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Width-k Eulerian polynomials of type A and B: The γ-positivity

Research Article

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Abstract: In this paper, we introduce some new generalizations of classical descent and inversion statistics on signed permutations that arise from the work of Sack and Úlfarsson [18], and called *k-width descents* and *k-width inversions* of type A ([8]). Using the aforementioned new statistics, we derive new generalizations of Eulerian polynomials of type A, B and D. We establish also the γ-positivity of the Eulerian *"width-k"* polynomials. Referring to Petersen's paper [16], we give a combinatorial interpretation of finite sequences associated with these new polynomials using quasi-symmetric functions and a partition P.

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1. Introduction

The main purpose of this paper is to extend some fundamental aspects of the theory of Eulerian polynomials on Coxeter groups and their unimodality, symmetry and *Gamma*-positivity. Many polynomials with combinatorial meanings have been shown to be unimodal (see [4] or [14] for example). Let $\mathcal{A} = \{a_i\}_{i=0}^d$ be a finite sequence of nonnegative numbers. Recall that a polynomial $g(x) = \sum_{i=0}^d a_i x^i$ of degree d is said to be positive and unimodal, if the coefficients are increasing and then decreasing, i.e., there is a certain index $0 \leq j \leq d$ such that

$$0 \leq a_0 \leq a_1 \leq \cdots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \cdots \geq a_d \geq 0.$$

We will say that $g(x)$ is *palindromic* (or symmetric) with center of symmetry at $\lfloor d/2 \rfloor$, if $a_i = a_{d-i}$ for all $0 \leq i \leq d$.

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The polynomial $g(x)$ is said to be *Gamma*-positive (or γ-positive) if

$$g(x) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i x^i (1+x)^{d-2i},$$

where d is the degree of $g(x)$ and nonnegative reals $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor d/2 \rfloor}$. So, both palindromic and unimodal are two necessary conditions for the *Gamma*-positivity of $g(x)$. One of the most important polynomials in combinatorics is the nth Eulerian polynomials, for the statistic "des^A" on $\mathfrak{S}_n$, defined as

$$A_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{des^A(\sigma)}.$$

Given a set of combinatorial objects τ, a combinatorial statistics is an integer for to every element of the set. In other words, it is a function $st : \tau \to \mathbb{N}$.

For a statistic st on symmetric group $\mathfrak{S}_n$, one may form the generating function:

$$F_n^{st}(x) = \sum_{\pi \in \mathfrak{S}_n} x^{st(\pi)}.$$

MacMahon [15] considered four different statistics for a permutation σ in the group of all permutations $\mathfrak{S}_n$ (it is also called type-A permutation) of the set $[n] := \{1, \ldots, n\}$; the number of descents ($des^A(\sigma)$), the number of excedances ($exc^A(\sigma)$), the number of inversions ($inv^A(\sigma)$) and the major index ($maj^A(\sigma)$). Given a permutation σ in $\mathfrak{S}_n$, we say that the pair $(i,j) \in [n]^2$ is an inversion of σ if $i < j$ and $\sigma(i) > \sigma(j)$, that $i \in \{1, 2, \ldots, n-1\}$ is a descent if $\sigma(i) > \sigma(i+1)$, and that $i \in \{1, \ldots, n\}$ is an excedance if $\sigma(i) > i$. The major index is the sum of all the descendants. These four statistics have many generalizations.

The symmetric group $\mathfrak{S}_n$ is generated by the set $\tau := \{\tau_i;\ 1 \leq i \leq n-1\}$, where $\tau_i := (1, \ldots, i-1, i+1, i, i+2, \ldots, n) \in \mathfrak{S}_n$, for $i \in [n-1]$. Therefore, the length function on $\mathfrak{S}_n$ is defined to be: for any $\sigma \in \mathfrak{S}_n$,

$$\ell^A(\sigma) := \min\{r \geq 0;\ \sigma = \tau_{x_1}\tau_{x_2}\cdots\tau_{x_r};\ \tau_{x_i} \in \tau,\ x_i \in [n-1]\}.$$

Then the length function can be written in terms of inversion number on $\mathfrak{S}_n$ (see [2], Proposition 3.1, for $\sigma \in \mathfrak{S}_n$) as follows

$$\ell^A(\sigma) = inv^A(\sigma).$$

The paper is organized as follows. We start with some definitions which generalized the *width-k descents* and *width-k inversions* statistics on classical permutations studied by Davis [8] into signed permutations. In Section 2, we will prove Proposition 2.9, in which we improve the combinatorial formulas of these last statistics in signed permutations and give some examples. In section 3 and 4, we will show Theorem 3.4, Theorem 4.2 and Theorem 4.4, in which we define the *width-k Eulerian* polynomials of type A and B. So, we give some recurrence relations concerning the coefficients of these polynomials. Then, we will study the γ-positivity by specifying the combinatorial values of γ. Finally, in Section 5, which is the same as section 3 and 4, we will define the *width-k Eulerian* polynomials of type D and we define two sets $WD_{n,k,p}$ and $W\bar{D}_{n,k,p}$ in order to find the recurrence relations for the coefficients of this polynomial. We will prove Theorem 5.6 by studying the necessary condition for this polynomial to be γ-positive.

2. Width-k descents and width-k inversions on signed permutations

Recently, Sack and Úlfarsson [18] introduced new generalizations of classical descents and inversions statistics for any permutation in $\mathfrak{S}_n$. They are called *width-k descents* and *width-k inversions* (see [8]).

For each $1 \leq k < n$, the sets of such statistics are defined as follows

$$\begin{aligned} Des_k^A(\sigma) &:= \{i \in [n-k]; \sigma(i) > \sigma(i+k)\}, \\ Inv_k^A(\sigma) &:= \{(i,j) \in [n]^2; \sigma(i) > \sigma(j) \ \ and \ j - i = mk, \ m > 0\}. \end{aligned}$$

Their cardinalities are denoted, respectively, by $des_k^A(\sigma)$ and $inv_k^A(\sigma)$.

In this paper, we study same analogues of these statistics on signed permutations. A signed permutation is a bijection of $[-n, n] := \{-n, \cdots, -1, 1, \cdots, n\}$ onto itself that satisfies $\pi(-i) = -\pi(i)$ for all $i \in [n]$. We denote by B_n the set of signed permutations of length n, which are known as hyperoctahedral groups. Let $D_n \subset B_n$ be the subset consisting of the signed permutations with even negative entries number. Adin, Brenti and Roichman [1] defined a permutation statistics known as signed descent number (or type-B descent number) and $flag$ descent number. A signed descent of $\pi = (\pi(1), \pi(2), \ldots, \pi(n)) \in B_n$ is an integer $0 \leq i \leq n-1$ satisfying $\pi(i) > \pi(i+1)$, where $\pi(0) = 0$. The signed descent of $\pi \in D_n$ (type-D descent) given by Chow [7] is a type-B descent restricted to D_n. The signed descent number of $\pi \in B_n$ is denoted by $des^B(\pi)$ and defined as follows

$$des^B(\pi) := |\{0 \leq i \leq n-1; \ \pi(i) > \pi(i+1)\}|.$$

The $flag$ descent statistics of a signed permutation π, denoted by $fdes^B(\pi)$, counts a descent at position 0 once and all other descents twice. In other words,

$$fdes^B(\pi) := des^B(\pi) + des^A(\pi).$$

The *nth Eulerian* polynomials of signed permutations and the *nth flag descents* polynomials are defined, respectively, by

$$\mathfrak{B}_n(x) = \sum_{\pi \in B_n} x^{des^B(\pi)},$$

$$F_n(x) = \sum_{\pi \in B_n} x^{fdes^B(\pi)}.$$

Definition 2.1. *Let π be a permutation in B_n. We define:*

1. *the inversion number in π by*

$$inv^A(\pi) := |\{(i,j); 1 \leq i < j \leq n \text{ and } \pi(i) > \pi(j)\}|,$$

2. *the negative integers number in π by*

$$neg(\pi) := |\{i \in [n]; \pi(i) < 0\}|,$$

3. *the negative pairs sum number on π by*

$$nsp(\pi) := |\{(i,j) \in [n]^2; i < j \text{ and } \pi(i) + \pi(j) < 0\}|,$$

4. *the type-B inversion's number of π by*

$$inv^B(\pi) := inv^A(\pi) + neg(\pi) + nsp(\pi).$$

As the Coxeter group, B_n is generated by the set $\tau^B := \{\tau_1^B, \ \tau_2^B, \ldots, \tau_{n-1}^B, \ \tau_0^B\}$, where $\tau_0^B := (-1, 2, \cdots, n)$ and $\tau_i^B := (1, 2, \cdots, i-1, i+1, i, i+2, \cdots, n)$, for $i \in [n-1]$ (for more details see again [2]). We define the length function $\ell^B(\cdot)$, on B_n, by

$$\ell^B(\pi) := \min\{r \geq 0; \ \pi = \tau_{x_1}^B \tau_{x_2}^B \cdots \tau_{x_r}^B; \ \tau_{x_i}^B \in \tau^B, \ 0 \leq x_i \leq n-1\}, \quad \forall \pi \in B_n.$$

Recently, Brenti ([2]) has proved that the function $\ell^B(\cdot)$ can be interpreted combinatorially as

$$\ell^B(\pi) = inv^B(\pi).$$

We extend the same definitions of the width-k statistics on classical permutations, introduced in [8], to signed permutations as follows:

Definition 2.2. *Let $1 \leq k \leq n$ and $\pi = (\pi(1), \pi(2), \cdots, \pi(n))$ be a permutation in B_n with $\pi(0) = 0$. We define:*

1. $Inv_k^A(\pi) := \{(i,j); 1 \leq i < j \leq n; \pi(i) > \pi(j) \text{ and } j - i = mk, m > 0\}$ *as the set of width-k type A inversion,*
2. $Inv_k^B(\pi) := \{(i,j); 0 \leq i < j \leq n;\ \pi(i) > \pi(j) \text{ and } j - i = mk, m > 0\} \cup \{(-i,j); 1 \leq i < j \leq n;\ \pi(-i) > \pi(j) \text{ and } j - i = mk, m > 0\}$ *as the set of width-k inversion,*
3. $Des_k^B(\pi) := \{0 \leq i \leq n-k;\ \pi(i) > \pi(i+k)\}$ *as the set of width-k descent,*
4. $des_k^B(\pi) := |Des_k^B(\pi)|$ *as the width-k descent number,*
5. $fdes_k^B := des_k^A(\pi) + des_k^B(\pi)$ *as the width-k flag descent number,*
6. $ndes_k^B(\pi) := |\{1 \leq i \leq n-k; \pi(-i) > \pi(i+k)\}|$ *as the width-k negative descent number,*
7. $inv_k^A(\pi) := |Inv_k^A(\pi)|$ *as the width-k type A inversion number,*
8. $neg_k(\pi) := |\{1 \leq i \leq \lfloor \frac{n}{k} \rfloor;\ \pi(ik) < 0\}|$ *as the width-k negative number,*
9. $nsp_k(\pi) := |\{(i,j) \in [n]^2; \pi(i) + \pi(j) < 0, j - i = mk, m > 0\}|$ *as the width-k negative sum pairs number,*
10. $inv_k^B(\pi) := inv_k^A(\pi) + neg_k(\pi) + nsp_k(\pi)$ *as the width-k inversion number.*

For the case $k = 1$, the set $Inv_1^B(\pi)$ can be also denoted by $Inv^B(\pi)$ and we will prove that $inv_k^B(\pi) = |Inv_k^B(\pi)|$, for all $1 \leq k \leq n$.

Let $des_k^D(\pi)$ be the set of all *width-k* descents of $\pi \in D_n$ given by

$$des_k^D(\pi) := |\{0 \leq i \leq n-k; \pi(i) > \pi(i+k)\}|, \quad (\pi(0) = 0).$$

Taking $k = 1$, we recover Borowiec's and Mlotkowski's definition of width-1 type D descents (see [3] for $\pi(0) = 0$). A similar definition was established by Brenti ([2]), in which $\pi(0){:=}{-\pi(2)}$.

Let $K \subseteq [n]$ $(\neq \emptyset)$ be the widths set under consideration. It holds that:

$$Des_K^B(\pi) := \bigcup_{k \in K} Des_k^B(\pi) \text{ and } des_K^B(\pi) = |Des_K^B(\pi)|,$$

$$Inv_K^A(\pi) := \bigcup_{k \in K} Inv_k^A(\pi) \text{ and } inv_K^A(\pi) = |Inv_K^A(\pi)|,$$

$$Inv_K^B(\pi) := \bigcup_{k \in K} Inv_k^B(\pi) \text{ and } inv_K^B(\pi) = |Inv_K^B(\pi)|.$$

Example 2.3. *For $\pi = (4, -1, -3, -6, 5, -7, 2) \in B_7$, we obtain:*

1. $Des_{\{2,3\}}^B(\pi) = \{0, 1, 2, 3, 4, 5\}$,

2. $Inv^A_{\{2,3\}}(\pi) = \{(1,3),(1,4),(1,7),(2,4),(2,6),(3,6),(4,6),(5,7)\}$,

3. $Inv^B_{\{2,3\}}(\pi) = \{(0,2),(0,4),(0,6),(1,3),(1,7),(2,4),(2,6),(3,7),(5,7),(-2,4),$
$(-2,6),(-3,7),(-5,7),(0,3),(1,4),(3,6),(-1,4),(-3,6),(-4,7)\}$.

In the following two propositions, we generalize the width-k inversion number of type A definition established in [8].

Proposition 2.4. *For any $k \in [n]$ and any $\pi \in B_n$, we have*

$$|Inv^B_k(\pi)| = inv^B_k(\pi) = \sum_{m\geq 1}(des^B_{mk}(\pi) + ndes^B_{mk}(\pi)). \tag{1}$$

Proof. For some $m > 0$, the elements of $Inv^B_k(\pi)$ are pairs of the form $(i, i+mk)$; $0 \leq i \leq n-1$ or $(-i, i+mk)$; $1 \leq i \leq n-1$. So, an element exists in $Inv^B_k(\pi)$ if and only if there is a width-mk descent of π at i or a width-mk negative descent of π at $(-i)$. Thus, $inv^B_k(\pi)$ just counts the number of descents of length mk for every possible m. Hence, the identity (1) holds true.

□

Proposition 2.5. *For any $K \subseteq [n]$ and any $\pi \in B_n$, we have*

$$inv^B_K(\pi) = \sum_{\emptyset \subsetneq K' \subseteq K} (-1)^{|K'|+1}\ inv^B_{lcm(K')}(\pi), \tag{2}$$

where $inv^B_{lcm(K')}(\pi) = 0$ for $lcm(K') \geq n+1$.

Proof. If $K = \{k\}$ then the identity (1) allows to conclude. For a general subset K, applying the same technics developed in [8]-Proposition 2, for the classical permutations, we can conclude.

□

Example 2.6. *Let π as in Example 2.3. Then, we have*

$$\begin{aligned}inv^B_2(\pi) &= \sum_{m\geq 1}(des^B_{2m}(\pi) + ndes^B_{2m})\\ &= (des^B_2(\pi) + ndes^B_2(\pi)) + (des^B_4(\pi) + ndes^B_4(\pi)) + (des^B_6(\pi) + ndes^B_6(\pi))\\ &= (5+2) + (2+2) + (2+0) = 13.\end{aligned}$$

Moreover, we have $inv^B_{\{2,3\}}(\pi) = 19$, where $inv^B_2 = 13$ and $inv^B_3 = 8$. Remark that $(0,6)$ and $(1,7)$ have both the width 2 and the width 3, so it must also have the width-lcm$(2,3)$. Thus, $inv^B_6(\pi) = \{(0,6),(1,7)\}$. Hence, it holds that $inv^B_{\{2,3\}}(\pi) = inv^B_2(\pi) + inv^B_3(\pi) - inv^B_6(\pi)$.

We aim, now, to generalize the search function described in [8], on the set of signed permutations. It helps to show the interaction between width-k statistics by changing its normalization map.

Let n, k be positive integers satisfying $n = dk + r$, for some $(d,r) \in \mathbb{N}^2$ with $0 \leq r < k$. For any π in B_n, we may associate the set of disjoint substrings $\gamma_{n,k}(\pi) = \{\gamma^1_{n,k}(\pi), \gamma^2_{n,k}(\pi), \ldots, \gamma^k_{n,k}(\pi)\}$, where

$$\gamma^i_{n,k}(\pi) = \begin{cases}(\pi(i), \pi(i+k), \pi(i+2k), \ldots, \pi(i+dk)), & \text{if } i \leq r;\\ (\pi(i), \pi(i+k), \pi(i+2k), \ldots, \pi(i+(d-1)k)), & \text{if } r < i \leq k.\end{cases}$$

Let ψ be the map defined as follows

$$\begin{aligned}\psi : B_n &\longrightarrow B^r_{d+1} \times B^{k-r}_d\\ \pi &\longmapsto (std\gamma^1_{n,k}(\pi), std\gamma^2_{n,k}(\pi), \ldots, std\gamma^k_{n,k}(\pi)).\end{aligned} \tag{3}$$

We denote by std the standardization map. For all $1 \leq i \leq k$, the permutation $std\gamma^i_{n,k}(\pi)$ is obtained by replacing the smallest integer in absolute value of $\gamma^i_{n,k}(\pi)$ by 1, the second smallest integer in absolute value by 2, etc... Then, for each element of $\gamma^i_{n,k}(\pi)$, we add " $-$ " at each $\pi(i + jk) < 0$, where $0 \leq j \leq d$. This yields that each $std\gamma^i_{n,k}(\pi)$ is a signed permutation of B_d or B_{d+1}.

Example 2.7. *Let π as in Example 2.3 and assume that $k = 3$. We have*

$$\begin{aligned}\gamma_{7,3}(\pi) &= (std\gamma^1_{7,3}(\pi), std\gamma^2_{7,3}(\pi), std\gamma^3_{7,3}(\pi))\\ &= (std(4, -6, 2), std(-1, 5), std(-3, -7))\\ &= ((2, -3, 1), (-1, 2), (-1, -2)).\end{aligned}$$

Let $\ell^B_k(.)$ be the *width-k*-analogue definition of the length function statistics on signed permutations, given by

$$\ell^B_k(\pi) := \sum_{i=1}^{k} \ell^B(std\gamma^i_{n,k}(\pi)).$$

In the following result, we give the explicit combinatorial description of $\ell^B_k(.)$.

Proposition 2.8. *Let $\pi \in B_n$. Then, we have*

$$\ell^B_k(\pi) = inv^B_k(\pi).$$

Proof. For any $\pi \in B_n$, it is straightforward to see that

$$\begin{aligned}\left|Inv^B_k(\pi)\right| &= \left|\bigcup_{i=1}^{k} Inv^B(std\gamma^i_{n,k}(\pi))\right|\\ &= \sum_{i=1}^{k} \left|Inv^B(std\gamma^i_{n,k}(\pi))\right|\\ &= \sum_{i=1}^{k} \ell^B(std\gamma^i_{n,k}(\pi)) = \ell^B_k(\pi).\end{aligned}$$

□

For $n, k \in \mathbb{N}$, $n = dk + r$ with $0 \leq r < k$, $d > 0$, we denote by $M_{n,k}$ the multinomial coefficient defined by

$$M_{n,k} = \binom{n}{(d+1)^r, d^{k-r}}, \tag{4}$$

where i^m indicates i repeated m times. Let $[n]_x$ be the x-analogue of the integer $n \geq 1$ given by

$$[n]_x := \frac{1 - x^n}{1 - x} = 1 + x + x^2 + \cdots + x^{n-1}.$$

Obviously, we obtain the x-analogue factorial

$$[n]_x! := [1]_x \cdots [n-1]_x[n]_x, \quad [0]! := 1 \quad \text{and} \quad [2n]_x!! := \prod_{i=1}^{n} [2i]_x, \quad [0]!! := 1.$$

We come now to one of the main results in this framework.

Theorem 2.9. *For $n \geq 1$, we have*

$$\sum_{\pi \in B_n} x^{inv^A(\pi)+nsp(\pi)} t^{neg(\pi)} = [n]_x! \prod_{i=0}^{n-1} (1+tx^i). \tag{5}$$

Proof. We use an induction argument to show (5). Notice that the result is obvious for $n = 1$. Assume that the result holds true for the step $n-1$ and we will show it for n $(n \geq 1)$, that is

$$\sum_{\pi \in B_{n-1}} x^{inv^A(\pi)+nsp(\pi)} t^{neg(\pi)} = [n-1]_x! \prod_{i=0}^{n-2} (1+tx^i). \tag{6}$$

For that, it suffices to take into account the number of *inversions*, *nsp* and *negatives* for any integer $\pi(n)$ in $\pi = (\pi(1), \pi(2), \ldots, \pi(n-1), \pi(n)) \in B_n$. So, if $\pi(n) = l$, for $1 \leq l \leq n$ and for all $j \in [-n, n]$, such that $\pi(j) = k$ where $l < k \leq n$, we have exactly one inversion or one nsp. Then $\pi(n)$ makes $(n-l)$ choices of *inversions* and *nsp*. This implies that the identity (6) is multiplied by x^{n-l}, for all $\pi(n) = l$. If $\pi(n) = -l$, $1 \leq l \leq n$, then for all $-l < k < l$ there exists $j \in [-n, n]$, such that $pi(j) = k$. In this case, we have exactly $(2l-2)$ *inversions* and *nsp*. And for all $l < k \leq n$ if $k > l$, we have exactly $(n-l)$ *inversions* or *nsp*. Thus, $\pi(n)$ makes $(n-2+l)$ choices of *inversions* and *nsp*. This gives the identity (6) multiplied by tx^{n-2+l}. Finally, the identity (6) is multiplied by $[n]_x(1+tx^{n-1})$. This completes the proof.

□

As an immediate consequence of Theorem 2.9, for $t = 1$, the identity (5) becomes

$$\frac{1}{2} \sum_{\pi \in B_n} x^{inv^A(\pi)+nsp(\pi)} = [n]_x[2(n-1)]_x!!\,, \tag{7}$$

and we recover the similar identity as in the OEIS [[20], A162206].

3. Width-k Eulerian polynomials of type A

In this section, we will introduce Davis's *width-k Eulerian polynomials* of type A and we will prove its *γ-positivity*. Thus, the concept of *γ-positivity* on classical Eulerian polynomials, one of the most important polynomials in combinatorics, appeared first in the works of Foata and Schützenberger [9] and then of Foata and Strehl ([11],[10]). *γ-positivity* is an elementary property that polynomials with symmetric coefficients can have, which directly implies their unimodality. Working on the statistics of *Eulerian descent* is in a way a generalization of the study of Eulerian numbers which counts the number of permutations with the same descent number. For a permutation $\sigma \in \mathfrak{S}_n$, an index $i \in [n]$ is a double descent of σ if $\sigma(i-1) > \sigma(i) > \sigma(i+1)$, where $\sigma(0) = \sigma(n+1) = \infty$. We also have a *left peak* (resp. *pic*) of $\sigma \in \mathfrak{S}_n$ is any index $i \in [n-1]$ (resp. $2 \leq i \leq n-1$) such that $\sigma(i-1) < \sigma(i) > \sigma(i+1)$, where $\sigma(0) := 0$.

In the following, we define such statistics.

Definition 3.1. *Let $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ be a permutation in $\mathfrak{S}_n$ and $1 \leq k < n$.*

1. *The set of double width-k descents in σ is given by*

$$Ddes_k^A(\sigma) := \{i \in [n]; \sigma(i-k) > \sigma(i) > \sigma(i+k)\}$$

where $\sigma(j) = \infty$, for all $j > n$ or $j \leq 0$. The number of double width-k descents in σ is

$$ddes_k^A(\sigma) := |Ddes_k^A(\sigma)|.$$

2. *The set of width-k peaks (called also interior width-k peaks) in σ is given by*

$$Peak_k(\sigma) := \{k+1 \le i \le n-k; \sigma(i-k) < \sigma(i) > \sigma(i+k)\}.$$

The number of width-k peaks is

$$peak_k(\sigma) := |Peak_k(\sigma)|.$$

3. *The set of width-k left peaks in σ is given by*

$$Lpeak_k(\sigma) := \{k \le i \le n-k; \sigma(i-k) < \sigma(i) > \sigma(i+k)\}$$

where $\sigma(0) = 0$. The number of width-k left peaks in σ is

$$lpeak_k(\sigma) := |Lpeak_k(\sigma)|.$$

The *width-k Eulerian* polynomials of type A, denoted by $\mathfrak{WA}_{n,k}(x)$, are with the following form

$$\mathfrak{WA}_{n,k}(x) := F_n^{des_k^A}(x) = \sum_{\sigma\in\mathfrak{S}_n} x^{des_k^A(\sigma)}. \tag{8}$$

We denote by $\mathrm{WA}_{n,k,p}$ the set $\mathrm{WA}_{n,k,p} := \{\sigma \in \mathfrak{S}_n; des_k^A(\sigma) = p\}$, and $a(n,k,p)$ its cardinal. Taking $k=1$, we recover the classical Eulerian polynomials, and its *nth* γ-positivity, $A_n(x) = \mathfrak{WA}_{n,1}(x)$, given by Foata and Schützenberger [[9], Theorem 5.6], with form

$$\mathfrak{WA}_{n,1}(x) = \sum_{p=0}^{\lfloor\frac{n-1}{2}\rfloor} |\Gamma_{n,p}|\; x^p(1+x)^{n-1-2p}, \tag{9}$$

where $\Gamma_{n,p}$ is the set of permutations $\sigma \in S_n$ with $des_1^A(\sigma) = p$ and $ddes_1^A(\sigma) = 0$. In the same context, an other interpretation was given by Shapiro et al. [19] by introducing the notion of slides, and showing that the γ-expansion coefficient $\gamma_{n,p}$ counts the number of n-permutations with p descents and p slides. Later, Chow generalized in [6] the notion of slides to types B and D.

The following proposition gives an identity that was originally established by Sack and Úlfarsson [18], but with a different notation. Later, Davis [8] gives a different proof.

Proposition 3.2. *For all $n \ge 1$ and $1 \le k \le n-1$, we have*

$$\mathfrak{WA}_{n,k}(x) = F_n^{des_k^A}(x) = M_{n,k}A_{d+1}^r(x)A_d^{k-r}(x), \tag{10}$$

where $M_{n,k}$ was defined in (4).

Let $\alpha(n,k,p)$ be the coefficients of the polynomials $\mathfrak{WA}_{n,k}(x)$ such that $a(n,k,p) = M_{n,k}\alpha(n,k,p)$. For a clearer observation, we give, in the table below, the first coefficients of $\alpha(n,k,p)$, for $1 \le n \le 6$, $1 \le k \le n-1$ and $0 \le p \le n-k$.

The polynomial $\mathfrak{WA}_{n,k}(x)$ (as the product of k unimodal, symmetric and γ-positive polynomials) is unimodal, symmetric with nonnegative coefficients, γ-positive with center of symmetry $\lfloor\frac{n-k}{2}\rfloor$ and $deg(\mathfrak{WA}_{n,k}(x)) = n-k$. For example,

$$\mathfrak{WA}_{6,2}(x) = M_{6,2}(1 + 8x + 18x^2 + 8x^3 + x^4)$$

is γ-positive since

$$\mathfrak{WA}_{6,2}(x) = 20x^0(1+x)^4 + 80x(1+x)^2 + 80x^2(1+x)^0.$$

The coefficient $\alpha(n,k,p)$ is a new integers sequence on the Coxeter group of type A. It is therefore natural to pose the following problem on the recurrence relation of this sequence which has been confirmed by the fact that any $n \ge 4$, k is the smallest positive integer such that $n = 2k + r$ with $0 \le r < k$ and $0 \le p \le n-k$, we have

$$\begin{aligned} \alpha(n,k,p) &= \alpha(n-2,k-1,p-1) + \alpha(n-2,k-1,p), \\ \text{with}\quad \alpha(n,k,0) &= \alpha(n,k,n-k) = 1 \quad\text{and}\quad \alpha(n,k,-1) = 0. \end{aligned}$$

Table 1. The first values of $\alpha(n,k,p)$.

n	k	p					
		0	1	2	3	4	5
1	1	1					
2	1	1	1				
3	1	1	4	1			
	2	1	1				
4	1	1	11	11	1		
	2	1	2	1			
	3	1	1				
5	1	1	26	66	26	1	
	2	1	5	5	1		
	3	1	2	1			
	4	1	1				
6	1	1	57	302	302	57	1
	2	1	8	18	8	1	
	3	1	3	3	1		
	4	1	2	1			
	5	1	1				

Problem 3.3. *Is it possible to find the recurrence relation of $\alpha(n,k,p)$, for all $1 \le k \le n$?*

We have the following result.

Theorem 3.4. *For all $n \ge 1$ and $1 \le k \le n-1$, we have*

$$\mathfrak{WA}_{n,k}(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{des_k^A(\sigma)} = \sum_{p=0}^{\lfloor \frac{n-k}{2} \rfloor} |\Gamma_{n,k,p}| \; x^p (1+x)^{n-k-2p},$$

where $\Gamma_{n,k,p}$ is the set of permutations $\sigma \in S_n$ with $des_k^A(\sigma) = p$ and $ddes_k^A(\sigma) = 0$.

Proof. Using the standardization map φ defined on $\mathfrak{S}_n$ (see [8]) and assuming that, for all $1 \le k \le n-1$, $\varphi(\sigma) = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ such that $\sigma_i = std\gamma^i_{n,k}(\sigma)$ for all i, we have

$$des_k^A(\sigma) = \sum_{i=1}^{k} des^A(\sigma_i).$$

So, it is clear that each *width-k* descent and *width-k* double descent in σ are usual descent and double descent in some unique σ_i and vice versa. Since $\mathfrak{WA}_{n,k}(x)$ is a γ-positive with center of symmetry $\lfloor \frac{n-k}{2} \rfloor$ and $deg(\mathfrak{WA}_{n,k}(x)) = n-k$, the desired claim follows from that.

□

For $1 \le n \le 6,\ 1 \le k \le n-1$ and $0 \le p \le \lfloor \frac{n-k}{2} \rfloor$, we give in the following tabular some values of $\gamma^A_{n,k,p}$.

4. Width-k Eulerian polynomials of type B

In this section, we give a new generalization of type B Eulerian polynomials and its γ-positivities. We introduce $\mathfrak{WB}_{n,k}(x)$ the *width-k Eulerian* polynomials of type B as follows

$$\mathfrak{WB}_{n,k}(x) = F_n^{des_k^B}(x) := \sum_{\pi \in B_n} x^{des_k^B(\pi)}.$$

n	k	p		
		0	1	2
1	1	1		
2	1	1		
3	1	1	2	
	2	3		
4	1	1	8	
	2	6	0	
	3	12		
5	1	1	22	16
	2	10	20	
	3	30	0	
	4	60		
6	1	1	52	136
	2	20	80	80
	3	90	0	
	4	180	0	
	5	360		

Table 2. The first values of $\gamma^A_{n,k,p}$.

Let $\mathrm{WB}_{n,k,p}$ be the set

$$\mathrm{WB}_{n,k,p} := \{\pi \in B_n; des_k^B(\pi) = p\}.$$

Its cardinal will be denoted by $b(n,k,p)$. It is clear that, for $k = 1$, we obtain the classical Eulerian polynomials of type B. The associated nth γ-positive is given by the following result.

Theorem 4.1. *([16],Proposition 4.15) For all $n \geq 1$, we have*

$$\mathfrak{WB}_{n,1}(x) = \sum_{p=0}^{\lfloor n/2 \rfloor} \gamma^B_{n,p}\; x^p (1+x)^{n-2p},$$

where $\gamma^B_{n,p}$ is the number of permutations $\sigma \in \mathfrak{S}_n$ with p left peaks, multiplied by 4^p.

For any $\pi \in B_n$, we define the statistics descent of type A over the set of signed permutations as follows

$$des^A(\pi) := |\{i \in [n-1]; \pi(i) > \pi(i+1)\}|.$$

The following identities hold true.

Theorem 4.2. *For any $n \geq k \geq 1$ and $d \geq 0$ such that $n = dk + r$, $0 \leq r < k$, we have*

$$F_n^{des_k^B}(x) = 2^{n-d} M_{n,k} B_d(x) A_d^{k-r-1}(x) A^r_{d+1}(x), \tag{11}$$

$$F_n^{fdes_k^B}(x) = 2^{n-d} M_{n,k}\; F_d(x)\; A_d^{k-r-1}(x^2)\; A^r_{d+1}(x^2), \tag{12}$$

$$F_n^{inv_k^B}(x) = 2^{k-1} M_{n,k}\; [d]_x^{k-r-1}\; [d+1]^r_x\; [2d]_x^{r+1}!!\; [2(d-1)]_x^{k-r-1}!!. \tag{13}$$

Proof. Let ψ be the map given by (3). We fix $k \in [n]$ and $\psi(\pi) = (\pi_1, \pi_2, \ldots, \pi_k) \in B^r_{d+1} \times B_d^{k-r}$ such that, $std\gamma^i_{n,k}(\pi) = \pi_i$ for all i. There are $M_{n,k}$ choices to partition $[n]$ into subsequences $\gamma^i_{n,k}(\pi)$. For $\pi \in B_n$, we define the map ϵ as follows

$$\epsilon(\pi) = \begin{cases} 1, & \text{if } \pi(1) < 0; \\ 0, & \text{otherwise.} \end{cases}$$

We note that

$$des_k^B(\pi) = des^B(\pi_k) + \sum_{i=1}^{k-1}(des^B(\pi_i) - \epsilon(\pi_i)).$$

It follows from that

$$\sum_{i=1}^{k-1}(des^B(\pi_i) - \epsilon(\pi_i)) = \sum_{i=1}^{k-1} des^A(\pi_i), \quad \pi \in B_n.$$

Hence,

$$\begin{aligned}
F_n^{des_k^B}(x) &= \sum_{\pi \in B_n} x^{des_k^B(\pi)} \\
&= M_{n,k} \sum_{\pi_k \in B_d,\ (\pi_1,\ldots,\pi_{k-1}) \in B_d^{k-r-1} \times B_{d+1}^r} x^{des^B(\pi_k)} x^{des^A(\pi_1)} \cdots x^{des^A(\pi_{k-1})} \\
&= M_{n,k}\ B_d(x)\ (2^d)^{k-r-1} A_d^{k-r-1}(x)\ (2^{d+1})^r\ A_{d+1}^r(x) \\
&= 2^{n-d}\ M_{n,k}\ B_d(x)\ A_d^{k-r-1}(x)\ A_{d+1}^r(x)
\end{aligned}$$

which proves the first identity.

Now, by using the definition of width-k descent, we get

$$\begin{aligned}
fdes_k^B(\pi) &= des_k^A(\pi) + des_k^B(\pi) \\
&= \sum_{i=1}^{k} des^A(\pi_i) + des^B(\pi_k) + \sum_{i=1}^{k-1} des^A(\pi_i) \\
&= \sum_{i=1}^{k-1} 2des^A(\pi_i) + des^A(\pi_k) + des^B(\pi_k) \\
&= fdes^B(\pi_k) + \sum_{i=1}^{k-1} 2des^A(\pi_i).
\end{aligned}$$

On the other hand, we have

$$\sum_{\pi \in B_n} x^{2des^A(\pi)} = 2^n A_n(x^2).$$

Hence,

$$\begin{aligned}
F_n^{fdes_k^B}(x) &= \sum_{\pi \in B_n} x^{fdes_k^B(\pi)} \\
&= M_{n,k} \sum_{\pi_k \in B_d\ ,(\pi_1,\ldots,\pi_{k-1}) \in B_d^{k-r-1} \times B_{d+1}^r} x^{fdes^B(\pi_k)} x^{2des^A(\pi_1)} \cdots x^{2des^A(\pi_{k-1})} \\
&= 2^{n-d}\ M_{n,k}\ F_d(x)\ A_d^{k-r-1}(x^2)\ A_{d+1}^r(x^2).
\end{aligned}$$

which proves the second identity. Now, observe that the width-k inversion number in $\pi \in B_n$ is the sum

$$inv_k^B(\pi) = inv^B(\pi_k) + \sum_{i=1}^{k-1}(inv^A(\pi_i) + nsp(\pi_i)).$$

The generating function of the inversion numbers can be presented as following (see, for instance [13], Section 3.15)

$$\sum_{\pi\in B_n} x^{inv^B(\pi)} = [2n]_x!!\,.$$

Hence, we obtain

$$\begin{aligned}
F_n^{inv_k^B}(x) &= \sum_{\pi\in B_n} x_k^{inv^B}(\pi) \\
&= M_{n,k} \sum_{\pi_k\in B_d,\ (\pi_1,\dots,\pi_{k-1})\in B_d^{k-r-1}\times B_{d+1}^r} x^{inv^B(\pi_k)} x^{(inv^A(\pi_1)+nsp(\pi_1))}\cdots x^{(inv^A(\pi_{k-1})+nsp(\pi_{k-1}))} \\
&= 2^{k-1} M_{n,k}[2d]_x^{r+1}!![2(d-1)]_x^{k-r-1}!![d]_x^{k-r-1}[d+1]_x^r
\end{aligned}$$

which improves the last identity.

□

By using the identity (11), we deduce that

$$\mathfrak{WB}_{n,k}(x) = 2^{n-d} M_{n,k} B_d(x) A_d^{k-r-1}(x) A_{d+1}^r(x).$$

Let $\beta(n,k,p)$ be the coefficient of the polynomial $\mathfrak{WB}_{n,k}(x)$ such that

$$b(n,k,p) = 2^{n-d} M_{n,k}\beta(n,k,p).$$

In the table below, we give some coefficients of $\beta(n,k,p)$, for $1 \le n \le 6, 1 \le k \le n$ and $0 \le p \le n-k+1$.

Table 3. The first values of $\beta(n,k,p)$.

n	k	p = 0	1	2	3	4	5	6
1	1	1	1					
2	1	1	6	1				
	2	1	1					
3	1	1	23	23	1			
	2	1	2	1				
	3	1	1					
4	1	1	76	230	76	1		
	2	1	7	7	1			
	3	1	2	1				
	4	1	1					
5	1	1	237	1682	1682	237	1	
	2	1	10	26	10	1		
	3	1	3	3	1			
	4	1	2	1				
	5	1	1					
6	1	1	722	10543	23548	10543	722	1
	2	1	27	116	116	27	1	
	3	1	8	14	8	1		
	4	1	3	3	1			
	5	1	2	1				
	6	1	1					

Since the coefficient $\beta(n,k,p)$ is a new integer sequence on the Coxeter group of type B, it is natural to pose the following problem on the recurrence relation of this sequence. It has been confirmed that any $n \ge 3$, k is the smallest positive integer such that $n+1 = 2k+r$ with $0 \le r < k$ and $0 \le p \le n-k+1$, we have the following recurrence relation

$$\begin{aligned}
\beta(n,k,p) &= \beta(n-2,k-1,p-1) + \beta(n-2,k-1,p), \\
\text{with}\quad \beta(n,k,0) &= \beta(n,k,n-k+1) = 1, \quad \text{and} \quad \beta(n,k,-1) = 0.
\end{aligned}$$

Problem 4.3. *Is it possible to find the recurrence relation of $\beta(n,k,p)$, for all $1 \leq k \leq n$?*

The polynomial $\mathfrak{WB}_{n,k}(x)$ is unimodal and symmetric with nonnegative coefficients. It is γ-positive (as the product of k γ-positive polynomials) with center of symmetry $\lfloor \frac{n-k+1}{2} \rfloor$ and $deg(\mathfrak{WB}_{n,k}(x)) = n-k+1$. For example, for $n=6$, $k=2$, we have

$$\mathfrak{WB}_{6,2}(x) = 2^3 M_{6,2}(1+27x+116x^2+116x^3+27x^4+x^5)$$

is γ-positive since

$$\mathfrak{WB}_{6,2}(x) = 160x^0(1+x)^5 + 3520x(1+x)^3 + 6400x^2(1+x).$$

So, we have the following theorem.

Theorem 4.4. *For any $1 \leq k \leq n$, the following identity holds true*

$$\mathfrak{WB}_{n,k}(x) = \sum_{\pi \in B_n} x^{des_k^B(\pi)} = \sum_{p=0}^{\lfloor \frac{n-k+1}{2} \rfloor} 2^{2p+k-1} |\Gamma_{n,k,p}^{(\ell)}| \; x^p (1+x)^{n-k+1-2p},$$

where $\Gamma_{n,k,p}^{(\ell)}$ is the set of permutations $\sigma \in S_n$ with p width-k left peaks.

To prove this theorem, we need to generalize some results on P-partitions and enriched P-partitions due to the work of Petersen [16].

In the partition theory, it is well known that there are two P-partition definitions. The first one is due to Stanley [22] which defined them as the order reversing maps. The second one is introduced by Gessel [12] and defined them as order preserving maps. In the current paper, we will adopt the second definition.

Let X be a subset of the positive integers and $\mathfrak{L}(P)$ be the set of all permutations of $[n]$ which extend a poset P with partial order $<_p$ to a total order. When X has a finite cardinality p, the number of P-partitions must also be finite. In this case, let define the order polynomial, denoted by $\Omega(P;p)$, to be the number of P-partitions $f : [n] \to X$. In our study of P-partitions, it is enough to consider P as a permutation and $\Omega_A(P,p)$ is the type A order polynomial. Recall that Chow [5] defined a type B poset P whose elements are 0, ± 1, $\ldots$, $\pm n$ such that if $i <_p j$ then $-j <_p -i$. So, the type B P-partitions differ from ordinary P-partition only in the property $f(-i) = -f(i)$.

For the group of signed permutations, the only difference with the symmetric group of type A is that if $\pi(1) < 0$, then 0 is a descent of π. Let $\Omega_B(\pi,p)$ be the order polynomial for any signed permutation. For fixed n, the polynomials of order turn out to be: $\Omega_A(i,x) = \Omega_B(i,x) = \binom{x+n-i}{n}$, for any permutation of type A or type B with $i-1$ descents. Similarly to corollary 2.4 of [16], we will give the relation between the polynomial of order width-k of a poset P and the sum of the polynomials of order width-k of its linear extensions in the following definition.

Definition 4.5. *We can say that each permutation $\pi \in \mathfrak{S}_n$ as a poset with the total order $\pi(s) < \pi(s+k)$, for all $s \in Des_k^A(\pi)$. Then, the width-k order polynomial of a poset P having $p \in \mathcal{N}$ descents, denoted by $\Omega_A(P,k,p)$, is the sum of the width-k order polynomials of its linear extensions:*

$$\Omega_A(P,k,p) = \sum_{\pi \in \mathfrak{L}(P)} \Omega_A(\pi,k,p),$$

where, for any permutation $\pi \in \mathfrak{S}_n$, the width-k order polynomial is

$$\Omega_A(\pi,k,p) := |\{f : [n-k+1] \to [p] / \tag{14}$$

$$\begin{gathered} 1 \leq f(\pi(1)) \leq f(\pi(1+k)) \leq f(\pi(1+2k)) \leq \cdots f(\pi(1+\tau_d k)) < \\ f(\pi(2)) \leq f(\pi(2+k)) \leq f(\pi(2+2k)) \leq \cdots f(\pi(2+\tau_d k)) < \cdots < \\ f(\pi(k)) \leq f(\pi(k+k)) \leq f(\pi(k+2k)) \leq \cdots \leq f(\pi(k+\tau_d k)) \leq p, \\ \textit{and } f(\pi(s)) < f(\pi(s+k)), \; \textit{if } \; s \in des_k^A(\pi)\}|, \end{gathered}$$

with

$$\tau_d = \begin{cases} d, & \text{if } i \leq r; \\ d-1, & \text{if } r < i \leq k, \end{cases}$$

and $n = dk + r$ *is the Euclidean division of* n *by* k.

In the same way, we define the type B width-k order polynomial of a poset P having p descents of type B, denoted by $\Omega_B(P,k,p)$, as the sum of the type B width-k order polynomials of its linear extensions

$$\Omega_B(P,k,p) = \sum_{\pi \in \mathfrak{L}(P)} \Omega_B(\pi,k,p),$$

where, for any permutation $\pi \in B_n$, the type B width-k order polynomial is

$$\Omega_B(\pi,k,p) := |\{f : [n-k+1] \to [p]/ \tag{15}$$

$$\begin{gathered} 1 \leq f(\pi(1)) \leq f(\pi(1+k)) \leq f(\pi(1+2k)) \leq \cdots f(\pi(1+\tau_d k)) < \\ f(\pi(2)) \leq f(\pi(2+k)) \leq f(\pi(2+2k)) \leq \cdots f(\pi(2+\tau_d k)) < \cdots < \\ f(\pi(k)) \leq f(\pi(k+k)) \leq f(\pi(k+2k)) \leq \cdots \leq f(\pi(k+\tau_d k)) \leq p, \\ \textit{and } f(\pi(s)) < f(\pi(s+k)),\ \textit{if } s \in des_k^B(\pi)\}|. \end{gathered}$$

In the following theorem, we give the analogous generating function type B polynomials order of Proposition 3.10 in [17] in terms of statistics of width-k descents.

Theorem 4.6. *For a given permutation* $\pi \in B_n$*, the generating function for width-*k *type* B *order polynomials is given by*

$$\sum_{p \geq 0} \Omega_B(\pi,k,p)x^p = \frac{x^{des_k^B(\pi)}}{(1-x)^{n-k+2}}. \tag{16}$$

Proof. Let $\mathfrak{L}(P) = \{\pi\}$, where π has width-k descents counting an extra width-k descent at the end (we assume $\pi(k+\tau_d k) > \pi(n+1)$). Then, $\Omega_B(\pi;k;p)$ is the number of solutions of Equation (15):

$$\begin{gathered} 1 \leq f(\pi(1)) \leq f(\pi(1+k)) \leq f(\pi(1+2k)) \leq \cdots f(\pi(1+\tau_d k)) < \\ f(\pi(2)) \leq f(\pi(2+k)) \leq f(\pi(2+2k)) \leq \cdots f(\pi(2+\tau_d k)) < \cdots < \\ f(\pi(k)) \leq f(\pi(k+k)) \leq f(\pi(k+2k)) \leq \cdots \leq f(\pi(k+\tau_d k)) \leq p - (des_k^B(\pi) - 1), \end{gathered}$$

which is equal to the number of ways choosing $n-k+1$ things from $p - des_k^B(\pi) + 1$ with repetitions. Therefore, the number is $\binom{p-des_k^B(\pi)+1+n-k}{n-k+1} = \binom{p-des_k^B(\pi)+n-k+1}{n-k+1}$. So, we obtain

$$\sum_{p \geq 0} \Omega_B(\pi,k,p)x^p = \sum_{p \geq 0} \binom{p - des_k^B(\pi) + n - k + 1}{n-k+1} x^p = \frac{x^{des_k^B(\pi)}}{(1-x)^{n-k+2}} .$$

□

Petersen [16] gave a relation between enriched P-partitions and quasisymmetric functions. Hence, we can see this relation in terms of width-k statistic which the connection helps us to prove Theorem 4.4. The basic theory of enriched P-partitions is due to Stembridge [23]. An enriched P-partitions and a left enriched P-partitions of type A are defined as follow: let $\mathbb{P}'$ denote the set of nonzero integers, totally ordered given by

$$-1 < +1 < -2 < +2 < -3 < +3 < \cdots ,$$

and $\mathbb{P}^{(\ell)}$ to be the integers with the following total order

$$0 < -1 < +1 < -2 < +2 < -3 < +3 < \cdots .$$

For any totally ordered set $X = \{x_1, x_2, \ldots\}$, let $\mathbb{X}'$ and $\mathbb{X}^{(\ell)}$ to be the sets given, respectively, by

$$\{-x_1, x_1, -x_2, \cdots\} \quad and \quad \{x_0, -x_1, x_1, -x_2, \cdots\},$$

with total order

$$x_0 < -x_1 < x_1 < -x_2 < x_2 < \cdots .$$

Definition 4.7 ([16], Definition 4.1)**.** *An enriched P-partition (resp. left enriched P-partition) is an order-preserving map $f : P \to \mathbb{X}'$ (resp. $\mathbb{X}^{(\ell)}$) such that, for all $i <_P j$ in P, we have*

1. *if $i <_P j$ in $\mathbb{Z}$ then $f(i) \leq^+ f(j)$,*
2. *if $i <_P j$ and $i > j$ in $\mathbb{Z}$ then $f(i) \leq^- f(j)$.*

For type B poset, an enriched P-partition of type B differs from type A only in the addition of a third condition, $f(-i) = -f(i)$ (for more details, see [[16], Definition 4.8]).

Let $\varepsilon(P)$ be the set of all enriched P-partitions and $\varepsilon^{(\ell)}(P)$ be the set of left enriched P-partitions. The number of (left) enriched P-partitions is finite if we assume that $|X| = p$. In this case, we define *the enriched order polynomial*, denoted by $\Omega'(P, p)$, to be the number of enriched P-partitions $f : P \to X'$ and *the left enriched order polynomial*, denoted by $\Omega^{(\ell)}(P, p)$, to be the number of left enriched P-partitions $f : P \to X^{(\ell)}$.

Following Gessel [12], a quasisymmetric function is a map for which the coefficient of $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_p}^{\alpha_p}$ is the same for all fixed tuples of integers $(\alpha_1, \alpha_2, \ldots, \alpha_p)$ and for all $i_1 < i_2 < \cdots < i_p$. For any subset $D = \{d_1 < d_2 < \cdots < d_{p-1}\}$ of $[n]$, the quasisymmetric functions are characterized by two common bases: the monomial quasisymmetric functions, denoted M_D, and the fundamental quasisymmetric functions, denoted F_D, given by

$$\begin{aligned} M_D &= \sum_{i_1 < i_2 < \cdots < i_p} x_{i_1}^{d_1} x_{i_2}^{d_2 - d_1} \cdots x_{i_p}^{n - d_{p-1}} = \sum_{i_1 < i_2 < \cdots < i_p} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_p}^{\alpha_p}, \\ F_D &= \sum_{D \subset T \subset [n-1]} M_T = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_p \\ d \in D \Rightarrow i_d < i_{d+1}}} \prod_{d=1}^{n} x_{i_d} = \Gamma_A(\pi), \end{aligned}$$

where $\Gamma_A(\pi)$ is the generating function for the type A P-partitions of a permutation with descent set D. It is possible to recover the order polynomial of π by specializing:

$$\Omega_A(\pi, p) = \Gamma_A(\pi)(1^p). \tag{17}$$

For all $D \subset [n-1]$, the functions M_D and F_D span the quasisymmetric functions of degree n, denoted $Qsym_n$. The ring of quasisymmetric functions is defined by

$$Qsym := \bigoplus_{n \geq 0} Qsym_n.$$

Moreover, the generating function for enriched P-partitions $f : P \to \mathbb{P}'$ is defined by

$$\Delta_A(P) := \sum_{f \in \varepsilon(P)} \prod_{i=1}^{n} x_{|f(i)|}.$$

Evidently, $\Delta_A(P)$ is a quasisymmetric function and it satisfies

$$\Omega'(P, p) = \Delta_A(P)(1^p).$$

Chow [5] gave a connection between ordinary type B P-partitions and type B quasisymmetric functions. Furthermore, Petersen [16] related the type B quasisymmetric functions to left enriched P-partitions and type B enriched P-partitions.

For a fixed n and for any subset $D = \{d_1 < d_2 < \cdots < d_{p-1}\}$ of $\{0, 1, \ldots, n\}$, the monomial N_D and the fundamental quasisymmetric functions of type B, L_D, are defined, respectively, by

$$\begin{aligned} N_D &= \sum_{0<i_2<\cdots<i_p} x_0^{d_1} x_{i_2}^{d_2-d_1} \cdots x_{i_p}^{n-d_{p-1}} = \sum_{0<i_2<\cdots<i_p} x_0^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_p}^{\alpha_p}, \\ L_D &= \sum_{D\subset T\subset[0,n-1]} N_T = \sum_{\substack{0\leq i_2\leq\cdots\leq i_p \\ d\in D\Rightarrow i_d<i_{d+1}}} \prod_{d=1}^{n} x_{i_d} = \Gamma_B(\pi), \end{aligned}$$

where $\Gamma_B(\pi)$ is the generating function for the ordinary type B P-partitions, of any signed permutation with descent set D. In particular, we have

$$\Omega_B(\pi, p) = \Gamma_B(\pi)(1^{p+1}). \tag{18}$$

Similar to type A, these functions form a basis for the type B quasisymmetric functions of degree n, defined by

$$BQsym := \bigoplus_{n\geq 0} BQsym_n.$$

The generating function for left enriched P-partitions $f\colon P \to \mathbb{P}^{(\ell)}$, is defined by

$$\Delta^{(\ell)}(P) := \sum_{f\in\varepsilon^{(\ell)}(P)} \prod_{i=1}^{n} x_{|f(i)|}.$$

It is also clear that $\Delta^{(\ell)}(P)$ is a quasisymmetric function and we have

$$\Omega^{(\ell)}(P, p) = \Delta^{(\ell)}(P)(1^{p+1}).$$

For any two subsets of integers D and T, define the set $D + k = \{d + k; d \in D\}$ with, $1 \leq k \leq n$ and define the symmetric set difference by

$$D\Delta T = (D \cup T)\backslash(D \cap T).$$

Using the standardization map φ on $\mathfrak{S}_n$ defined in Section 2, we can write the width-k descent as in the following form

$$des_k^B(\pi) = des^B(\pi_k) + \sum_{i=1}^{k-1}(des^B(\pi_i) - \epsilon(\pi_i)).$$

So, we count 0 as a descent just in π_k, and all other permutations, we see it as a descent of type A. In addition, we can define $\Omega_B(D, k, p)$ by the ordinary type B order polynomial of any signed permutation with width-k descent set D by

$$\Omega_B(D, k, p) = \Omega_B(D_k, p_k) \prod_{i=1}^{k-1} \Omega_B^{'}(D_i, p_i), \tag{19}$$

with $\Omega_B(D_k, p_k)$ is the ordinary type B order polynomial of signed permutation with descent set D_k and $\Omega_B^{'}(D_i, p_i)$ is the ordinary type B order polynomial of signed permutation with descent set $D_i\backslash\{0\}$, where D_i is the set of descents in each π_i and $p_i = p - k + i$, for any $1 \leq i \leq k$. Thus,

$$\bigcup_{i=1}^{k} D_i = D.$$

To study the left enriched P-partitions, $\Omega^{(\ell)}(P,p)$, it is sufficient to consider the case where P is a permutation. So, it is possible to characterize the set of all left enriched π-partition in terms of descent set. Left peaks in $\mathfrak{S}_n$ are a special case of type B peaks, which are naturally related to type B descents. Thus, in terms of width-k descent sets, we can define the set of all width-k left enriched π-permutation, denoted by $\Omega^{(\ell)}(\pi,k,p)$, in which we consider 0 as a descent only in π_k. Then, for $\pi \in \mathfrak{S}_n$, we can write

$$\Omega^{(\ell)}(\pi,k,p) = \Omega^{(\ell)}(\pi_k,p_k)\prod_{i=1}^{k-1}\Omega^{(\ell)'}(\pi_i,p_i), \tag{20}$$

where $\Omega^{(\ell)'}(\pi_i,p_i)$ is the left enriched order polynomial with $Des(\pi_i)\subset[\tau_d]$. It is important to observe that

$$Lpeak_k(\pi) = Lpeak(\pi_k)\bigcup_{i=1}^{k-1}Peak(\pi_i).$$

Thus,

$$lpeak_k(\pi) = lpeak(\pi_k) + \sum_{i=1}^{k-1}peak(\pi_i).$$

The generating function for enriched π-partitions depends on the set of peaks. Moreover, the generating function for left enriched π-partitions depends on the set of left peaks. We can write $\Delta^{(\ell)}(\pi)$ according to the monomial and fundamental quasisymmetric functions of type B. In this case, by using the map φ introduced in Section 2, we can define the width-k left enriched P-partitions by

$$\Delta^{(\ell)}(\pi,k) = \Delta^{(\ell)}(\pi_k)\prod_{i=1}^{k-1}\Delta_A(\pi_i),$$

with nonnegative coefficients. This coefficient is equal to the product number of enriched and left enriched π-partitions f satisfying

$$(|f(\pi_1(1))|,|f(\pi_1(2))|,\ldots,|f(\pi_1(1+\tau_d k))|) = (1,\ldots 1,\ldots,p_1,\ldots p_1),$$

$$(|f(\pi_2(1))|,|f(\pi_2(2))|,\ldots,|f(\pi_2(1+\tau_d k))|) = (1,\ldots 1,\ldots,p_2,\ldots p_2),$$

$$\vdots$$

$$(|f(\pi_k(1))|,|f(\pi_k(2))|,\ldots,|f(\pi_k(1+dk))|) = (0,\ldots,0,1,\ldots 1,\ldots,p_k,\ldots p_k).$$

Applying Proposition 3.5 in [23], Theorem 6.6 in [16] and referring to Equation (19), we obtain

$$\Delta^{(\ell)}(\pi,k) = 2^{lpeak(\pi_k)}\sum_{\substack{D_k\subset[0,d-1]\\ Lpeak(\pi_k)\subset D_k\Delta(D_k+1)}}L_{D_k}\prod_{i=1}^{k-1}2^{peak(\pi_i)+1}\sum_{\substack{D_i\subset[\tau_d]\\ Peak(\pi_i)\subset D_i\Delta(D_i+1)}}F_{D_i}.$$

We can also write

$$\begin{aligned}\Omega^{(\ell)}(\pi,k,p) &= 2^{lpeak(\pi_k)}\sum_{\substack{D_k\subset[0,d-1]\\ Lpeak(\pi_k)\subset D_k\Delta(D_k+1)}}\Omega_B(D_k,p_k)\prod_{i=1}^{k-1}2^{peak(\pi_i)+1}\sum_{\substack{D_i\subset[\tau_d]\\ Peak(\pi_i)\subset D_i\Delta(D_i+1)}}\Omega'_B(D_i,p_i),\\ &= 2^{lpeak_k(\pi)+k-1}\sum_{\substack{D\subset[0,n-k]\\ Lpeak_k(\pi)\subset D\Delta(D+k)}}\Omega_B(D,k,p).\end{aligned}$$

Theorem 4.8. *The following generating function for width-k left enriched order polynomials holds true*

$$\sum_{p\geq 0}\Omega^{(\ell)}(\pi,k,p)x^p = 2^{k-1}\frac{(1+x)^{n-k+1}}{(1-x)^{n-k+2}}\left(\frac{4x}{(1+x)^2}\right)^{lpeak_k(\pi)}.$$

Proof. For any permutation π in $\mathfrak{S}_n$, we have

$$\sum_{p\geq 0}\Omega^{(\ell)}(\pi,k,p)x^p = \sum_{p\geq 0}2^{lpeak_k(\pi)+k-1}\sum_{\substack{D\subset[0,n-k]\\ Lpeak_k(\pi)\subset D\Delta(D+k)}}\Omega_B(D,k,p)x^p.$$

Applying the generating function for width-k type B order polynomials defined in Theorem 4.6, we obtain

$$\sum_{p\geq 0}\Omega^{(\ell)}(\pi,k,p)x^p = \frac{2^{lpeak_k(\pi)+k-1}}{(1-x)^{n-k+2}}\sum_{\substack{D\subset[0,n-k]\\ Lpeak_k(\pi)\subset D\Delta(D+k)}}x^{|D|}.$$

It is not complicated to specify the generation function for D sets by size. For all $s\in Lpeak_k(\pi)$, we have precisely s or $s-k$ is in D. So, there is still $n-k+1-2lpeak_k(\pi)$ elements in $\{0,1,\dots,n-k\}$ that can be contained in D or not. Thus, we obtain

$$\begin{aligned}\sum_{\substack{D\subset[0,n-k]\\ Lpeak_k(\pi)\subset D\Delta(D+k)}}x^{|D|} &= \underbrace{(x+x)(x+x)\cdots(x+x)}_{lpeak_k(\pi)}\underbrace{(1+x)(1+x)\cdots(1+x)}_{n-k+1-2lpeak_k(\pi)}\\ &= (2x)^{lpeak_k(\pi)}(1+x)^{n-k+1-2lpeak_k(\pi)}.\end{aligned}$$

By combining them all together, we deduce the desired identity

$$\begin{aligned}\sum_{p\geq 0}\Omega^{(\ell)}(\pi,k,p)x^p &= \frac{2^{lpeak_k(\pi)+k-1}}{(1-x)^{n-k+2}}(2x)^{lpeak_k(\pi)}(1+x)^{n-k+1-2lpeak_k(\pi)}\\ &= 2^{k-1}\frac{(1+x)^{n-k+1}}{(1-x)^{n-k+2}}\left(\frac{4x}{(1+x)^2}\right)^{lpeak_k(\pi)}.\end{aligned}$$

□

Recall that the number of permutations of $\mathfrak{S}_n$ with p width-k left peaks is $|\Gamma^{(\ell)}_{n,k,p}|$. So, the width-$k$ left peak polynomial can be defined as follows

$$W^{(\ell)}_{n,k}(x) = \sum_{\pi\in\mathfrak{S}_n}x^{lpeak_k(\pi)} = \sum_{p=0}^{\lfloor\frac{n-k+1}{2}\rfloor}|\Gamma^{(\ell)}_{n,k,p}|x^p. \tag{21}$$

The following identity for the type B Eulerian polynomials is due to Stembridge, Proposition 7.1.b in [21],

$$\sum_{p\geq 0}(2p+1)^n x^p = \frac{\mathfrak{B}_n(x)}{(1-x)^{n+1}}.$$

In the following result, we establish the relation between the width-k left peak polynomials and the width-k Eulerian polynomials of type B.

Proposition 4.9. *The following identity holds true:*

$$W^{(\ell)}_{n,k}\left(\frac{4x}{(1+x)^2}\right) = \frac{\mathfrak{W}\mathfrak{B}_{n,k}(x)}{2^{k-1}(1+x)^{n-k+1}}. \tag{22}$$

Proof. Using Equation (20), the number $\mathcal{N}$ of width-k left enriched P-partitions f is given by

$$\mathcal{N} = (2p+1)^d \left((2p+1)^{(d-1)}\right)^r \left((2p+1)^d\right)^{k-r-1}.$$

Remark that $\pi_k \in \mathfrak{S}_d$, $(\pi_1, \ldots, \pi_{k-1}) \in \mathfrak{S}_{d+1}^r \times \mathfrak{S}_d^{k-r-1}$ and the number of $f : [n-k+1] \to [p]^{(\ell)}$ is $(2p+1)^{n-k+1}$. Consequently, $\Omega^{(\ell)}(\pi, k, p) = (2p+1)^{n-k+1}$. Let P_B be a type B poset whose elements are $\{-n, -n+1, \ldots, -1, 1, \ldots, n\}$. In the present case, P_B is an antichain of $[-n+k-1, n-k+1]$. Thus, the order polynomial $\Omega_B(\pi, k, p)$ is the same as of $\Omega^{(\ell)}(\pi, k, p)$. On the other hand, we have

$$\sum_{p\geq 0}(2p+1)^{n-k+1}x^p = \frac{\mathfrak{WB}_{n,k}(x)}{(1-x)^{n-k+2}}.$$

Using Theorem 4.8, we obtain

$$\begin{aligned} 2^{k-1}\frac{(1+x)^{n-k+1}}{(1-x)^{n-k+2}}W_{n,k}^{(\ell)}\left(\frac{4x}{(1+x)^2}\right) &= \sum_{p\geq 0}(2p+1)^{n-k+1}x^p \\ &= \frac{\mathfrak{WB}_{n,k}(x)}{(1-x)^{n-k+2}}. \end{aligned}$$

The reorganization of the terms gives the desired identity, that is

$$2^{k-1}W_{n,k}^{(\ell)}\left(\frac{4x}{(1+x)^2}\right) = \frac{\mathfrak{WB}_{n,k}(x)}{(1+x)^{n-k+1}}.$$

□

Now, we are able to prove Theorem 4.4.

Proof. (Proof of Theorem 4.4) Substituting x by $\frac{4x}{(1+x)^2}$ in Equation (21) and referring to Equation (4), we obtain

$$\sum_{p=0}^{\lfloor\frac{n-k+1}{2}\rfloor} \gamma_{n,k,p}^B\, x^p(1+x)^{n-k+1-2p} = \sum_{p=0}^{\lfloor\frac{n-k+1}{2}\rfloor} 2^{2p+k-1}|\Gamma_{n,k,p}^{(\ell)}|x^p(1+x)^{n-k+1-2p}$$

which allows to conclude. □

For $1 \leq k \leq n \leq 6$ and $0 \leq p \leq \lfloor\frac{n-k+1}{2}\rfloor$, we record a few values of $\gamma_{n,k,p}^B$ in the following table.

5. Width-k Eulerian polynomials of type D

In this section, we aim to study the γ-positivity of the width-k Eulerian polynomials on the set D_n.

Borowiec and Mlotkowski [3] have introduced a new array of type D Eulerian numbers. They found, in particular, a recurrence relation for this array. Our purpose is to generalize these numbers and to give a new extensions of the Eulerian polynomials of type D, $D_n(x)$, and its γ-positivity with the statistics *width-k descent*. The case $k=1$ corresponds to the Eulerian polynomials $D_n(x)$ in the sense of Borowiec and Mlotkowski [3].

We start by defining the width-k Eulerian polynomials of type D as follows

$$\mathfrak{WD}_{n,k}(x) := F_n^{des_k^D}(x) = \sum_{\pi\in D_n} x^{des_k^D(\pi)}.$$

The following theorem is the mean result of this section.

Table 4. The first few values of $\gamma^B_{n,k,p}$.

n	k	p = 0	1	2	3
1	1	1			
2	1	1	4		
	2	4			
3	1	1	20		
	2	12	0		
	3	24			
4	1	1	72	0	
	2	24	96		
	3	96	0		
	4	192			
5	1	1	232	976	
	2	80	480	640	
	3	480	0		
	4	960	0		
	5	1920			
6	1	1	716	7664	3904
	2	160	3520	6400	
	3	1440	5760	0	
	4	5760	0		
	5	11520	0		
	6	23040			

Theorem 5.1. *Let $n \geq 1$. Then, the following statements hold true:*

1. *If $k = 1$, then we have*

$$\mathfrak{WD}_{n,1}(x) = D_n(x),$$

2. *If $2 \leq k \leq n$, then we have*

$$\mathfrak{WD}_{n,k}(x) = 2^{n-d-1} M_{n,k} B_d(x) A_d^{k-r-1}(x) A_{d+1}^{r}(x). \tag{23}$$

Proof. If $2 \leq k \leq n$, we use the standardization map ψ defined in Section 2 and restrained to the set D_n. So, the desired identity follows completely in the same way as the identity (11) of Theorem 4.2, by applying the same reasoning and taking into account that $|D_n| = \frac{|B_n|}{2}$. □

Denote $\bar{D_n} = B_n \backslash D_n$ and,

$$\mathrm{WD}_{n,k,p} = \{\pi \in D_n; des_k^D(\pi) = p\},$$

$$\mathrm{W\bar{D}}_{n,k,p} = \{\pi \in \bar{D}_n; des_k^{\bar{D}}(\pi) = p\}.$$

So, $\mathrm{WD}_{n,k,p} = \mathrm{WB}_{n,k,p} \bigcap D_n$ and $\mathrm{W\bar{D}}_{n,k,p} = \mathrm{WB}_{n,k,p} \backslash D_n$. The cardinals of these sets will be denoted by $d(n,k,p)$ and $\bar{d}(n,k,p)$, respectively. Since $\mathrm{WB}_{n,k,p} = \mathrm{WD}_{n,k,p} \bigcup \mathrm{W\bar{D}}_{n,k,p}$, we have

$$b(n,k,p) = d(n,k,p) + \bar{d}(n,k,p). \tag{24}$$

Now, we give the following result in which we generalize Proposition 4.1 in [3] (for $k = 1$).

Proposition 5.2. *Let $0 \leq p \leq n$. Then, the following statements are satisfied:*

1. *If n is even with $1 \leq k \leq n$ or n is odd with $2 \leq k \leq n$, then*

$$\delta(n,k,p) = \delta(n,k,n-k+1-p) \text{ and } \bar{\delta}(n,k,p) = \bar{\delta}(n,k,n-k+1-p). \tag{25}$$

2. If n is odd with $k = 1$, then

$$\delta(n, 1, p) = \bar{\delta}(n, 1, n-p) \text{ and } \bar{\delta}(n, 1, p) = \delta(n, 1, n-p).$$

Proof. Let $\xi : B_n \to B_n$ be a map satisfying $\xi(\pi) = (-\pi(1), \cdots, -\pi(n))$, for any $\pi = (\pi(1), \cdots, \pi(n)) \in B_n$. It is easy to see that ξ is bijective from $\mathrm{WD}_{n,k,p}$ onto $WD_{n,k,n-k+1-p}$ and from $\mathrm{W\bar{D}}_{n,k,p}$ onto $\mathrm{W\bar{D}}_{n,k,n-k+1-p}$. So, Equations (25) hold true if n is even with $1 \le k \le n$. Moreover, ξ is bijective from $\mathrm{WD}_{n,k,p}$ onto $\mathrm{W\bar{D}}_{n,k,n-k+1-p}$ if n is odd with $2 \le k \le n$. By using Equation (24), we obtained the desired claim.

□

For $k = 1$, Borowiec and Mlotkowski [3] showed the following recurrence relations.

Proposition 5.3. *Let $0 \le p \le n$. Then, the following identities hold true:*

$$\begin{aligned} d(n,1,p) &= (2p+1)d(n-1,1,p) + (2n-2p+1)d(n-1,1,p-1) + (-1)^p\binom{n-1}{p-1}, \\ \bar{d}(n,1,p) &= (2p+1)\bar{d}(n-1,1,p) + (2n-2p+1)\bar{d}(n-1,1,p-1) - (-1)^p\binom{n-1}{p-1}. \end{aligned}$$

Referring to Equation (23), we give in the table 5 some coefficients of the product polynomials $B_d(x)A_d^{k-r-1}(x)A_{d+1}^r(x)$, denoted by $\delta(n, k, p)$, for $1 \le n \le 6$, $1 < k \le n$ and $0 \le p \le n-k+1$. Moreover, in the table 6, we give the first values of $\bar{\delta}(n, k, p)$.

Table 5. The first values of $\delta(n, k, p)$.

n	k	p=0	1	2	3	4	5	6
1	1	1	1					
2	1	1	2	1				
	2	1	1					
3	1	1	10	13	0			
	2	1	2	1				
	3	1	1					
4	1	1	36	118	36	1		
	2	1	7	7	1			
	3	1	2	1				
	4	1	1					
5	1	1	116	846	836	121	0	
	2	1	10	26	10	1		
	3	1	3	3	1			
	4	1	2	1				
	5	1	1					
6	1	1	358	5279	11764	5279	358	1
	2	1	27	116	116	27	1	
	3	1	8	14	8	1		
	4	1	3	3	1			
	5	1	2	1				
	6	1	1					

Remark that, for $0 \le p \le n-k+1$, with k being the smallest positive integer such that $n+1 = 2k+r$ $(0 \le r < k)$, we obtain the following recurrence relations

$$\begin{aligned} \delta(n,k,p) &= \delta(n-2,k-1,p-1) + \delta(n-2,k-1,p), \quad n \ge 3, \\ \bar{\delta}(n,k,p) &= \bar{\delta}(n-2,k-1,p-1) + \bar{\delta}(n-2,k-1,p), \quad n \ge 4, \end{aligned}$$

Table 6. The first values of $\bar{\delta}(n, k, p)$.

n	k	p						
		0	1	2	3	4	5	6
1	1	0	1					
2	1	0	4	0				
	2	1	1					
3	1	0	13	10	1			
	2	1	2	1				
	3	1	1					
4	1	0	40	112	40	0		
	2	1	7	7	1			
	3	1	2	1				
	4	1	1					
5	1	0	121	836	846	116	1	
	2	1	10	26	10	1		
	3	1	3	3	1			
	4	1	2	1				
	5	1	1					
6	1	0	364	5264	11784	5264	364	0
	2	1	27	116	116	27	1	
	3	1	8	14	8	1		
	4	1	3	3	1			
	5	1	2	1				
	6	1	1					

where,

$$\begin{aligned}
\delta(n,k,0) &= \bar{\delta}(n,k,0) = \delta(n,k,n-k+1) = \bar{\delta}(n,k,n-k+1) = 1, \text{ for } 2 \le k \le n,\\
\delta(n,k,-1) &= \bar{\delta}(n,k,-1) = 0,\\
\delta(n,1,0) &= 1, \quad \bar{\delta}(n,1,0) = 0
\end{aligned}$$

and

$$\delta(n,1,n) = \begin{cases} 1, & \text{if n is even;} \\ 0, & \text{if n is odd,} \end{cases}, \qquad \bar{\delta}(n,1,n) = \begin{cases} 0, & \text{if n is even;} \\ 1, & \text{if n is odd.} \end{cases}$$

Problem 5.4. *Is it possible to find the recurrence relations of $\delta(n, k, p)$ and $\bar{\delta}(n, k, p)$ for all $1 \le k \le n$?*

Remark 5.5. *Let n be a positive integer. Then, the following statements hold:*

1. *If n is even, then we obtain*

$$deg(\mathfrak{WD}_{n,k}(x)) = n - k + 1 \quad for\ all\ 1 \le k \le n.$$

2. *If n is odd, then we obtain*

$$deg(\mathfrak{WD}_{n,k}(x)) = \begin{cases} n-1, & if\ k = 1; \\ n-k+1, & if\ 2 \le k \le n. \end{cases}$$

Moreover, according to Table 5, we can deduce that if n is odd, with $k = 1$, then the polynomial $\mathfrak{WD}_{n,k}(x)$ is not unimodal and not symmetric. For example, we have

$$\mathfrak{WD}_{5,1}(x) = 1 + 116x + 846x^2 + 836x^3 + 121x^4.$$

Theorem 5.6. *Let n, k be positive integers with $2 \leq k \leq n$, and n is not odd. Then, the following identity holds true*

$$\mathfrak{WD}_{n,k}(x) = \sum_{\pi \in D_n} x^{des_k^D(\pi)} = \sum_{p=0}^{\lfloor \frac{n-k+1}{2} \rfloor} \frac{\gamma_{n,k,p}^B}{2} x^p (1+x)^{n-k+1-2p},$$

.

Proof. If n is odd and $k = 1$, there are no permutations π in D_n such that $des_1^D(\pi) = n$. Thus, $\mathfrak{WD}_{n,1}(x)$ is not symmetric, and therefore $\mathfrak{WD}_{n,1}(x)$ is not γ-positive.

If n is even, then for all $1 \leq k \leq n$, the number of permutations whose $des_k^D(\pi) = n - k + 1 - p$ is equal to the number of permutations of which $des_k^D(\pi) = p$ with $0 \leq p \leq \lfloor \frac{n-k+1}{2} \rfloor$. Thus, $\mathfrak{WD}_{n,k}(x)$ is symmetric with center of symmetry $\lfloor \frac{n-k+1}{2} \rfloor$. Moreover, it is easy to see that $\mathfrak{WD}_{n,k}(x)$ is unimodal.

Therefore, since for all $2 \leq k \leq n$ the width-k Eulerian polynomials of type D is γ-positive, and the cardinal of this group is equal to $\frac{|B_{n,k}|}{2}$, thus $\gamma_{n,k,p}^D = \frac{\gamma_{n,k,p}^B}{2}$. □

For $2 \leq n \leq 6$, $1 \leq k \leq n$ and $0 \leq p \leq \lfloor \frac{n-k+1}{2} \rfloor$, we give in Table 7 some values of $\gamma_{n,k,p}^D$.

Table 7. The first values of $\gamma_{n,k,p}^D$.

n	k	p = 0	p = 1	p = 2	p = 3
2	1	1	0		
	2	2			
3	1				
	2	6	0		
	3	12			
4	1	1	32	48	
	2	12	48		
	3	48	0		
	4	96			
5	1				
	2	40	240	320	
	3	240	0		
	4	480	0		
	5	960			
6	1	1	352	3856	1920
	2	80	1760	3200	
	3	720	2880	0	
	4	2880	0		
	5	5760	0		
	6	11520			

Problem 5.7. *Is it possible to find the recurrence relation of $\gamma_{n,1,p}^D$, if n is even ($k = 1$)?*

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