental annihilator ring (for short $\ell.e.a.$ ring) was defined analogously.

Proposition 2.5. *Let R be a ring. If R is a r.e.a. (respectively, ℓ.e.a.) ring, then R has Property $(a.c.)$ on the right (respectively, on the left).*

Proof. Suppose that R is a $r.e.a.$ ring, and let I be a finitely generated right ideal of R. It is well known that $r_R(I)$ is a two-sided ideal, thus there exists an element $a \in R$ such that $r_R(I) = r_R(a)$. It is clear that $r_R(aR) \subseteq r_R(a)$. For $b \in r_R(a)$, we have $Rb \subseteq r_R(a)$ because $r_R(a)$ is a two-sided ideal of R. Then, $aRb = 0$ and then $b \in r_R(aR)$. Equivalently, $r_R(aR) = r_R(a)$, therefore $r_R(I) = r_R(aR)$ and R has Property $(a.c.)$ on the right. By the same argument, we prove that $\ell.e.a.$ ring has Property $(a.c.)$ on the left. □

For a two-sided ideal I of a ring R, R/I does not have necessarily Property $(a.c.)$, even if R does. The ring $\mathbb{Z}_2[x, y]$ is a nontrivial graded ring, hence it has Property $(a.c.)$. However, $R = \mathbb{Z}_2[x, y]/\langle x^2, y^2\rangle$ does not have Property $(a.c.)$. Moreover, $R/\bar{x}R$ has Property $(a.c.)$ while R does not (see [16, Example 2.6]). In the following, we will study some cases when R/I will have Property $(a.c.)$.

Proposition 2.6. *Let R be a right chain ring. Then, for any two-sided ideal I of R, R/I has Property $(a.c.)$ on the right.*

Proof. Let I be an ideal of R and $\bar{a}, \bar{b} \in \bar{R} = R/I$. Proposition 2.4, assure the existence of an element $c \in R$ verifying $r_R(aR + bR) = r_R(cR)$. Moreover, we would have either $c = a$ or $c = b$, which implies that $cR = aR \cup bR$.

$$\begin{aligned} \bar{d} \in r_{\bar{R}}(\bar{a}\bar{R} + \bar{b}\bar{R}) &\Leftrightarrow \bar{a}\bar{R}\bar{d} = \bar{0} \text{ and } \bar{b}\bar{R}\bar{d} = \bar{0} \\ &\Leftrightarrow aRd \subseteq I \text{ and } bRd \subseteq I \\ &\Leftrightarrow cRd = aRd \cup bRd \subseteq I \\ &\Leftrightarrow \bar{c}\bar{R}\bar{d} = \bar{0} \\ &\Leftrightarrow \bar{d} \in r_{\bar{R}}(\bar{c}\bar{R}). \end{aligned}$$

Therefore, R/I has Property $(a.c.)$ on the right. □

Remark 2.7. *Proposition 2.6 still true if, replace the word "right" with "left".*

Proposition 2.8. *Let R be a ring having Property $(a.c.)$ and S is a nonempty subset of R. If $\ell_R(S)$ (respectively, $r_R(S)$) is a two-sided ideal of R, then $R/\ell_R(S)$ (respectively, $R/r_R(S)$) has Property $(a.c.)$ on the right (respectively, on the left).*

Proof. Assume that $\ell_R(S)$ is a two-sided ideal of R and let $\bar{a}$, $\bar{b} \in R/\ell_R(S)$. There exists $c \in R$ such that $r_R(aR + bR) = r_R(cR)$.

Let $\bar{d} \in R/\ell_R(S)$.

$$\begin{aligned} \bar{d} \in r_{R/\ell_R(S)}(\bar{a}R/\ell_R(S) + \bar{b}R/\ell_R(S)) &\Leftrightarrow \bar{a}R/\ell_R(S)\bar{d} = \bar{0} \text{ and } \bar{b}R/\ell_R(S)\bar{d} = \bar{0} \\ &\Leftrightarrow aRd \subseteq \ell_R(S) \text{ and } bRd \subseteq \ell_R(S) \\ &\Leftrightarrow aRdS = bRdS = 0 \\ &\Leftrightarrow dS \subseteq r_R(aR + bR) = r_R(cR) \\ &\Leftrightarrow cRdS = 0 \\ &\Leftrightarrow cRd \subseteq \ell_R(S) \\ &\Leftrightarrow \bar{c}R/\ell_R(S)\bar{d} = \bar{0} \\ &\Leftrightarrow \bar{d} \in r_{R/\ell_R(S)}(\bar{c}R/\ell_R(S)). \end{aligned}$$

Therefore, $R/\ell_R(S)$ has Property $(a.c.)$ on the right. □

Corollary 2.9. *Let R be a ring and I a two-sided ideal of R. If R is a ℓ.e.a. (respectively, r.e.a.) ring, then R/I has Property $(a.c.)$ on the right (respectively, on the left).*

Proposition 2.10. *Let R be a ring and I a two-sided ideal of R. If I is a prime or a maximal ideal, then R/I has Property $(a.c.)$.*

Proof. Assume that I is a prime ideal and let $\bar{a}, \bar{b} \in R/I$. Suppose that $a, b \notin I$. Then

$$\begin{aligned} \bar{d} \in r_{R/I}(\bar{a}R/I + \bar{b}R/I) &\Leftrightarrow \bar{a}R/I\bar{d} = \bar{0} \text{ and } \bar{b}R/I\bar{d} = \bar{0} \\ &\Leftrightarrow aRd \subseteq I \text{ and } bRd \subseteq I \\ &\Leftrightarrow d \in I \text{ (because } I \text{ is a prime ideal)} \\ &\Leftrightarrow Rd \subseteq I \\ &\Leftrightarrow 1Rd \subseteq I \\ &\Leftrightarrow \bar{1}R/I\bar{d} = \bar{0} \\ &\Leftrightarrow \bar{d} \in r_{R/I}(\bar{1}R/I). \end{aligned}$$

Therefore, R/I has Property $(a.c.)$ on the right. By the same arguments, we prove the left side. Suppose that I is a maximal two-sided ideal of R and $a, b \in R$. Since I is maximal than R/I is a simple ring. In other words, R/I has no proper nonzero ideals. Consequently, $r_{R/I}(\bar{a}R/I + \bar{b}R/I)$ is either equal to $\bar{0}$ or equal to R/I. Thus, $r_{R/I}(\bar{a}R/I + \bar{b}R/I) = r_{R/I}(\bar{1}R/I)$ or $r_{R/I}(\bar{a}R/I + \bar{b}R/I) = r_{R/I}(\bar{0}R/I)$. Therefore, R/I has Property $(a.c.)$ on the right. The left side can be proved similarly. □

In the next, we will introduce the definition of Property $(a.c.)$ for a nonempty subset S of a ring R.

Definition 2.11. *A nonempty subset S of R has Property $(a.c.)$ on the right (respectively, on the left) over R if, for all $a, b \in S$, there exists $c \in S$ such that $r_R(aR+bR) = r_R(cR)$ (respectively, $\ell_R(Ra+Rb) = \ell_R(Rc)$).*

Proposition 2.12. *Let S be a nonempty subset of a ring R. Assume for all $a \in R$ there exists $b \in S$ such that $r_R(aR) = r_R(bR)$. Then R has Property $(a.c.)$ if and only if S has Property $(a.c.)$.*

Proof. Suppose that R has Property $(a.c.)$ and let $a, b \in S$. Then there exists $c \in R$ such that $r_R(aR + bR) = r_R(cR)$. Moreover, there exits $d \in S$ such that $r_R(cR) = r_R(dR)$. Consequently, $r_R(aR + bR) = r_R(dR)$ with $d \in S$. Therefore, S has Property $(a.c.)$ on the right. For the other implication, suppose that S has Property $(a.c.)$. Let $a, b \in R$, then there exists $c, d \in S$ such that $r_R(aR) = r_R(cR)$ and $r_R(bR) = r_R(dR)$. As S has Property $(a.c.)$ over R, then there exists $e \in S$ such that $r_R(aR + bR) = r_R(cR + dR) = r_R(eR)$. Therefore, R has Property $(a.c.)$ on the right. By the same arguments, we prove the left side. □

In 2019, Bajor and Ziembowsi [2], proved that Property $(a.c.)$ does not pass to polynomial rings and power series rings, by constructing two finite-dimensional algebras A and A', such that A has Property $(a.c.)$ on the right, but neither $A[x]$ nor $A[[x]]$ do has; and $A'[x]$ has Property $(a.c.)$ on the right but A' does not. Later, in 2021, Dube and Taherifar [10], defined a class of rings in which Property $(a.c.)$ passes to polynomial rings. This class of rings was defined as follows: a ring R is called a right strongly quasi-Armendariz (for short, s.q.-Armendariz) ring if, for each $p(x) \in R[x]$, there exists $a \in R$ such that $r_{R[x]}(p(x)R[x]) = r_R(aR)R[x]$.

Proposition 2.13. *If R is a s.q.-Armendariz ring, then for all $p(x) \in R[x]$, there exists $a \in R$ such that $r_{R[x]}(p(x)R[x]) = r_{R[x]}(aR[x])$.*

Proof. Suppose that R is s.q.-Armendariz, then for all $p(x) \in R[x]$, there exists an element $a \in R$ such that $r_{R[x]}(p(x)R[x]) = r_R(aR)R[x]$. We claim that $r_{R[x]}(aR[x]) = r_R(aR)R[x]$ for all $a \in R$. It is well know that every s.q.-Armendariz ring is quasi-Armnedariz, so by [10, Lemma 2.1], we have $r_{R[x]}(aR[x]) = r_R(aR)R[x]$.

Therefore, $r_{R[x]}(p(x)R[x]) = r_R(aR)R[x] = r_{R[x]}(aR[x])$. We conclude that for all $p(x) \in R[x]$, there exists $a \in R$ such that $r_{R[x]}(p(x)R[x]) = r_{R[x]}(aR[x])$. □

Corollary 2.14 ([10, Theorem 2.9])**.** *Let R be a s.q.-Armendariz ring. Then, $R[x]$ has Property $(a.c.)$ if and only if R so has.*

Proof. Clearly from Propositions 2.12 and 2.13. □

3. Skew polynomial rings and skew power series rings having Property $(a.c.)$

In this section, we study ordinary polynomial rings, skew polynomial rings and power series rings that have Property $(a.c.)$. Following Baser et al. [3], a ring R is said to be α-sps Armendariz if, whenever $p(x)q(x) = 0$ for $p(x) = \sum_{i=0} p_i x^i$, $q(x) = \sum_{j=0} q_j x^j \in R[[x, \alpha]]$, then $p_i q_j = 0$ for each i, j. Paykan and Mussavi [20], generalized this definition for a ring R with a strictly ordered monoid $(M, \leqslant)$ and a monoid homomorphism $\sigma\colon\ M \to End(R)$, as follows: R is said to be (M, σ)-quasi-Armendariz if whenever the product $pR[[M, \sigma]]q$ is zero, for $p, q \in R[[M, \sigma]]$ then so is $p(m)R\sigma_m(q(n))$ for all $m, n \in M$. We will reformulate this definition for the case when $M = \mathbb{N} \cup \{0\}$ and $\sigma(1) = \alpha$.

Definition 3.1. *A ring R is called a α-sps quasi-Armendariz ring if for any $p(x) = \sum_{i=0} p_i x^i$, $q(x) = \sum_{j=0} q_j x^j \in R[[x, \alpha]]$, $p(x)R[[x, \alpha]]q(x) = 0$ implies $p_i R\alpha^i(q_j) = 0$ for each i, j. When α is the identity mapping, then R is called a ps quasi-Armendariz ring.*

In the next, we will see that α-skew quasi-Armendariz property is a sufficient condition to have Property $(a.c.)$ on the left for skew polynomial rings.

Proposition 3.2. *If R is α-skew quasi-Armendariz (respectively, α-sps quasi-Armendariz) ring, then $R[x, \alpha]$ (respectively, $R[[x, \alpha]]$) has Property $(a.c.)$ on the left.*

Proof. Suppose that R is α-skew quasi-Armendariz ring. For $p(x) = \sum_{i=0}^{n} p_i x^i$, $q(x) = \sum_{j=0}^{m} q_j x^j \in R[x, \alpha]$, let $h(x) = \sum_{i=0}^{m+n+1} h_i x^i$, with $h_i = p_{\frac{i}{2}}$ when i is even and $h_i = q_{\frac{i-1}{2}}$ when i is odd, for all i. We claim that

$$\ell_{R[x,\alpha]}(R[x,\alpha]p(x) + R[x,\alpha]q(x)) = \ell_{R[x,\alpha]}(R[x,\alpha]h(x)).$$

Let $l(x) = \sum_{i=0}^{u} l_i x^i \in \ell_{R[x,\alpha]}(R[x,\alpha]p(x) + R[x,\alpha]q(x))$, then $l(x)R[x,\alpha]p(x) = 0$ and $l(x)R[x,\alpha]q(x) = 0$, since the ring R is a α-skew quasi-Armendariz, we have for all i, j; $l_i R\alpha^i(p_j) = l_i R\alpha^i(q_j) = 0$. Moreover, by [15, Remark 3], we have $l_i R\alpha^{i+t}(p_j) = l_i R\alpha^{i+t}(q_j) = 0$ for all positive integer t. Thus, $l_i R\alpha^{i+t}(h_j) = 0$ for all i, j and all positive integer t. Let $w(x) = \sum_{i=0}^{v} w_i x^i \in R[x,\alpha]$, we have $l(x)w(x)h(x) = \sum_i \sum_{j+k=i} \sum_{t+r=j} l_t \alpha^t(w_r)\alpha^j(h_k)x^i = 0$. Then,

$$\ell_{R[x,\alpha]}(R[x,\alpha]p(x) + R[x,\alpha]q(x)) \subseteq \ell_{R[x,\alpha]}(R[x,\alpha]h(x)).$$

For the other inclusion, let $l(x) \in \ell_{R[x,\alpha]}(R[x,\alpha]h(x))$, then $l(x)R[x,\alpha]h(x) = 0$, as R is α-skew quasi-Armendariz, so for all i, j and for all positive integer t, we have $l_i R\alpha^{i+t}(h_j) = 0$ and so $l_i R\alpha^{i+t}(p_j) = l_i R\alpha^{i+t}(q_j) = 0$. Let $w(x) = \sum_{i=0}^{v} w_i x^i \in R[x,\alpha]$, we have

$$l(x)w(x)p(x) = \sum_i \sum_{j+k=i} \sum_{t+r=j} l_t \alpha^t(w_r)\alpha^j(p_k)x^i = 0$$

and

$$l(x)w(x)q(x) = \sum_i \sum_{j+k=i} \sum_{t+r=j} l_t \alpha^t(w_r)\alpha^j(q_k)x^i = 0$$

so $l(x) \in \ell_{R[x,\alpha]}(R[x,\alpha]p(x) + R[x,\alpha]q(x))$. Equivalently,

$$\ell_{R[x,\alpha]}(R[x,\alpha]p(x) + R[x,\alpha]q(x)) = \ell_{R[x,\alpha]}(R[x,\alpha]h(x)).$$

Therefore, $R[x,\alpha]$ has Property $(a.c.)$ on the left. Similar proof when R is α-sps quasi-Armendariz ring. □

Corollary 3.3. *If R is a quasi-Armendariz (respectively, ps quasi-Armendariz) ring, then $R[x]$ (respectively, $R[[x]]$) has Property $(a.c.)$ on the left.*

While the property of being α-skew quasi-Armendariz insures Property $(a.c.)$ on the left, it does not guarantee Property $(a.c.)$ on the right. The following example confirm this fact.

Example 3.4. *Let $R = \mathbb{Z}_2[x,y]/\langle x^2, y^2\rangle$ and α is the homomorphism of R defined by $\alpha(p(x,y)) = p(0,0)$. The skew polynomial ring $R[z,\alpha]$ does not have Property $(a.c.)$ on the right but it does have the property on the left (see [16, Example 1.2]). We will prove that R is a $\alpha-skew$ quasi-Armendariz ring. Let $p(z) = \sum_{i=0}^{n} p_i z^i$, $q(z) = \sum_{j=0}^{m} q_j z^j \in R[z,\alpha]$, such that $p(z)R[z,\alpha]q(z) = 0$. Suppose that $p(z)$, and $q(z)$ are nonzero polynomials. We have*

$$p(z)R[z,\alpha]q(z) = 0 \Rightarrow p(z)q(z) = 0 \Rightarrow \sum_{i=0}^{m+n} \sum_{j+k=i} p_j \alpha^j(q_k)x^i = 0.$$

We denote $\alpha_i = \sum_{j+k=i} p_j \alpha^j(q_k) = p_i\alpha^i(q_0) + p_{i-1}\alpha^{i-1}(q_1) + \cdots + p_0 q_i = 0$, for all i. We will use a similar method as in [14, Example 5]. Suppose that there exists $s \geq 0$ such that $p_s \neq 0$ and $p_i = 0$ for all $i < s$. Equivalently, $p_i\alpha^i(q_j) = 0$ for all $i < s$. We have

$$\begin{aligned} \alpha_s = 0 &\Leftrightarrow p_s\alpha^s(q_0) + p_{s-1}\alpha^{s-1}(q_1) + \cdots + p_0 q_s = 0 \\ &\Leftrightarrow p_s\alpha^s(q_0) = 0 \\ &\Leftrightarrow q_0(0,0) = 0 \end{aligned}$$

By the same method, we have

$$\begin{aligned}\alpha_{s+1}=0 &\Leftrightarrow p_{s+1}\alpha^s(q_0)+p_s\alpha^s(q_1)+\cdots+p_0q_{s+1}=0\\ &\Leftrightarrow p_s\alpha^s(q_1)=0\\ &\Leftrightarrow q_1(0,0)=0\end{aligned}$$

Continuing this process, we obtain $q_0(0,0)=q_1(0,0)=\cdots=q_m(0,0)=0$. *Therefore* $p_i\alpha^i(q_j)=0$ *for all* i *and* j.
As R *is a commutative ring, then* $p_iR\alpha^i(q_j)=0$ *for all* i,j. *This prove that* R *is a* α*-skew quasi-Armendariz ring.*

According to Baser and Kwak [4], a ring R is called α quasi-Armendariz if, for any $p(x)=\sum_{i=0}^n p_ix^i$, $q(x)=\sum_{j=0}^m q_jx^j\in R[x,\alpha]$ if, $p(x)R[x,\alpha]q(x)=0$, then $p_iR\alpha^j(q_k)=0$ for all i,j,k. In the next, we will extend this definition to skew power series rings.

Definition 3.5. *A ring* R *is called a* α*-ps quasi-Armendariz ring if, for any* $p(x)=\sum_{i\geq 0}p_ix^i$, $q(x)=\sum_{j\geq 0}q_jx^j\in R[[x,\alpha]]$ *if* $p(x)R[[x,\alpha]]q(x)=0$, *then* $p_iR\alpha^j(q_k)=0$ *for all* i,j,k.

Proposition 3.6. *If* R *is a* α *quasi-Armendariz (respectively,* α*-ps quasi-Armendariz) ring, then* $R[x,\alpha]$ *(respectively,* $R[[x,\alpha]]$*) has Property (a.c.) on the right.*

Proof. Suppose that R is a α quasi-Armendariz ring. For $p(x)=\sum_{i=0}^n p_ix^i$, $q(x)=\sum_{j=0}^m q_jx^j\in R[x,\alpha]$, let $h(x)=\sum_{i=0}^{n+m+1}h_ix^i\in R[x,\alpha]$ with $h_i=p_{\frac{i}{2}}$ when i is even and $h_i=q_{\frac{i-1}{2}}$ when i is odd, for all i. We claim that

$$r_{R[x,\alpha]}(p(x)R[x,\alpha]+q(x)R[x,\alpha])=r_{R[x,\alpha]}(h(x)R[x,\alpha]).$$

Let $l(x)=\sum_{i=0}^u l_ix^i\in r_{R[x,\alpha]}(p(x)R[x,\alpha]+q(x)R[x,\alpha])$, then $p(x)R[x,\alpha]l(x)=0$ and $q(x)R[x,\alpha]l(x)=0$, as R is a α quasi-Armendariz ring, so we have $p_iR\alpha^j(l_k)=q_iR\alpha^j(l_k)=0$, for all i,j,k. Thus $h_iR\alpha^j(l_k)=0$, for all i,j,k. Let $w(x)=\sum_{i=0}^v w_ix^i\in R[x,\alpha]$, we have $h(x)w(x)l(x)=\sum_i\sum_{j+k=i}\sum_{t+r=j}h_t\alpha^t(w_r)\alpha^j(l_k)x^i=0$. Then,

$$r_{R[x,\alpha]}(p(x)R[x,\alpha]+q(x)R[x,\alpha])\subseteq r_{R[x,\alpha]}(h(x)R[x,\alpha]).$$

For the other inclusion, let $l(x)\in r_{R[x,\alpha]}(h(x)R[x,\alpha])$, then $h(x)R[x,\alpha]l(x)=0$, as R is a α quasi-Armendariz ring, so we have $h_iR\alpha^j(l_k)=0$ for all i,j,k and so $p_iR\alpha^j(l_k)=q_iR\alpha^j(l_k)=0$, for all i,j,k. Let $w(x)=\sum_{i=0}^v w_ix^i\in R[x,\alpha]$, we have

$$p(x)w(x)l(x)=\sum_i\sum_{j+k=i}\sum_{t+r=j}p_t\alpha^t(w_r)\alpha^j(l_k)x^i=0$$

and

$$q(x)w(x)l(x)=\sum_i\sum_{j+k=i}\sum_{t+r=j}q_t\alpha^t(w_r)\alpha^j(l_k)x^i=0,$$

equivalently, $l(x)\in r_{R[x,\alpha]}(p(x)R[x,\alpha]+q(x)R[x,\alpha])$. Equivalently,

$$r_{R[x,\alpha]}(p(x)R[x,\alpha]+q(x)R[x,\alpha])=r_{R[x,\alpha]}(h(x)R[x,\alpha]).$$

Therefore, $R[x,\alpha]$ has Property $(a.c.)$ on the right. By the same arguments, we prove that $R[x,\alpha]$ also has Property $(a.c.)$ on the right when R is α-ps quasi-Armendariz. □

Corollary 3.7. *If* R *is a quasi-Armendariz (respectively, ps quasi-Armendariz) ring, then* $R[x]$ *(respectively,* $R[[x]]$*) has Property (a.c.) on the right.*

Recall that a ring R is called to be α-compatible if, for all $a,b\in R$, $ab=0\Leftrightarrow a\alpha(b)=0$ [11].

Corollary 3.8. *Let* R *be a* α*-compatible ring. If* R *is* α*-skew quasi-Armendariz (respectively,* α*-sps quasi-Armendariz), then* $R[x,\alpha]$ *(respectively,* $R[[x,\alpha]]$*) has Property (a.c.) on the right.*

Proof. Assume that R is α-skew quasi-Armendariz. Let $p(x)=\sum_{i=0}^{n}p_ix^i,\ q(x)=\sum_{j=0}^{m}q_jx^j\in R[x,\alpha]$, such that $p(x)R[x,\alpha]q(x)=0$, then $p_iR\alpha^i(q_k)=0$ for all i,k. As R is α-compatible, we have $p_iR\alpha^j(q_k)=0$, for all i,j,k. Therefore, R is a α quasi-Armendariz ring and by Proposition 3.6, it has Property $(a.c.)$ on the right. The same proof for $R[[x,\alpha]]$.

□

Hong et al. [15], showed that when α is an epimorphism, then every semiprime ring is α-skew quasi-Armendariz. Moreover, for all $p(x)=\sum_{i=0}^{n}p_ix^i,\ q(x)=\sum_{j=0}^{m}q_jx^j\in R[[x,\alpha]]$, whenever $p(x)R[[x,\alpha]]q(x)=0$ then $p_iR\alpha^{i+t}(q_j)=0$ for all i,j and for all positive integer t. In particular for $t=0$, $R[[x,\alpha]]$ is α-sps quasi-Armendariz ring.

Corollary 3.9. *Let R be a semiprime ring and α is an epimorphism. If R is α-compatible, then $R[x,\alpha]$ and $R[[x,\alpha]]$ have Property $(a.c.)$.*

It was mentioned before that Property $(a.c.)$ does not pass to polynomials and power series [2]. However, Property $(a.c.)$ can be transferred to skew polynomial rings (respectively, power series rings) for some kind of rings, as shown by the following result. It is the case of α-skew quasi-Armendariz rings and α-sps quasi-Armendariz rings.

Proposition 3.10. *Let R be a ring having Property $(a.c.)$ on the right. If R is α-skew quasi-Armendariz (respectively, α-sps quasi-Armendariz), then $R[x,\alpha]$ (respectively, $R[[x,\alpha]]$) has Property $(a.c.)$ on the right.*

Proof. Let $p(x)=\sum_{i=0}^{n}p_ix^i,\ q(x)=\sum_{j=0}^{m}q_jx^j\in R[x,\alpha]$, and let $\eta=\max\{n,\ m\}$, then we can write $p(x)=\sum_{i=0}^{\eta}p_ix^i$ and $q(x)=\sum_{j=0}^{\eta}q_jx^j$ with $p_i=q_j=0$ for $n<i$ and $m<j$.
As R has Property $(a.c.)$ on the right, then for all $i\in\{0,\dots,\eta\}$, there exists h_i an element of R such that $r_R(p_iR+q_iR)=r_R(h_iR)$. Let $h(x)=\sum_{i=0}^{\eta}h_ix^i$, we claim that

$$r_{R[x,\alpha]}(p(x)R[x]+q(x)R[x,\alpha])=r_{R[x,\alpha]}(h(x)R[x,\alpha]).$$

Let $l(x)=\sum_{i=0}^{u}l_ix^i\in r_{R[x,\alpha]}(p(x)R[x,\alpha]+q(x)R[x,\alpha])$, then $p(x)R[x,\alpha]l(x)=0$ and $q(x)R[x,\alpha]l(x)=0$, as R is a α-skew quasi-Armendariz ring, then for all i,j and for all integer $t\geq 0$ we have $p_iR\alpha^{i+t}(l_j)=q_iR\alpha^{i+t}(l_j)=0$, thus $h_iR\alpha^{i+t}(l_j)=0$. Let $w(x)=\sum_{i=0}^{v}w_ix^i\in R[x,\alpha]$, we have $h(x)w(x)l(x)=\sum_i\sum_{j+k=i}\sum_{t+r=j}h_t\alpha^t(w_r)\alpha^j(l_k)x^i=0$. Equivalently, $h(x)R[x,\alpha]l(x)=0$. Therefore,

$$r_{R[x,\alpha]}(p(x)R[x,\alpha]+q(x)R[x,\alpha])\subseteq r_{R[x,\alpha]}(h(x)R[x,\alpha]).$$

For the other inclusion, let $l(x)\in r_{R[x,\alpha]}(h(x)R[x,\alpha])$, then $h(x)R[x,\alpha]l(x)=0$, as R is a α-skew quasi-Armendariz ring then for all i,j and for all integer $t\geq 0$, we have $h_iR\alpha^{i+t}(l_j)=0$ and so $p_iR\alpha^{i+t}(l_j)=q_iR\alpha^{i+t}(l_j)=0$. Let $w(x)=\sum_{i=0}^{v}w_ix^i\in R[x,\alpha]$, we have

$$p(x)w(x)l(x)=\sum_i\sum_{j+k=i}\sum_{t+r=j}p_t\alpha^t(w_r)\alpha^j(l_k)x^i=0$$

and

$$q(x)w(x)l(x)=\sum_i\sum_{j+k=i}\sum_{t+r=j}q_t\alpha^t(w_r)\alpha^j(l_k)x^i=0,$$

equivalently, $p(x)R[x,\alpha]l(x)=q(x)R[x,\alpha]l(x)=0$. Therefore, $r_{R[x,\alpha]}(p(x)R[x,\alpha]+q(x)R[x,\alpha])=r_{R[x,\alpha]}(h(x)R[x,\alpha])$ and so $R[x,\alpha]$ has Property $(a.c.)$ on the right. By the same method, we prove that $R[[x,\alpha]])$ has Property $(a.c.)$ on the right. □

Ben Yakoub and Louzari [5], introduced the condition $\mathcal{C}_\alpha$ for a ring R while $\alpha\in End(R)$, as follows: a ring R satisfied the condition $\mathcal{C}_\alpha$ if, whenever $a\alpha(b)=0$, then $ab=0$ with $a,b\in R$. The condition $\mathcal{C}_\alpha$ generalizes the α-rigid condition. Moreover, they coincide when R is reduced (see [5, Lemma 2.2]).

Proposition 3.11. *Let R be a right p.q.-Baer ring and $\alpha \in End(R)$ such that Re is α-stable for all $e \in \mathcal{S}_l(R)$. If R verifies the condition $\mathcal{C}_\alpha$, then $R[x,\alpha]$ has Property $(a.c.)$ on the right.*

Proof. From [5, Proposition 3.1], $R[x,\alpha]$ is a right p.q.-Baer ring, therefore $R[x,\alpha]$ has Property $(a.c.)$ on the right by Proposition 2.1. □

Corollary 3.12. *If R is a p.q.-Baer ring, then $R[x]$ has Property $(a.c.)$ on the right.*

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References

[1] E. P. Armendariz, A note on extensions of Baer and P. P. rings, J. Austral. Math, Soc., 18(4) (1974) 470–473.
[2] G. Bajor, M. Ziembowski, Annihilator condition does not pass to polynomials and power series, J. Pure Appl. Algebra, 223(9) (2019) 3869–3878.
[3] M. Baser, A. Harmanci, T. K. Kwak, Generalized semicommutative rings and their extensions, Bull. Korean Math. Soc., 45(2) (2008) 285–297.
[4] M. Baser, T. K. Kwak, Quasi-Armendariz property for skew polynomial rings. Comm. Korean Math. Soc., 26(4) (2011) 557-573.
[5] L. Ben Yakoub, M. Louzari, Ore extensions of principally quasi-Baer rings, JP J. Algebra Number Theory Appl., 13(2) (2009) 137–151.
[6] G. F. Birkenmeier, J. Y. Kim, J. K. Park, Principally quasi-Baer rings, Comm. Algebra., 29(2) (2001) 639–660.
[7] G. F. Birkenmeier, J. K. Park, S. T. Rizvi, Extensions of rings and modules, Springer New York, NY (2013).
[8] S. Chhawchharia, M. B. Rege, Armendariz rings, Proc. Japan. Acad. Ser. A Math. Sci., 73(1) (1997) 14–17.
[9] W. E. Clark, Twisted matrix units semigroup algebras. Duke Math. J., 34(3) (1967) 417–424.
[10] T. Dube, A. Taherifar, On the lattice of annihilator ideals and its applications, Comm. Algebra, 49(6) (2021) 2444–2456.
[11] E. Hashemi, A. Moussavi, Polynomial extensions of quasi-Baer rings, Acta Math. Hungar., 107(3) (2005) 207–224.
[12] M. Henriksen, M. Jerison, The space of minimal prime ideals of a commutative ring, Trans. Amer. Math. Soc., 115 (1965) 110–130.
[13] Y. Hirano, On annihilator ideals of a polynomial ring over a noncommutative ring, J. Pure Appl. Algebra., 168(1) (2002) 45–52.
[14] C. Y. Hong, N. K. Kim, T. K. Kwak, On skew Armendariz rings, Comm. Algebra., 31(1) (2003) 103–122.
[15] C. Y. Hong, N. K. Kim, Y. Lee, Skew polynomial rings over semiprime rings, J. Korean Math. Soc., 47(5) (2010) 879–897.
[16] C. Y. Hong, N. K. Kim, Y. Lee, P. P. Nielsen, The minimal prime spectrum of rings with annihilator conditions, J. Pure Appl. Algebra, 213(7) (2009) 1478–1488.
[17] J. A. Huckaba, J. M. Keller, Annihilation of ideals in commutative rings, Pacific J. Math., 83(2) (1979) 375–379.
[18] T. G. Lucas, Two annihilator conditions: property (A) and $(a.c.)$, Comm. Algebra, 14(3) (1986) 557–580.
[19] R. Mazurek, G. Törner, Comparizer ideals of rings, Comm. Algebra, 32(12) (2004) 4653–4665.
[20] A. Moussavi, K. Paykan, Quasi-Armendariz generalized power series rings, J. Algebra Appl., 15(5) (2016) 1650086.

[21] C. R. Yohe, On rings in which every ideal is the annihilator of an element, Proc. Amer. Math. Soc., 19(6) (1968) 1346–1348.