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New good large (n,r)-arcs in PG(2,29) and PG(2,31)

Research Article

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Abstract: An (n, r)-arc is a set of n points of a projective plane such that some r, but no $r + 1$ of them, are collinear. The maximum size of an (n, r)-arc in $\mathrm{PG}(2, q)$ is denoted by $m_r(2, q)$. In this article a $(477, 18)$-arc, a $(596, 22)$-arc, a $(697, 25)$-arc in PG(2,29) and a $(598, 21)$-arc, a $(664, 23)$-arc, a $(699, 24)$-arc, a $(769, 26)$-arc, a $(838, 28)$-arc in PG(2,31) are presented. The constructed arcs improve the respective lower bounds on $m_r(2, 29)$ and $m_r(2, 31)$ in [6]. As a consequence, there exist eight new three-dimensional linear codes over the respective finite fields.

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1. Introduction

1.1. Optimal linear codes

Let $\mathrm{V}(n, q)$ be the vector space of all ordered n-tuples over $\mathrm{GF}(q)$. Every k-dimensional subspace of $\mathrm{V}(n, q)$ is an $[n, k]_q$ linear code. The elements of the code are called *codewords*. The number of nonzero positions in a codeword x is called *Hamming weight* of x. The *distance* between two codewords is the number of positions in which they differ. The smallest of the distances between any two different codewords of the code is called *minimum distance*. A code is called an $[n, k, d]_q$ code if its minimum distance is d. The minimum distance of a linear code is equal to the smallest of the weights of the nonzero codewords.

We notice that each linear code over $\mathrm{GF}(q)$ has three basic parameters - length, dimension and minimum distance. If we fix two of them and the finite field and optimize the third parameter, we will get the following three optimization problems:

1. For a fixed dimension and a minimum distance, find the smallest length for which such a code exists. This is the function $n_q(k, d)$.

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2. For a fixed length and dimension, find the largest minimum distance for which such a code exists. This is the function $d_q(n,k)$.

3. For a fixed length and a minimum distance, find the largest dimension for which such a code exists. This is the function $k_q(n,d)$

The codes that satisfy the conditions of these three problems are called *optimal*. For codes and optimal codes we refer to [24], [23].

The study of the third function $k_q(n,d)$ is very complex and therefore the functions that are considered by researchers are the first two.

The problem of finding the parameters of the optimal codes was studied in details by Dodunekov [21] and Hill [23]. Over the years it has been shown that it is very complex and difficult. For its complete solution it is necessary to solve the following two tasks:

First task: At a given length and dimension, to construct new linear codes with increasing minimum distance.

This task is usually solved using a computer or by applying many different and well-known constructions over appropriate codes.

Second task: Proving the non-existence of codes with certain parameters.

The second task requires proving the non-existence of codes for which the length and dimension are fixed and the minimum distance decreases. In its final stage, it is very complex.

It follows from the results of Dodunekov and Hill that all linear codes over a fixed finite field in the first two dimensions $k = 1$ and $k = 2$ are optimal. The expectations that the problem will be easily solved in the next dimension $k = 3$ turn out to be incorrect. After more than 70 years of research, it has been solved only for $q \leq 9$ and the problem remains open for $q \geq 11$ (see [26]).

The next fact is well known: there exists a projective $[n, 3, d]_q$ code over GF(q) if and only if there exists an $(n, n-d)$-arc in PG(2,q) ([23], Theorem 5.2). By this reason a lot of research has been done over GF(q), $q \leq 31$ by means of projective geometry.

1.2. Arcs and blocking sets in PG(2,q)

Let GF(q) denote the Galois field of q elements and V$(3,q)$ be the vector space of row vectors of length three with entries in GF(q). Let PG$(2,q)$ be the corresponding projective plane. The *points* (x_1, x_2, x_3) of PG$(2,q)$ are the 1-dimensional subspaces of V$(3,q)$. Subspaces of dimension two are called *lines*. The number of points and the number of lines in PG$(2,q)$ is q^2+q+1. There are $q+1$ points on every line and $q+1$ lines through every point.

Definition 1.1. *An (n,r)-arc is a set of n points of a projective plane such that some r, but no $r+1$ of them, are collinear.*

Definition 1.2. *An (l,t)-blocking set S in PG$(2,q)$ is a set of l points such that every line of PG$(2,q)$ intersects S in at least t points, and there is a line intersecting S in exactly t points.*

An (n,r)-arc is the complement of a $(q^2+q+1-n, q+1-r)$-blocking set in a projective plane and conversely.

Definition 1.3. *Let M be a set of points in any plane. An i-secant is a line meeting M in exactly i points. Define τ_i as the number of i-secants to a set M.*

In terms of τ_i the definitions of an (n,r)-arc and an (l,t)-blocking set become the following: An (n,r)-arc is a set of n points of a projective plane for which $\tau_i \geq 0$ for $i < r$, $\tau_r > 0$ and $\tau_i = 0$ when $i > r$. An (l,t)-blocking set is a set of l points of a projective plane for which $\tau_i = 0$ for $i < t$, $\tau_t > 0$ and $\tau_i \geq 0$ when $i > t$.

Table 1. Lower bounds on $m_r(2,29)$ in [6], [7]

r	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$m_r(2,29)$	30	44	70	94	126	148	181	208	234	262	300	325	364	407
r	16	17	18	19	20	21	22	23	24	25	26	27	28	
$m_r(2,29)$	436	452	476	507	534	565	595	628	662	697	725	755	784	

Table 2. Bounds on $m_r(2,31)$ in [6], [7]

r	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$m_r(2,31)$	32	46	75	100	132	158	193	217	252	282	312	348	378	423
r	16	17	18	19	20	21	22	23	24	25	26	27	28	29
$m_r(2,31)$	466	497	514	539	567	597	631	663	698	734	768	805	837	869

A survey with the best known results for (n,r)-arcs was presented for the first time in [25]. In the following years many improvements were obtained in [10], [13] and [8]. Summarizing these improvements, Ball and Hirschfeld [3] presented a new table with bounds on $m_r(2,q)$ for $q \leq 19$. As we can see from these tables the exact values of $m_r(2,q)$ are known only for $q \leq 9$. The new improvements in recent years are published periodically in the online table for $m_r(2,q)$, $q \leq 19$, maintained by S. Ball [1]. New good results and tables with the best known lower and upper bounds on $m_r(2,q)$ for $q = 23, 25, 27, 29, 31$ are presented in [14], [15], [20], [6] and [7].

The lower bounds on $m_r(2,29)$ and $m_r(2,31)$ in Tables 1 and 2 are given in [6], [7]. In these tables mainly results of A. Kohnert, Daskalov and Metodieva and M. Braun are included. The $(30,2)$-arc, $(407,15)$-arc, $(436,16)$-arc in PG(2,29) and $(32,2)$-arc, $(466,16)$-arc, $(497,17)$-arc in PG(2,31) are optimal (see [5], [4], [2]).

We say that an (n,r)-arc is large, if $r > (q+3)/2$, and an (l,t)-blocking set is small, if $t < (q-1)/2$. All constructed, in this article, arcs are large. Constructing record-breaking large (n,r)-arcs is a hard problem and computationally it makes more sense to construct their corresponding small (l,t)-blocking sets. We can divide (l,t)-blocking sets into two types - those that contain at least one line, and those that do not contain any lines. The following two theorems hold for (l,t)-blocking sets in $\mathrm{PG}(2,q)$, q-prime:

Theorem 1.4. [2] *If an (l,t)-blocking set in $\mathrm{PG}(2,q)$, q-prime, contains a $(q+1)$-secant, then $l \geq (t+1)q$*

Theorem 1.5. [11] *Let B be an (l,t)-blocking set in $\mathrm{PG}(2,q)$, $q \leq 31$, prime. If $2 < t < (q-1)/2$, then $l \geq (t+1)q + t - (q-3)/2$.*

Theorem 2 shows that for $2 < t < (q-1)/2$ the cardinality of blocking sets satisfying Theorem 1 can be improved by $t - (q-3)/2$, but in practice it has proved difficult to improve their cardinality even by one.

In Tables 3 and 4 (which hold from Tables 1 and 2) the parameters of the best known small (l,t)-blocking sets in finite projective planes of order 29 and 31 are presented. From these tables we can see that in PG(2, 29) three of the best known small (l,t)-blocking sets are $((t+1)q,t)$-blocking sets and these sets cannot be improved by using lines in $\mathrm{PG}(2,29)$. In PG(2,31) only one of the best known small (l,t)-blocking sets is a $((t+1)q,t)$-blocking set and this set also cannot be improved by using lines. All of

Table 3. The best known small (l,t)-blocking sets in $\mathrm{PG}(2,29)$

t	2	3	4	5	6	7	8	9	10	11	12
l	$3q$	$4q$	$5q+1$	$6q$	$7q+6$	$8q+11$	$9q+15$	$10q+16$	$11q+18$	$12q+16$	$13q+18$

Table 4. The best known small (l,t)-blocking sets in $\mathrm{PG}(2,31)$

t	2	3	4	5	6	7	8	9	10	11
l	$3q-1$	$4q$	$5q+1$	$6q+2$	$7q+8$	$8q+11$	$9q+16$	$10q+20$	$11q+22$	$12q+24$

the remaining ones have worse parameters and only one example with better parameters is known. This is a $(3q-1,2)$-blocking set in PG(2,31), given in [9].

In order to present the results in more concise form, the points in PG(2,29) and PG(2,31) are in lexicographic order and each point is associated with its number. The first point in PG(2,29) is $(0,0,1)$, the point $(0,4,8)$ has number 155, the point $(1,12,25)$ has number 404, the point $(1,28,9)$ has number 852. The first point in PG(2,31) is $(0,0,1)$, the point $(0,1,1)$ has number 3, the point $(1,8,15)$ has number 296, the point $(1,27,29)$ has number 899.

2. About our approach

To obtain good (l,t)-blocking sets we apply local search techniques. The neighborhood structure is simple one. Given an blocking set, then its neighborhood consists of all blocking sets that can be obtained from the given blocking set by adding new points or deleting some points. The choice of a starting solution is based on some heuristic observations. The cost function is chosen to favor as local optima blocking sets with a small number of t-secants. The computation times are in order of several minutes up to a few hours on a PC. Similar techniques are employed for construction of (n,r)-arcs. In the last 17 years many new record-breaking (n,r)-arcs and (l,t)-blocking sets in $\mathrm{PG}(2,q)$, $(13 \le q \le 31)$ have been constructed, applying this non-exhaustive local computer search (see [10-11, 13-20]).

In this paper we present a new version of our method for blocking sets that contain some lines. In the new version of our approach the choice of a starting solution is not based on heuristic observations.

Our approach to construct good blocking sets of the first type is based on the following strategy:

1. We generate a large number (hundreds of thousands) of combinations of $(t+1)$ lines in general position and for each combination we compute the secant distribution of the resulting blocking set.

2. We divide the generated blocking sets into as many groups as distinct secant distributions.

3. We try to extend the blocking sets in each group to new record-breaking ones by adding and removing points of $\mathrm{PG}(2,q)$.

4. The process in 3 gets optimized by choosing at each step the blocking set that has the smallest number of shortest secants.

In the next example we will show how starting from blocking sets consisting of some lines in general position, we improve the parameters of the best-known (276,8)-blocking set in PG(2,29), constructed in [6].

In PG(2,29) nine lines in general position have $9.30 - 36 = 234$ points. We are looking for new (275, 8)-blocking set. The distance between 275 and 234 is large, and by this reason, we will start the search with 10 lines in general position.

1. We generate a large number of combinations of 10 lines. Any combination of 10 lines in general position has 255 points.

2. For each combination we generate the respective $(255, t)$-blocking set.

3. Thus, we obtain more than 15 different groups of (255,6)-blocking sets. Ten of these sets have the next secant distributions:

$$\begin{array}{l}
\tau_6 = 1, \tau_7 = 101, \tau_8 = 321, \tau_9 = 311, \tau_{10} = 127, \tau_{32} = 10,\\
\tau_6 = 1, \tau_7 = 102, \tau_8 = 318, \tau_9 = 314, \tau_{10} = 126, \tau_{32} = 10,\\
\tau_6 = 1, \tau_7 = 103, \tau_8 = 315, \tau_9 = 317, \tau_{10} = 125, \tau_{32} = 10,\\
\tau_6 = 1, \tau_7 = 104, \tau_8 = 312, \tau_9 = 320, \tau_{10} = 124, \tau_{32} = 10,\\
\tau_6 = 1, \tau_7 = 106, \tau_8 = 306, \tau_9 = 326, \tau_{10} = 122, \tau_{32} = 10,\\
\tau_6 = 1, \tau_7 = 107, \tau_8 = 303, \tau_9 = 329, \tau_{10} = 121, \tau_{32} = 10,\\
\tau_6 = 1, \tau_7 = 108, \tau_8 = 300, \tau_9 = 332, \tau_{10} = 120, \tau_{32} = 10,\\
\tau_6 = 2, \tau_7 = \ \ 96, \tau_8 = 330, \tau_9 = 304, \tau_{10} = 129, \tau_{32} = 10,\\
\tau_6 = 3, \tau_7 = \ \ 94, \tau_8 = 330, \tau_9 = 306, \tau_{10} = 128, \tau_{32} = 10,\\
\tau_6 = 3, \tau_7 = \ \ 96, \tau_8 = 324, \tau_9 = 312, \tau_{10} = 126, \tau_{32} = 10,
\end{array}$$

4. We choose a representative blocking set with $\tau_6 = 1$, $\tau_7 = 101$ and begin a process of extending it by adding points.

5. By adding 20 points to it, we obtain a (275,7)-blocking set B_1 with $\tau_7 = 4$.

6. From B_1 we remove 7 points to get a (268, 7)-blocking set B_2 with $\tau_7 = 21$.

7. We add 7 new points to B_2 to produce a (275, 7)-blocking set B_3 with $\tau_7 = 2$.

8. We then remove 5 points from B_3 to obtain a (270,7)-blocking set B_4 with $\tau_7 = 13$.

9. Finally, we add the last 5 new points to B_4 and produce a new (275, 8)-blocking set with $\tau_8 = 296$.

In this paper we improve the lower bounds on $m_r(2, q)$ in PG(2,29) for $r = 18, 22, 25$ and in PG(2,31) for $r = 21, 23, 24, 26$ and 28.

3. The new arcs in PG(2, 29)

Theorem 3.1. *There exist a* $(477, 18)$*-arc, a* $(596, 22)$*-arc and at least three* $(697, 25)$*-arcs in* PG(2,29).

Proof. 1. The set of points having numbers 3 7 12 14 15 19 20 21 22 27 28 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 62 68 69 71 73 74 75 76 78 80 81 87 91 96 98 100 101 104 106 107 109 111 112 116 119 120 123 125 126 128 133 136 137 139 140 142 145 147 148 150 151 154 158 161 163 165 169 172 173 175 176 178 181 183 185 188 193 196 198 200 203 204 205 206 209 211 218 219 220 221 222 228 230 233 234 237 238 242 243 244 246 249 253 254 255 259 264 267 268 270 274 275 277 278 280 281 285 287 288 291 292 294 297 298 302 305 308 311 315 316 317 319 326 328 329 330 333 334 337 338 340 341 342 343 345 353 355 359 360 361 363 364 365 366 368 369 370 374 376 380 381 383 388 389 392 393 394 395 398 399 404 406 407 408 411 413 414 415 417 422 423 428 430 432 434 440 441 444 445 449 450 451 453 454 458 459 462 463 467 469 472 473 475 476 477 484 485 486 492 494 495 497 498 501 503 509 510 516 517 518 521 522 528 529 530 532 535 536 541 542 543 545 547 548 549 555 557 560 561 564 566 569 574 575 576 578 580 583 584 585 586 590 591 593 600 602 603 607 609 610 611 613 615 621 622 625 626 629 630 635 636 638 641 642 643 646 650 652 656 657 663 666 667 668 669 670 673 676 677 678 679 688 689 691 694 697 699 704 708 709 710 712 713 715 717 721 723 726 727 728 729 732 735 739 744 748 751 753 754 755 756 759 763 766 767 768 769 773 774 775 778 782 787

788 789 790 791 794 797 802 805 808 809 810 811 812 815 817 819 820 822 827 828 829 830 835 836 837 838 840 842 843 848 849 851 852 856 859 861 863 864 866 867 forms a (394,12)-blocking set in PG(2,29) with secant distribution $\tau_{12} = 275$, $\tau_{13} = 282$, $\tau_{14} = 205$, $\tau_{15} = 71$, $\tau_{16} = 10$, $\tau_{18} = 1$, $\tau_{26} = 6$, $\tau_{27} = 7$, $\tau_{28} = 11$, $\tau_{29} = 2$ and $\tau_{30} = 1$.

The complement of this blocking set is a new $(477, 18)$-arc in PG(2, 29).

2. The set of points having numbers 2 10 11 15 16 18 20 23 29 33 39 40 42 43 44 45 48 58 66 71 72 74 75 77 80 82 87 89 90 96 98 101 105 106 116 117 123 124 125 127 130 135 137 139 140 151 152 153 158 159 164 165 170 172 179 181 185 188 193 196 201 203 206 209 210 212 217 222 227 229 240 244 246 250 251 253 257 262 267 271 273 275 279 280 282 283 288 292 295 297 304 309 310 316 318 320 324 328 333 334 335 336 337 338 342 343 350 354 362 363 365 367 370 373 376 381 382 384 386 391 392 395 396 397 401 402 410 411 419 420 424 425 434 437 438 447 449 451 454 455 456 458 472 476 478 481 483 484 489 494 495 497 502 505 507 512 517 521 529 532 533 536 538 541 543 544 547 549 557 559 565 566 567 568 570 578 579 581 583 585 586 589 590 594 595 597 599 603 611 612 613 623 628 632 638 639 640 648 650 652 654 655 657 660 661 666 671 675 681 685 686 695 696 697 702 709 710 713 715 716 722 723 728 730 732 733 739 744 746 750 751 752 760 764 766 767 768 773 777 780 793 794 797 798 802 804 806 809 815 816 819 823 826 831 835 836 838 841 845 849 854 855 858 860 861 863 865 866 413 213 25 306 832 70 52 forms a (275,8)-blocking set in PG(2,29) with secant distribution $\tau_8 = 296$, $\tau_9 = 254$, $\tau_{10} = 182$, $\tau_{11} = 83$, $\tau_{12} = 36$, $\tau_{13} = 8$, $\tau_{14} = 1$, $\tau_{15} = 1$, $\tau_{29} = 2$ and $\tau_{30} = 8$.

The complement of this blocking set is a new $(596, 22)$-arc in PG(2, 29).

3. The set of points having numbers 1 2 3 14 19 30 31 32 43 48 59 60 61 72 77 94 101 106 113 125 130 135 140 146 157 159 164 166 178 188 193 203 217 219 220 222 237 246 251 260 272 275 280 283 300 304 309 313 325 333 338 346 362 363 366 367 379 380 381 382 383 384 385 386 387 388 389 390 391 392 393 394 395 396 397 398 399 400 401 402 403 404 405 406 407 419 420 425 426 443 449 454 460 472 478 483 489 506 507 512 513 524 525 526 527 528 529 530 531 532 533 534 535 536 537 538 539 540 541 542 543 544 545 546 547 548 549 550 551 552 565 566 569 570 586 594 599 607 619 623 628 632 649 652 657 660 672 681 686 695 710 712 713 715 729 739 744 754 766 768 773 775 792 797 802 807 819 826 831 838 843 844 855 860 forms a (174,5)-blocking set in PG(2,29) with secant distribution $\tau_5 = 341$, $\tau_6 = 352$, $\tau_7 = 159$, $\tau_8 = 7$, $\tau_9 = 6$, and $\tau_{30} = 6$.

The complement of this blocking set is a new $(697, 25)$-arc in PG(2, 29).

The secant distributions of the other two (174,5)-blocking sets in PG(2,29) are:

$$\tau_5 = 341, \tau_6 = 354, \tau_7 = 153, \tau_8 = 13, \tau_9 = 4, \tau_{30} = 6,$$

$$\tau_5 = 345, \tau_6 = 342, \tau_7 = 165, \tau_8 = 9, \tau_9 = 4, \tau_{30} = 6.$$

The $(697, 25)$-arc in PG (2, 29), the corresponding (174,5)-blocking set of which has the third secant distribution has been presented in [7]. □

4. The new arcs in PG(2, 31)

Theorem 4.1. *There exist a* $(598, 21)$-*arc, a* $(664, 23)$-*arc, a* $(699, 24)$-*arc, a* $(769, 26)$-*arc, and a* $(838, 28)$-*arc in* PG(2,31).

Proof. 1. The set of points having numbers 6 7 8 13 16 20 21 22 27 28 29 37 39 41 45 46 47 50 51 56 58 59 60 65 66 69 73 75 76 78 83 84 86 90 92 94 96 99 100 102 104 109 110 111 112 116 119 121 125 127 132 134 135 136 138 139 145 147 149 153 156 157 159 160 162 163 167 171 176 178 183 185 188 189 195 203 204 207 208 212 213 215 218 219 222 223 224 226 230 231 238 239 240 243 245 256 258 260 265 266 267

271 273 274 276 279 281 284 290 292 293 294 295 296 297 300 305 314 315 320 323 326 329 330 332 335 342 347 349 356 361 364 365 367 368 370 371 372 374 375 379 380 382 383 385 390 392 396 400 404 405 408 409 410 420 421 428 430 431 432 433 440 444 445 446 447 449 452 456 457 458 459 463 464 467 468 469 471 474 475 476 479 481 484 486 489 491 503 505 510 511 513 514 516 517 518 521 524 527 531 534 536 537 540 541 542 547 548 549 552 553 558 560 562 564 567 569 577 582 583 584 587 589 590 594 601 602 606 607 609 611 613 618 619 622 623 625 627 628 630 638 645 649 650 652 654 658 664 666 668 671 673 675 676 678 679 683 686 690 693 694 697 702 706 708 709 711 712 716 717 718 723 725 726 728 729 733 735 738 743 744 749 753 755 758 759 760 761 762 763 764 766 770 774 779 782 783 784 785 787 792 798 800 801 802 806 808 812 814 815 820 828 829 832 833 834 835 839 840 841 843 846 851 854 855 858 864 866 868 869 870 872 873 875 884 887 888 889 891 895 896 899 902 905 907 909 910 911 913 914 920 923 926 936 939 942 946 947 949 954 959 962 964 966 972 977 980 982 983 989 991 992 993 227 608 951 945 321 416 187 532 488 849 754 734 568 838 455 152 191 18 173 217 175 767 forms a (395,11)-blocking set in PG(2,31) with secant distribution $\tau_{11} = 292$, $\tau_{12} = 274$, $\tau_{13} = 227$, $\tau_{14} = 102$, $\tau_{15} = 57$, $\tau_{16} = 19$, $\tau_{17} = 4$, $\tau_{18} = 2$, $\tau_{20} = 1$, $\tau_{31} = 2$, $\tau_{32} = 13$.

The complement of this blocking set is a new $(598, 21)$-arc in PG(2, 31).

2. The set of points having numbers 7 15 16 19 20 22 25 27 28 29 37 39 44 45 51 56 58 59 65 67 75 76 78 81 84 86 90 93 94 96 99 102 104 105 109 110 111 121 125 127 132 134 135 136 138 139 144 147 149 153 159 162 163 167 171 175 176 182 183 188 189 190 192 195 208 215 217 218 219 222 224 238 239 240 243 244 245 252 256 257 258 260 265 267 271 279 281 284 290 293 296 297 298 305 309 313 315 320 322 323 325 326 329 330 332 340 349 352 353 356 367 368 370 371 372 375 379 385 387 390 392 396 397 400 404 406 408 409 410 413 420 421 428 430 432 435 436 440 444 445 446 447 452 454 457 458 464 467 469 471 475 481 484 489 490 491 503 505 510 511 513 517 518 521 524 527 531 534 536 541 542 545 548 552 553 555 558 560 562 564 568 572 577 582 589 590 594 599 601 602 607 609 612 613 619 622 625 626 627 630 636 638 645 647 649 652 653 659 664 666 669 671 673 675 676 683 690 693 694 697 702 708 711 712 713 723 725 726 729 733 736 738 743 744 746 749 755 758 759 761 762 764 765 770 774 779 782 783 784 792 794 798 806 808 811 815 819 827 828 829 832 834 835 836 839 840 843 846 851 855 858 859 864 866 872 873 875 876 882 884 887 888 889 895 902 905 909 910 911 913 914 923 924 926 928 931 936 939 941 946 947 958 959 962 964 966 980 982 983 985 989 992 248 364 312 35 801 942 917 268 544 207 549 993 793 335 233 334 422 forms a (329,9)-blocking set in PG(2,31) with secant distribution $\tau_9 = 276$, $\tau_{10} = 330$, $\tau_{11} = 207$, $\tau_{12} = 109$, $\tau_{13} = 49$, $\tau_{14} = 8$, $\tau_{15} = 2$, $\tau_{29} = 1$, $\tau_{31} = 1$, $\tau_{32} = 10$.

The complement of this blocking set is a new $(664, 23)$-arc in PG(2, 31).

3. The set of points having numbers 6 7 13 20 21 22 25 28 37 39 41 44 50 51 56 58 65 67 69 76 83 86 92 94 96 99 104 109 112 119 121 122 125 132 135 136 138 144 147 148 151 153 156 157 160 162 167 171 174 178 183 189 190 195 200 203 204 207 218 223 226 230 231 238 243 244 246 252 256 257 258 265 267 273 274 276 279 284 290 294 295 299 300 305 309 313 314 320 323 326 329 335 341 349 352 356 361 364 365 367 368 370 372 374 379 383 385 387 390 392 394 400 405 408 409 413 420 421 428 433 436 439 440 444 447 452 456 457 463 464 469 471 474 475 476 479 489 490 491 496 500 505 511 513 514 516 521 522 524 527 531 536 540 541 542 547 548 549 558 560 567 569 574 577 582 584 590 595 600 602 609 611 613 618 619 622 623 626 627 628 636 637 638 649 654 658 659 664 673 676 678 679 683 690 693 694 697 706 709 713 716 717 726 728 729 732 735 736 743 744 749 759 761 762 763 764 766 774 779 782 784 785 787 798 800 801 812 813 820 827 828 833 834 835 836 839 840 846 854 855 859 868 869 872 873 875 880 882 891 895 896 905 907 911 913 922 923 926 933 942 946 947 948 949 959 962 966 974 975 977 980 982 983 991 992 10 78 143 275 362 388 397 477 509 572 634 653 662 699 723 724 952 917 forms a (294,8)-blocking set in PG(2,31) with secant distribution $\tau_8 = 289$, $\tau_9 = 357$, $\tau_{10} = 203$, $\tau_{11} = 86$, $\tau_{12} = 39$, $\tau_{13} = 7$, $\tau_{14} = 2$, $\tau_{32} = 10$.

The complement of this blocking set is a new $(699, 24)$-arc in PG(2, 31).

4. The set of points having numbers 10 15 16 27 28 29 38 44 45 46 51 58 60 73 78 83 84 90 94 96 100 104 105 110 117 119 122 127 130 132 136 144 149 159 162 163 169 174 184 185 188 194 208 212 217 218 219 222 231 239 244 245 247 255 266 267 271 275 281 288 293 294 297 298 300 320 323 325 326 332 333 345 347 349 350 352 371 372 374 375 379 380 385 389 406 419 430 431 432 433 444 446 456 458 459 464 481 484 486 489 490 491 497 503 505 506 510 513 518 531 536 540 544 545 547 548 560 562 567 572 583 587 591 594 599 604 607 612 619 626 630 637 645 648 652 653 661 662 666 669 671 679 687 697 705

Table 5. **The new best small (l,t)-blocking sets in** $\mathrm{PG}(2,29)$

t	2	3	4	**5**	6	7	**8**	9	10	11	**12**
l	$3q$	$4q$	$5q+1$	$6q$	$7q+6$	$8q+11$	$9q+14$	$10q+16$	$11q+18$	$12q+16$	$13q+17$

Table 6. **The new best known small (l,t)-blocking sets in** $\mathrm{PG}(2,31)$

t	2	3	**4**	5	**6**	7	**8**	**9**	10	**11**
l	$3q-1$	$4q$	$5q$	$6q+2$	$7q+7$	$8q+11$	$9q+15$	$10q+19$	$11q+22$	$12q+23$

706 708 711 716 718 723 725 733 738 743 749 755 760 762 765 768 770 784 787 792 793 794 806 814 818 819 829 832 833 841 843 846 858 863 873 876 880 884 888 899 902 910 919 920 924 926 931 933 936 947 949 953 958 966 974 977 980 985 992 993 56 88 192 205 241 252 329 382 473 532 535 577 602 605 682 707 721 849 916 forms a (224,6)-blocking set in PG(2, 31) with secant distribution $\tau_6 = 352$, $\tau_7 = 354$, $\tau_8 = 196$, $\tau_9 = 61$, $\tau_{10} = 17$, $\tau_{11} = 5$, $\tau_{12} = 1$, $\tau_{32} = 7$.

The complement of this blocking set is a new $(769, 26)$-arc in PG(2, 31).

5. The set of points having numbers 10 13 15 16 29 41 44 46 52 60 66 73 83 100 111 117 122 125 127 130 136 156 169 170 174 178 185 189 208 212 215 218 229 231 239 247 255 266 273 274 275 284 288 293 294 320 326 332 333 345 347 368 372 374 379 380 389 392 406 419 421 432 433 446 451 458 459 463 474 486 490 496 497 503 505 510 513 516 540 544 547 555 558 560 567 569 583 591 594 604 611 614 622 628 630 648 652 664 669 673 679 687 705 706 708 716 717 718 732 733 746 755 759 760 762 787 791 794 801 806 812 814 819 833 836 841 850 854 863 876 880 895 896 899 907 909 919 920 926 933 949 953 954 958 966 968 977 980 991 85 240 414 504 610 781 forms a (155,4)-blocking set in PG(2, 31) with secant distribution $\tau_4 = 291$, $\tau_5 = 555$, $\tau_6 = 132$, $\tau_7 = 10$, $\tau_{31} = 1$, $\tau_{32} = 4$.

The complement of this blocking set is a new $(838, 28)$-arc in PG(2, 31). □

Remark 1: It follows from the second distributions that the new blocking sets, presented in this article, contain 1 line, 8 lines, 6 lines, 13 lines, 10 lines, 10 lines, 7 lines and 4 lines, respectively.

Remark 2: The (174,5)-blocking set in PG(2,29) is a ($6q$,5)-blocking set. The new (155,4)-blocking set in PG(2,31) is a ($5q$,4)-blocking set. According to Theorem 1 these two results are very good.

In Table 5 and Table 6 the parameters of the new best-known small (l,t)-blocking sets in $\mathrm{PG}(2,29)$ and $\mathrm{PG}(2,31)$ are given.

5. Back to codes

The well-known lower bound for the function $n_q(k,d)$ is the Griesmer bound [22], [27]

$$n_q(k,d) \geq g_q(k,d) = \sum_{j=0}^{k-1} \lceil \frac{d}{q^j} \rceil$$

($\lceil x \rceil$denotes the smallest integer $\geq x$).

An $[n,k,d]_q$ code is a Griesmer code if $n = g_q(k,d)$. Note that $n_q(k,d) = g_q(k,d)$ for all d when $k = 1$ or 2 [23]. The problem of finding $n_q(k,d)$ for all d has been solved only in the next cases (see [26]):

- $k \le 8$ for codes over GF(2),
- $k \le 5$ for codes over GF(3),
- $k \le 4$ for codes over GF(4),
- $k = 3$ for codes over GF(q), $5 \le q \le 9$.

Let B be a $((t+1).q, t)$-blocking set. The complement of this blocking set is a $(q^2+q+1-(t+1).q, q+1-t)$-arc A. The code C, corresponding to A, has parameters

$$[q^2+1-t.q,\ 3,\ q^2-q-(q-1).t]_q.$$

We will now prove that this code reaches the Griesmer bound.

Theorem 5.1. *The code C, having parameters*

$$[q^2+1-t.q,\ 3,\ q^2-q-(q-1).t]_q$$

is a Griesmer code.

Proof. For our 3-dimensional code the Griesmer bound is

$$g_q(3,d) = \sum_{i=0}^{2} \left\lceil \frac{d}{q^i} \right\rceil = d + \left\lceil \frac{d}{q} \right\rceil + \left\lceil \frac{d}{q^2} \right\rceil$$

The minimum distance is $d = q^2 - q - (q-1).t$.
So we have:

$$g_q(3,d) = q^2 - q - (q-1).t + \left\lceil \frac{q^2-q-(q-1).t}{q} \right\rceil + \left\lceil \frac{q^2-q-(q-1).t}{q^2} \right\rceil =$$

$$= q^2 - q - (q-1).t + \left\lceil q - 1 - t + \frac{1}{q}.t \right\rceil + \left\lceil 1 - \frac{1}{q} - \frac{q-1}{q^2}.t \right\rceil =$$

$$= q^2 - q - (q-1).t + q - t + 1 =$$

$$= q^2 - q - q.t + t + q - t + 1 = q^2 + 1 - tq = n$$

□

From Theorem 5 and the connection between $(n, n-d)$-arcs in PG$(2,q)$ and projective $[n,3,d]_q$ codes over GF(q) the next corollary holds.

Corollary 5.2. *There exist* $[477,3,459]_{29}$, $[596,3,574]_{29}$, $[598,3,577]_{31}$, $[664,3,641]_{31}$, $[699,3,675]_{31}$, $[769,3,743]_{31}$ *projective codes and* $[697,3,672]_{29}$, $[838,3,810]_{31}$ *Griesmer codes.*

6. How to check the results established in Theorems 3 and 4

In this section we give a simple protocol on how to verify the results established in the article, which may be of use to readers who are not experts in the field of coding theory or in combinatorial configurations in finite projective geometries.

1. Generate in lexicographical order the points of the finite projective plane.

2. Consider the complement of the given (l,t)-blocking set. This complement consists of $q^2+q+1-l$ points and forms a $(q^2+q+1-l, q+1-t)$-arc in the projective plane.

3. Construct a matrix whose column vectors are the points of this complement. This matrix is a generator matrix of a code.

4. Calculate the minimum weight or the weight distribution of this code, using for example Magma. The result must be a $[q^2+q+1-l, 3, q^2+t-l]_q$ code.

Since the finite planes PG(2,29) and PG(2,31) contain a large number of points, we will give an example in PG(2,11)$(a=10)$.

The set of points having numbers 1 3 5 11 13 14 15 16 17 18 19 20 21 22 23 24 27 29 36 38 41 43 46 47 48 51 55 56 58 60 65 67 72 74 76 83 84 85 86 88 91 92 94 100 101 103 105 108 112 117 119 121 130 131 132 133 forms a (56,4)-blocking set in PG(2, 11) with secant distribution $\tau_4 = 57$, $\tau_5 = 49$, $\tau_6 = 18$, $\tau_7 = 3$, $\tau_{10} = 1$, $\tau_{12} = 5$.

The complement of this blocking set consists 77 points in PG(2, 11). These points have numbers 2 4 6 7 8 9 10 12 25 26 28 30 31 32 33 34 35 37 39 40 42 44 45 49 50 52 53 54 57 59 61 62 63 64 66 68 69 70 71 73 75 77 78 79 80 81 82 87 89 90 93 95 96 97 98 99 102 104 106 107 109 110 111 113 114 115 116 118 120 122 123 124 125 126 127 128 129 and are (0,1,0), (0,1,2), (0,1,4), (0,1,5), (0,1,6), (0,1,7), (0,1,8), (0,1,a), (1,1,1), (1,1,2), (1,1,4), (1,1,6), (1,1,7), (1,1,8), (1,1,9), (1,1,a), (1,2,0), (1,2,2), (1,2,4), (1,2,5), (1,2,7), (1,2,9), (1,2,a), (1,3,3), (1,3,4), (1,3,6), (1,3,7), (1,3,8), (1,4,0), (1,4,2), (1,4,4), (1,4,5), (1,4,6), (1,4,7), (1,4,9), (1,5,0), (1,5,1), (1,5,2), (1,5,3), (1,5,5), (1,5,7), (1,5,9), (1,5,a), (1,6,0), (1,6,1), (1,6,2), (1,6,3), (1,6,8), (1,6,a), (1,7,0), (1,7,3), (1,7,5), (1,7,6), (1,7,7), (1,7,8), (1,7,9), (1,8,1), (1,8,3), (1,8,5), (1,8,6), (1,8,8), (1,8,9), (1,8,a), (1,9,1), (1,9,2), (1,9,3), (1,9,4), (1,9,6), (1,9,8), (1,9,a), (1,a,0), (1,a,1), (1,a,2), (1,a,3), (1,a,4), (1,a,5), (1,a,6).

Putting these points as column vectors of the next matrix

$$G = \begin{pmatrix} 00000000111111111111111111111111111111111111111111111111111111111111111111111 \\ 1111111111111111222222233333444444455555555666666777777788888889999999aaaaaaa \\ 0245678a1246789a024579a34678024567901235 79a01238a0356789135689a123468a0123456 \end{pmatrix}$$

we obtain a generator matrix of a $[77,3,69]_{11}$ code with weight distribution 0^1 69^{570} 70^{490} 71^{180} 72^{30} 75^{10} 77^{50}.

All this can be obtained by the online Calculator of Magma Algebra System. We submit:

```
> M := KMatrixSpace(FiniteField(11), 3, 77);
> G := M ! [0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,3,3,3,3,3,4,4,4,4,4,4,4,5,5,5,5,5,5,5,5,
6,6,6,6,6,6,7,7,7,7,7,7,7, 8,8,8,8,8,8,8,9,9,9,9,9,9,9,10,10,10,10,10,10,10,
0,2,4,5,6,7,8,10,1,2,4,6,7,8,9,10,0,2,4,5,7,9,10,3,4,6,7,8,0,2,4,5,6,7,9,0,1,2,3,5,7,9,
10,0,1,2,3,8,10,0,3,5,6, 7,8,9,1,3,5,6,8,9,10,1,2,3,4,6,8,10,0,1,2,3,4,5,6];
> G;
> C := LinearCode(G);
> C;
> WeightDistribution(C);
```

and the result is:

```
[0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 3 3 3 3 3 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5 6
6 6 6 6 6 7 7 7 7 7 7 7 8 8 8 8 8 8 8 9 9 9 9 9 9 9 10 10 10 10 10 10 10]
[0 2 4 5 6 7 8 10 1 2 4 6 7 8 9 10 0 2 4 5 7 9 10 3 4 6 7 8 0 2 4 5 6 7 9 0 1 2 3 5 7 9 10
0 1 2 3 8 10 0 3 5 6 7 8 9 1 3 5 6 8 9 10 1 2 3 4 6 8 10 0 1 2 4 5 6]

[77, 3, 69] Linear Code over GF(11)
Generator matrix:
```

```
[1 0 10 4 9 3 8 7 0 5 4 3 8 2 7 1 7 6 5 10 9 8 2 1 6 5 10 4 9 8 7 1 6 0 10 10 4 9 3 2 1 0
0 5 10 4 7 6 1 5 4 9 3 8 2 7 6 5 10 9 3 8 8 2 7 1 0 10 9 4 9 3 8 2 7 1]
[0 1 2 8 3 9 4 5 0 6 7 8 3 9 4 10 5 6 7 2 3 4 10 1 7 8 3 9 5 6 7 2 8 3 4 5 0 6 1  3 4 10
5 0 6 1 9 10 5 1 2 8 3 9 4 0 1 2 8 9 4 10 0 6 1 7 8 9 10 5 0 6 1 7 2 8]
[0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]

[ <0, 1>, <69, 570>, <70, 490>, <71, 180>, <72, 30>, <75, 10>, <77, 50> ]
```

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