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The relation between constants in generic and degenerate subspaces of free unital associative complex algebra*

Research Article

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Abstract: From the study of the constants in the generic and the degenerate weight subspaces of the free unitary associative complex algebra $\mathcal{B}$, it follows that the constants in the degenerate weight subspaces of the algebra $\mathcal{B}$ can be constructed from the corresponding constants in the generic case by a certain specialization procedure. Here we consider that each constant in each generic weight subspace of the algebra $\mathcal{B}$ can be expressed by certain iterated $\boldsymbol{q}$-commutators.

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1. Introduction

In this paper we first consider the multiparametric quon algebra $\mathcal{B}$ and its $\boldsymbol{q}$-differential structure. Following the works [6, 9], we recall an explicit formula for computing constants in the generic subspaces of the algebra $\mathcal{B}$. Our motivation is to explain that the constants in any degenerate subspace of the algebra $\mathcal{B}$ can be computed from suitable constants in the corresponding generic subspace. Therefore, we consider here a free unitary associative complex algebra $\mathcal{B} = \mathbb{C}\langle e_{i_1}, \ldots, e_{i_N}\rangle$ (generated by N generators, each of degree one), equipped with $\boldsymbol{q}$-differential structure by $\boldsymbol{q}$-differential operators $\partial_i \colon \mathcal{B} \to \mathcal{B}$, $i \in \mathcal{N}$, where $\mathcal{N} = \{i_1, i_2, \ldots, i_N\}$ is a fixed subset of nonnegative integers. The $\boldsymbol{q}$-differential operators ∂_i are recursively defined by

$$\partial_i(e_j x) = \delta_{ij} x + q_{ij} e_j \partial_i(x) \tag{1}$$

for each $x \in \mathcal{B}$, $i, j \in \mathcal{N}$ with $\partial_i(1) = 0$ and $\partial_i(e_j) = \delta_{ij}$, where δ_{ij} denotes a standard Kronecker delta and the q_{ij} are complex numbers. According to the formula (1), each passage of ∂_i through e_i (from the left) is characterized by an additional factor q_{ij}.

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Example 1.1. *The actions of the operators* ∂_i, $i = 1, 2, 3, \dots$ *on the monomial* $e_{131212} = e_1 e_3 e_1 e_2 e_1 e_2$ *are given by*

$$\begin{aligned}
\partial_1(e_{131212}) &= e_{31212} + q_{11}q_{13}\, e_{13212} + q_{11}^2 q_{13} q_{12}\, e_{13122} \\
\partial_2(e_{131212}) &= q_{21}^2 q_{23}\, e_{13112} + q_{21}^3 q_{23} q_{22}\, e_{13121} \\
\partial_3(e_{131212}) &= q_{31}\, e_{11212} \\
\partial_i(e_{131212}) &= 0 \quad \textit{for all } i \geq 4.
\end{aligned}$$

Therefore, ∂_i is a kind of generalized i-th partial derivative. In particular, if all q_{ij} are equal to one, then ∂_i coincides with an ordinary i-th partial derivative, e.g. $\partial_i(e_i^n) = n \cdot e_i^{n-1}$. We recall that the free unitary associative complex algebra $\mathcal{B}$ is naturally ordered by total degree and, more generally, can be viewed as multi-degree. The algebra $\mathcal{B}$ has a direct sum decomposition into the generic subspace $\mathcal{B}^{\text{gen}}$ spanned by all multilinear monomials and the degenerate subspace $\mathcal{B}^{\text{deg}}$ spanned by all monomials which are nonlinear in at least one variable, which can be written as $\mathcal{B} = \mathcal{B}^{\text{gen}} \oplus \mathcal{B}^{\text{deg}}$ with $\mathcal{B}^{\text{gen}} = \bigoplus_{Q \text{ a set}} \mathcal{B}_Q$, $\mathcal{B}^{\text{deg}} = \bigoplus_{Q \text{ a multiset (not set)}} \mathcal{B}_Q$. Thus we distinguish generic and degenerate subspaces of the algebra $\mathcal{B}$. We consider here that $Q = \{k_1^{n_1}, \dots, k_p^{n_p}\}$ denotes the multiset of cardinality $n = n_1 + \cdots + n_p$, where $k_i \neq k_j$ for each $1 \leq i < j \leq p$ and there exists at least one n_j such that $n_j \neq 1$. Note that n_j is considered as the repetition frequency of the element k_j in the multiset Q. In particular, if all n_j are equal to one, then Q is a set of cardinality n and we write it in the form $Q = \{l_1, \dots, l_n\}$, where $l_i \neq l_j$ for each $1 \leq i < j \leq n$. We call the weight subspace $\mathcal{B}_Q$ *generic* if Q is a set, otherwise *degenerate*. An arbitrary weighted subspace $\mathcal{B}_Q$ corresponding to a multiset Q of cardinality n is given by

$$\mathcal{B}_Q = span_{\mathbb{C}} \left\{ e_{j_1 \dots j_n} = e_{j_1} \cdots e_{j_n} \mid j_1 \dots j_n \in \widehat{Q} \right\}, \tag{2}$$

where $\widehat{Q}$ denotes the set of all unique permutations of the multiset Q. Thus, $\dim \mathcal{B}_Q = |\widehat{Q}|$, where $|\widehat{Q}|$ denotes the cardinality of the set $\widehat{Q}$. In other words, the dimension of a weighted subspace $\mathcal{B}_Q$ is equal to the cardinality of the set of all permutations of the multiset Q.

2. The constants in the algebra $\mathcal{B}$

Of special interest in the algebra $\mathcal{B}$ are the elements, called constants, which are annihilated by all $\boldsymbol{q}$-differential operators ∂_i, $i \in \mathcal{N}$. We consider here linearly independent constants which we call *basic constants*. A nonzero basic constant is called a nontrivial basic constant. The space of all constants in the algebra $\mathcal{B}$ is denoted by $\mathcal{C} = \{C \in \mathcal{B} \mid \partial_i(C) = 0,\ i \in \mathcal{N}\}$. Note that all $\boldsymbol{q}$-differential operators ∂_i, $i \in \mathcal{N}$ can be considered as operators of degree -1, so we introduce an operator $\partial \colon \mathcal{B} \to \mathcal{B}$, of degree zero, by the formula $\partial = \sum_{i \in \mathcal{N}} e_i \partial_i$, where $e_i \colon \mathcal{B} \to \mathcal{B}$ are considered as operators on $\mathcal{B}$, see [6]. Then we obtain that $\partial C = 0$ if and only if $\partial_i C = 0$ for all $i \in \mathcal{N}$. This implies that $\mathcal{C} = \ker \partial$, where $\ker \partial$ denotes the kernel of the operator ∂. The operator ∂ preserves the direct sum decomposition of the algebra $\mathcal{B}$. Considering that $\partial^Q \colon \mathcal{B}_Q \to \mathcal{B}_Q$ denotes the restriction of the operator ∂ to the subspace $\mathcal{B}_Q$, it follows that $\partial^Q x = \partial x$ for any $x \in \mathcal{B}_Q$. Moreover, $\mathcal{C}_Q = \ker \partial^Q$, where $\mathcal{C}_Q$ denotes the space of all constants belonging to the subspace $\mathcal{B}_Q$. The space $\mathcal{C}$ also inherits the direct sum decomposition into subspaces $\mathcal{C}_Q$, therefore the problem of determining the space of all constants in the algebra $\mathcal{B}$ turns on determining the finite-dimensional spaces $\mathcal{C}_Q$ for all Q (multisets and sets) over $\mathcal{N}$ of cardinality $n \geq 2$. Note that $\mathcal{C}_Q = \{0\}$ for $|Q| = 1$. The decisive role in the computation of the constants in the algebra $\mathcal{B}$ is played by the action of ∂_i on the monomial $e_{\underline{j}} = e_{j_1 \dots j_n}$ in the monomial basis $\mathsf{B}_Q = \{e_{\underline{j}} \mid \underline{j} \in \widehat{Q}\}$ of the subspace $\mathcal{B}_Q$, given by an explicit formula:

$$\partial_i(e_{\underline{j}}) = \sum_{1 \leq k \leq n,\, j_k = i} q_{ij_1} \cdots q_{ij_{k-1}} e_{j_1 \dots \widehat{j_k} \dots j_n} \tag{3}$$

for each $\underline{j} \in \widehat{Q}$ with $\underline{j} = j_1 \dots j_n$, where we have applied (1). Here $\widehat{j_k}$ denotes the omission of the corresponding index j_k, see Example 1.1. The number of terms in the sum (3) is equal to the number of occurrences of the generator e_i in the monomial $e_{\underline{j}}$. In particular, an important special case is $\partial_i(e_i^n) = [n]_{q_{ii}}\, e_i^{n-1}$, where

$$[n]_q = \sum_{k=0}^{n-1} q^k = 1 + q + \cdots + q^{n-1} \tag{4}$$

is an q analogue of a natural number n. On the other hand, it follows from (3) that the action of the operator ∂^Q on the monomial $e_{\underline{j}}$ in the monomial basis $\mathtt{B}_Q$ of $\mathcal{B}_Q$ is given by

$$\partial^Q(e_{\underline{j}}) = \sum_{1 \le m \le n} q_{j_m j_1} \cdots q_{j_m j_{m-1}} e_{j_m j_1 \dots \widehat{j_m} \dots j_n}, \qquad \text{for each } \underline{j} \in \widehat{Q}. \tag{5}$$

Example 2.1. *Considering the monomial e_{131212} from Example 1.1, we obtain by applying (5) that the action of the operator ∂^Q on this monomial is given by*

$$\begin{aligned}\partial^Q(e_{131212}) = e_{131212} &+ q_{11}q_{13}\, e_{113212} + q_{11}^2 q_{13} q_{12}\, e_{113122} \\ &+ q_{21}^2 q_{23}\, e_{213112} + q_{21}^3 q_{23} q_{22}\, e_{213121} + q_{31}\, e_{311212}.\end{aligned}$$

The problem of determining the space $\mathcal{C}_Q$ of all constants belonging to the subspace $\mathcal{B}_Q$ amounts to determining the kernel of the operator ∂^Q for any multiset (i.e., set) Q of cardinality n. So we first introduce the simpler operators

$$T_{m,1}\, e_{\underline{j}} = q_{j_m j_1} \cdots q_{j_m j_{m-1}} e_{j_m j_1 j_2 \dots \widehat{j_m} \dots j_n} \tag{6}$$

for each $\underline{j} \in \widehat{Q}$, $1 \le m \le n$ acting on $\mathcal{B}_Q$, where $T_{1,1} = id$ (i.e., in general $T_{m,m} = id$), and then we rewrite the operator ∂^Q (c.f. (5)) in terms of the operators (6) as follows $\partial^Q = \sum_{m=1}^{n} T_{m,1}$. Then we get

$$\partial^Q = D_{Q,n} \cdot C_{Q,n}^{-1} \tag{7}$$

with

$$C_{Q,n} = (id - T_{n,1}) \cdots (id - T_{2,1}) = \overleftarrow{\prod_{2 \le m \le n}} (id - T_{m,1}), \tag{8}$$

$$D_{Q,n} = \left(id - T_1^2\, T_{n,2}\right) \cdots \left(id - T_1^2\, T_{2,2}\right) = \overleftarrow{\prod_{2 \le m \le n}} \left(id - T_1^2\, T_{m,2}\right) \tag{9}$$

(c.f. [9]), where the action of the operators $T_{m,1}$, $2 \le m \le n$ on $\mathcal{B}_Q$ is given by (6) and the action of the operators $T_1^2\, T_{m,2}$, $2 \le m \le n$ on $\mathcal{B}_Q$ is given by

$$T_1^2\, T_{m,2}\, e_{\underline{j}} = \sigma_{j_1 j_m} q_{j_m j_2} \cdots q_{j_m j_{m-1}} e_{j_1 j_m j_2 \dots \widehat{j_m} \dots j_n} \tag{10}$$

with $\sigma_{ij} := q_{ij} q_{ji}$. Here we use the notation $T_1^2\, T_{m,2} := T_{2,1}^2\, T_{m,2}$. Note that (7) is a special case of the braid factorization from [1, Proposition 4.7] (c.f. with [5]).

Remark 2.2. *Let us denote by $\mathbf{B}_Q$ the matrix of the operator ∂^Q, by $\mathbf{C}_{Q,n}$, $\mathbf{D}_{Q,n}$ the corresponding matrices of the operators $C_{Q,n}$, $D_{Q,n}$, and also by $\mathbf{T}_{m,1}$, $\mathbf{T}_1^2\, \mathbf{T}_{m,2}$, $2 \le m \le n$ the corresponding matrices of the operators $T_{m,1}$, $T_1^2\, T_{m,2}$ with respect to the monomial basis $\mathtt{B}_Q$ of a subspace $\mathcal{B}_Q$ (considered with Johnson-Trotter order on permutations, see [10]), where we denote by $\mathbf{I}$ the unit matrix corresponding* *to the operator $T_{1,1} = id$ (i.e., $T_{m,m} = id$). Then the rows and columns of all introduced matrices are indexed by the elements of the monomial basis of $\mathcal{B}_Q$. Thus, these matrices are square matrices whose order is equal to $\dim \mathcal{B}_Q = |\widehat{Q}|$. Now (7) can be rewritten in matrix notation as $\mathbf{B}_Q = \mathbf{D}_{Q,n} \cdot (\mathbf{C}_{Q,n})^{-1}$, which implies*

$$\det \mathbf{B}_Q = \frac{\det \mathbf{D}_{Q,n}}{\det \mathbf{C}_{Q,n}}. \tag{11}$$

Example 2.3. *We briefly explain the above matrices for a weighted subspace $\mathcal{B}_Q$ corresponding first to a set $Q = \{l_1, l_2, l_3\}$ and then to a multiset $Q' = \{k_1^2, k_2\}$, see also Example 3.1.*

1. *Let $Q = \{l_1, l_2, l_3\}$ be a set of cardinality 3. Then the monomial basis of a subspace $\mathcal{B}_Q$ is given by $\mathtt{B}_Q = \{e_{j_1j_2j_3}, e_{j_1j_3j_2}, e_{j_3j_1j_2}, e_{j_3j_2j_1}, e_{j_2j_3j_1}, e_{j_2j_1j_3}\}$. Here the matrix $\mathbf{B}_Q$ of the operator ∂^Q is given by $\mathbf{B}_Q = \mathbf{T}_{3,1} + \mathbf{T}_{2,1} + \mathbf{I}$, where the matrix $\mathbf{T}_{m,1}$, $1 \le m \le 3$ corresponds to the operator $T_{m,1}$ with $\mathbf{I} = \mathbf{T}_{1,1}$, see (6). Applying (8) and (9), we get $\mathbf{C}_{Q,3} = (\mathbf{I} - \mathbf{T}_{3,1}) \cdot (\mathbf{I} - \mathbf{T}_{2,1})$, $\mathbf{D}_{Q,3} = \left(\mathbf{I} - \mathbf{T}_1^2\mathbf{T}_{3,2}\right) \cdot \left(\mathbf{I} - \mathbf{T}_1^2\right)$, where $\mathbf{T}_{2,2} = \mathbf{I}$. Thus, using (6) and (10), we obtain*

$$\mathbf{B}_Q = \begin{matrix} e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & q_{j_1j_2}q_{j_1j_3} & q_{j_1j_2} \\ 0 & 1 & q_{j_1j_3} & q_{j_1j_2}q_{j_1j_3} & 0 & 0 \\ q_{j_3j_1}q_{j_3j_2} & q_{j_3j_1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & q_{j_3j_2} & q_{j_3j_1}q_{j_3j_2} \\ 0 & 0 & q_{j_2j_1}q_{j_2j_3} & q_{j_2j_3} & 1 & 0 \\ q_{j_2j_1} & q_{j_2j_1}q_{j_2j_3} & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{I} - \mathbf{T}_{3,1} = \begin{matrix} e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -q_{j_1j_2}q_{j_1j_3} & 0 \\ 0 & 1 & 0 & -q_{j_1j_2}q_{j_1j_3} & 0 & 0 \\ -q_{j_3j_1}q_{j_3j_2} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -q_{j_3j_1}q_{j_3j_2} \\ 0 & 0 & -q_{j_2j_1}q_{j_2j_3} & 0 & 1 & 0 \\ 0 & -q_{j_2j_1}q_{j_2j_3} & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{I} - \mathbf{T}_{2,1} = \begin{matrix} e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -q_{j_1j_2} \\ 0 & 1 & -q_{j_1j_3} & 0 & 0 & 0 \\ 0 & -q_{j_3j_1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -q_{j_3j_2} & 0 \\ 0 & 0 & 0 & -q_{j_2j_3} & 1 & 0 \\ -q_{j_2j_1} & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{I} - \mathbf{T}_1^2\mathbf{T}_{3,2} = \begin{matrix} e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{matrix} \begin{bmatrix} 1 & -\sigma_{j_1j_2}q_{j_2j_3} & 0 & 0 & 0 & 0 \\ -\sigma_{j_1j_3}q_{j_3j_2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\sigma_{j_1j_3}q_{j_1j_2} & 0 & 0 \\ 0 & 0 & -\sigma_{j_2j_3}q_{j_2j_1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\sigma_{j_2j_3}q_{j_3j_1} \\ 0 & 0 & 0 & 0 & -\sigma_{j_1j_2}q_{j_1j_3} & 1 \end{bmatrix}$$

$$\mathbf{I} - \mathbf{T}_1^2 = \begin{matrix} e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{matrix} \begin{bmatrix} 1-\sigma_{j_1j_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\sigma_{j_1j_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\sigma_{j_1j_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\sigma_{j_2j_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\sigma_{j_2j_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\sigma_{j_1j_2} \end{bmatrix}$$

where $\det(\mathbf{I}-\mathbf{T}_{3,1}) = (1-\sigma_{j_1j_2j_3})^2$, $\det(\mathbf{I}-\mathbf{T}_{2,1}) = (1-\sigma_{j_1j_2})\cdot(1-\sigma_{j_1j_3})\cdot(1-\sigma_{j_2j_3})$, $\det\left(\mathbf{I}-\mathbf{T}_1^2\mathbf{T}_{3,2}\right) = (1-\sigma_{j_1j_2j_3})^3$, $\det\left(\mathbf{I}-\mathbf{T}_1^2\right) = (1-\sigma_{j_1j_2})^2\cdot(1-\sigma_{j_1j_3})^2\cdot(1-\sigma_{j_2j_3})^2$ *from which we obtain by applying (11)* $\det\mathbf{B}_Q = \frac{\det\mathbf{D}_{Q,3}}{\det\mathbf{C}_{Q,3}} = \frac{\det\left(\mathbf{I}-\mathbf{T}_1^2\mathbf{T}_{3,2}\right)\cdot\det\left(\mathbf{I}-\mathbf{T}_1^2\right)}{\det(\mathbf{I}-\mathbf{T}_{2,1})\cdot\det(\mathbf{I}-\mathbf{T}_{3,1})}$, *i.e.,*

$$\det\mathbf{B}_Q = (1-\sigma_{j_1j_2})\cdot(1-\sigma_{j_1j_3})\cdot(1-\sigma_{j_2j_3})\cdot(1-\sigma_{j_1j_2j_3})\,.$$

Consider the given set $Q=\{l_1,l_2,l_3\}$ *of cardinality* 3 *and all its subsets* $T_1=\{l_1,l_2\}$, $T_2=\{l_1,l_3\}$ *and* $T_3=\{l_2,l_3\}$ *of cardinality* 2, *we conclude that* $\det\mathbf{B}_Q$ *can be written as*

$$\det\mathbf{B}_Q = (1-\sigma_{l_1l_2})\cdot(1-\sigma_{l_1l_3})\cdot(1-\sigma_{l_2l_3})\cdot(1-\sigma_{l_1l_2l_3}) \tag{12}$$

or in the shorter form $\det\mathbf{B}_Q = (1-\sigma_{T_1})\cdot(1-\sigma_{T_2})\cdot(1-\sigma_{T_3})\cdot(1-\sigma_Q)$, *i.e., in the following form* $\det\mathbf{B}_Q = \prod_{\substack{T\subseteq Q\\ 2\le|T|\le 3}}(1-\sigma_T)$, *see (15). Note that here all* $(|T|-2)!\cdot(3-|T|)!$ *are equal to one for each* $2\le|T|\le 3$.

2. *Let* $Q'=\{k_1^2,k_2\}$ *be a multiset of cardinality* 3. *Then, similarly as above, the monomial basis of a subspace* $\mathcal{B}_{Q'}$ *is given by* $\mathtt{B}_{Q'}=\{e_{i_1i_1i_2},e_{i_1i_2i_1},e_{i_2i_1i_1}\}$ *and the matrix* $\mathbf{B}_{Q'}$ *of the operator* $\partial^{Q'}$ *is given by* $\mathbf{B}_{Q'}=\mathbf{T}_{3,1}+\mathbf{T}_{2,1}+\mathbf{I}$, *hence* $\mathbf{C}_{Q',3}=(\mathbf{I}-\mathbf{T}_{3,1})\cdot(\mathbf{I}-\mathbf{T}_{2,1})$ *and* $\mathbf{D}_{Q',3}=\left(\mathbf{I}-\mathbf{T}_1^2\mathbf{T}_{3,2}\right)\cdot\left(\mathbf{I}-\mathbf{T}_1^2\right)$. *By the use of (6), (10) we then obtain*

$$\mathbf{B}_{Q'} = \begin{matrix} e_{i_1i_1i_2}\\ e_{i_1i_2i_1}\\ e_{i_2i_1i_1}\end{matrix}\begin{bmatrix} 1+q_{i_1i_1} & q_{i_1i_1}q_{i_1i_2} & 0\\ 0 & 1 & q_{i_1i_2}(1+q_{i_1i_1})\\ q_{i_2i_1}^2 & q_{i_2i_1} & 1\end{bmatrix}$$

$$\mathbf{I}-\mathbf{T}_{3,1} = \begin{matrix} e_{i_1i_1i_2}\\ e_{i_1i_2i_1}\\ e_{i_2i_1i_1}\end{matrix}\begin{bmatrix} 1 & -q_{i_1i_1}q_{i_1i_2} & 0\\ 0 & 1 & -q_{i_1i_1}q_{i_1i_2}\\ -q_{i_2i_1}^2 & 0 & 1\end{bmatrix}$$

$$\mathbf{I}-\mathbf{T}_{2,1} = \begin{matrix} e_{i_1i_1i_2}\\ e_{i_1i_2i_1}\\ e_{i_2i_1i_1}\end{matrix}\begin{bmatrix} 1-q_{i_1i_1} & 0 & 0\\ 0 & 1 & -q_{i_1i_2}\\ 0 & -q_{i_2i_1} & 1\end{bmatrix}$$

$$\mathbf{I}-\mathbf{T}_1^2\mathbf{T}_{3,2} = \begin{matrix} e_{i_1i_1i_2}\\ e_{i_1i_2i_1}\\ e_{i_2i_1i_1}\end{matrix}\begin{bmatrix} 1 & -q_{i_1i_1}^2q_{i_1i_2} & 0\\ -\sigma_{i_1i_2}q_{i_2i_1} & 1 & 0\\ 0 & 0 & 1-\sigma_{i_1i_2}q_{i_1i_1}\end{bmatrix}$$

$$\mathbf{I}-\mathbf{T}_1^2 = \begin{matrix} e_{i_1i_1i_2}\\ e_{i_1i_2i_1}\\ e_{i_2i_1i_1}\end{matrix}\begin{bmatrix} 1-q_{i_1i_1}^2 & 0 & 0\\ 0 & 1-\sigma_{i_1i_2} & 0\\ 0 & 0 & 1-\sigma_{i_1i_2}\end{bmatrix}$$

see also Example 3.1. Note that $\sigma_{ij}=q_{ij}q_{ji}$ *implies* $\sigma_{i_1i_1}=q_{i_1i_1}^2$. *Now it is easy to verify that* $\det(\mathbf{I}-\mathbf{T}_{3,1}) = 1-q_{i_1i_1}^2\sigma_{i_1i_2}^2$, $\det(\mathbf{I}-\mathbf{T}_{2,1}) = (1-q_{i_1i_1})\cdot(1-\sigma_{i_1i_2})$, $\det\left(\mathbf{I}-\mathbf{T}_1^2\mathbf{T}_{3,2}\right) = \left(1-q_{i_1i_1}^2\sigma_{i_1i_2}^2\right)\cdot(1-q_{i_1i_1}\sigma_{i_1i_2})$, $\det\left(\mathbf{I}-\mathbf{T}_1^2\right) = \left(1-q_{i_1i_1}^2\right)\cdot(1-\sigma_{i_1i_2})^2$ *from which we obtain by applying (11)* $\det\mathbf{B}_{Q'} = \frac{\det\mathbf{D}_{Q',3}}{\det\mathbf{C}_{Q',3}} = \frac{\left(1-q_{i_1i_1}^2\sigma_{i_1i_2}^2\right)\cdot(1-q_{i_1i_1}\sigma_{i_1i_2})\cdot(1-q_{i_1i_1})\cdot(1+q_{i_1i_1})\cdot(1-\sigma_{i_1i_2})^2}{(1-q_{i_1i_1})\cdot(1-\sigma_{i_1i_2})\cdot\left(1-q_{i_1i_1}^2\sigma_{i_1i_2}^2\right)}$, *i.e.,*

$$\det\mathbf{B}_{Q'} = (1+q_{i_1i_1})\cdot(1-\sigma_{i_1i_2})\cdot(1-q_{i_1i_1}\sigma_{i_1i_2})\,.$$

If we now consider the given multiset $Q'=\{k_1^2,k_2\}$ *of cardinality* 3 *and also its two subsets* $T_1'=\{k_1^2\}$ *and* $T_2'=\{k_1,k_2\}$ *of cardinality* 2, *we obtain*

$$\det\mathbf{B}_{Q'} = (1+q_{k_1k_1})\cdot(1-\sigma_{k_1k_2})\cdot(1-q_{k_1k_1}\sigma_{k_1k_2}) \tag{13}$$

which can be written in the following shorter form $\det\mathbf{B}_{Q'} = \beta_{T_1'}\cdot\beta_{T_2'}\cdot\beta_{Q'}$, *where* $\beta_{T_1'}=1+q_{k_1k_1}$, $\beta_{T_2'}=1-\sigma_{k_1k_2}$, $\beta_{Q'}=1-q_{k_1k_1}\sigma_{k_1k_2}$.

In general, the entries of the matrix $\mathbf{B}_Q$ (Q is a multiset) are polynomials in q_{ij}'s, therefore its determinant is also a polynomial in q_{ij}'s. Considering the factorizations (8) and (9) of the operators $C_{Q,n}$ and $D_{Q,n}$ and the given matrix notation, we obtain that, the polynomial $\det \mathbf{B}_Q$ (c.f. (11)) can be factorized by the factors β_T for each $T \subseteq Q$, $|T| \geq 2$, where each β_T has the corresponding polynomial form. Thus, from the identity $\det \mathbf{B}_Q = 0$ it follows that β_T vanishes for at least one $T \subseteq Q$. Of particular interest are the actual values of parameters q_{ij} (called *singular values* or *singular parameters*) for which at least one $\beta_T = 0$ holds. We say that parameters q_{ij} are singular if $\det \mathbf{B}_Q = 0$, otherwise they are regular (called parameters in general position). Thus, if the parameters q_{ij} are regular, then there are no constants in $\mathcal{B}_Q$. In other words, there are constants in $\mathcal{B}_Q$, i.e., the space $\mathcal{C}_Q$ is nonzero only for singular parameters, c.f. [2]. We distinguish two types of singular parameters: *Q-cocycle condition* and *$(Q;T)$-cocycle condition* for the fixed $T \subset Q$ given in [6], but we consider here only the Q-cocycle condition

$$c_Q = \{\beta_Q = 0, \quad \beta_T \neq 0 \text{ for all } T \subset Q\}, \tag{14}$$

because it is the only one that plays a key role in the calculation of the constants. It is shown that the constants under the $(Q;T)$-cocycle condition can be obtained from the corresponding constants under the Q-cocycle condition by a special specialization procedure. From this we conclude that the space $\mathcal{C}_Q$ is nonzero only for the singular parameters q_{ij} for which $\det \mathbf{B}_Q$ vanishes, and that all constants in $\mathcal{B}_Q$ can be obtained from those under the Q-cocycle condition. The Q-cocycle condition (14) is sometimes written in the form $\beta_Q = 0$.

In particular, if $Q = \{l_1, \dots, l_n\}$ is a set (the generic case) of cardinality n, then the entries of the matrix $\mathbf{B}_Q$ are monomials in q_{ij}'s, therefore its determinant (11) is given by an explicit expression in terms of the product of the binomial factors $1 - \sigma_T$ for each $T \subseteq Q$, as follows

$$\det \mathbf{B}_Q = \prod_{\substack{T \subseteq Q \\ 2 \leq |T| \leq n}} (1 - \sigma_T)^{(|T|-2)! \cdot (n-|T|)!}, \tag{15}$$

where $|T|$ indicates the cardinality of T, see [6, 9]. If $|T| = k$ for every $2 \leq k \leq n$, then there are $\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$ terms of binomial factors $1 - \sigma_T$, where each term $1 - \sigma_T$ corresponds to the corresponding subset $T \subseteq Q$, $|T| = k$ with

$$\sigma_T = \prod_{\{a,b\} \subseteq T} \sigma_{ab} = \prod_{a \neq b \in T} q_{ab}. \tag{16}$$

We consider here the above identity $\sigma_{ab} = q_{ab} q_{ba}$. Then, in the generic case, the Q-cocycle condition (14) has the form

$$c_Q = \{1 - \sigma_Q = 0, \quad 1 - \sigma_T \neq 0 \text{ for all } T \subset Q\}. \tag{17}$$

Thus, in the generic weighted subspace $\mathcal{B}_Q \subseteq \mathcal{B}$ there are constants if the Q-cocycle condition (17) is satisfied.

Example 2.4. *Considering the obtained matrix $\mathbf{B}_Q$, $Q = \{l_1, l_2, l_3\}$ and its determinant from Example 2.3, it follows, that the corresponding Q-cocycle condition (17) is given by $1 - \sigma_{l_1 l_2 l_3} = 0$, see (12). It goes without saying that $1 - \sigma_{l_1 l_2} \neq 0$, $1 - \sigma_{l_1 l_3} \neq 0$, $1 - \sigma_{l_2 l_3} \neq 0$. Similarly, with respect to the matrix $\mathbf{B}_{Q'}$, $Q' = \{k_1^2, k_2\}$ and its determinant from Example 2.3, the corresponding Q'-cocycle condition (14) is given by $1 - q_{k_1 k_1} \sigma_{k_1 k_2} = 0$, see (13), where $1 + q_{k_1 k_1} \neq 0$ and $1 - \sigma_{k_1 k_2} \neq 0$.*

Recall now that any weighted subspace $\mathcal{B}_Q$ corresponding to a multiset (or set) Q of cardinality n is given by (2), and observe, that if the Q-cocycle condition is satisfied, then, there are constants in the subspace $\mathcal{B}_Q$ and there are no constants in the subspaces $\mathcal{B}_T$ for any proper subset $T \subset Q$. This is directly related to the fact that the operator $(id - T_1^2 T_{n,2})$ is not invertible, but all the operators $(id - T_1^2 T_{m,2})$ for $m = 2, \dots, n-1$ are invertible, so the identity (7) under the Q-cocycle condition can be written in the following form by using (9)

$$\partial^Q \cdot C_{Q,n} \cdot \prod_{2 \leq m \leq n-1} \left(id - T_1^2 T_{m,2}\right)^{-1} = \left(id - T_1^2 T_{n,2}\right).$$

Therefore, for each $Z \in \mathcal{B}$ we get

$$\partial^Q \cdot C_{Q,n} \cdot \prod_{2 \le m \le n-1} \left(id - T_1^2\, T_{m,2}\right)^{-1} \cdot Z = \left(id - T_1^2\, T_{n,2}\right) \cdot Z \tag{18}$$

where we can establish a relationship $\ker\left(id - T_1^2\, T_{n,2}\right) \subset \mathcal{B}_Q$ to $\ker \partial^Q$. We recall that $\ker \partial^Q = \mathcal{C}_Q$, where $\mathcal{C}_Q$ is the space of all constants in $\mathcal{B}_Q$. Then for any $U_{\underline{j}} \in \ker\left(id - T_1^2\, T_{n,2}\right)$ the right-hand side of (18) is zero, so the corresponding vector

$$X = C_{Q,n} \cdot \prod_{2 \le m \le n-1} \left(id - T_1^2\, T_{m,2}\right)^{-1} \cdot U_{\underline{j}} \tag{19}$$

belongs to $\ker \partial^Q$. Then X is a constant in $\mathcal{B}_Q$, see [9, Proposition 2]. Thus, the vectors in the kernel of the operator $(id - T_1^2\, T_{n,2})$ have a crucial importance in determining the constants in $\mathcal{B}_Q$. So the problem of computing the constants in $\mathcal{B}_Q$ boils down to the following two questions: first, how to write the vectors spanning the kernel $\ker\left(id - T_1^2\, T_{n,2}\right)$, and second, how to find a basis?

For the generic case, the above questions are solved in [9], where it is shown that under the Q-cocycle condition (17) all vectors $U_{\underline{j}} \in \ker\left(id - T_1^2\, T_{n,2}\right)$, $\underline{j} \in \widehat{Q}$ are given by

$$U_{\underline{j}} = \left(id - T_1^2\, T_{n,2}\right)^{-1} \cdot (1 - \sigma_Q)\; e_{\underline{j}}. \tag{20}$$

Let us now denote by $Q_{a\,b}$, $1 \le a, b \le n$ a diagonal operator on $\mathcal{B}_Q$ (c.f. (2)) defined by

$$Q_{a\,b}\, e_{\underline{j}} = q_{j_a j_b}\, e_{\underline{j}} \tag{21}$$

$\underline{j} = j_1 j_2 \dots j_n \in \widehat{Q}$. Then we denote a diagonal operator $Q_{\{a,b\}} = Q_{a\,b} \cdot Q_{b\,a}$, $1 \le a, b \le n$ on $\mathcal{B}_Q$, which, by applying (21) and the previously defined identity $\sigma_{j_a j_b} = q_{j_a j_b} q_{j_b j_a}$, is given by

$$Q_{\{a,b\}}\, e_{\underline{j}} = \sigma_{j_a j_b}\, e_{\underline{j}}. \tag{22}$$

Note that $Q_{a\,b} \cdot Q_{c\,d} = Q_{c\,d} \cdot Q_{a\,b}$. Similarly, we denote by $Q_{\{1,2,\dots,k\}} = \prod_{\{a,b\} \subseteq \{1,2,\dots,k\}} Q_{\{a,b\}}$ diagonal operator on $\mathcal{B}_Q$ for each $2 \le k \le n$ given by

$$Q_{\{1,2,\dots,k\}}\, e_{\underline{j}} = \prod_{\{a,b\} \subseteq \{1,2,\dots,k\}} \sigma_{j_a j_b}\, e_{\underline{j}}. \tag{23}$$

where we have used (22). We note that the right-hand side of (23) is connected with the identity (16), therefore we denote by

$$Q_{\{1,2,\dots,k\}}\, e_{j_1 j_2 \dots j_n} = \sigma_{j_1 j_2 \dots j_k}\, e_{j_1 j_2 \dots j_n}, \tag{24}$$

where $\sigma_{j_1 j_2 \dots j_k}$ is equal to the right-hand side of (23).
Then, for the set Q of cardinality n we get $(1 - \sigma_Q)\, e_{j_1 \dots j_n} = \left(id - Q_{\{1,2,\dots,n\}}\right) e_{j_1 \dots j_n}$, so (20) we can write as $U_{\underline{j}} = \left(id - T_1^2\, T_{n,2}\right)^{-1} \cdot \left(id - Q_{\{1,2,\dots,n\}}\right)\, e_{\underline{j}}$, from which it follows that the vector $X_{\underline{j}} \in \ker \partial^Q$ (c.f. (19)) is given by

$$X_{\underline{j}} = C_{Q,n} \cdot (D_{Q,n})^{-1} \cdot \left(id - Q_{\{1,2,\dots,n\}}\right)\, e_{\underline{j}} \tag{25}$$

for each $\underline{j} \in \widehat{Q}$, where we used (9). Thus, with the expression (25), the formula for computing the constants in $\mathcal{B}_Q$ is given when the Q-cocycle condition is satisfied, but here an additional problem of determining the inverse of the operator $D_{Q,n}$ arises. This problem is solved in [9], where first, assuming that $\sigma_T \ne 1$ holds for all $T \subseteq Q$, the inverse of the operator $D_{Q,n}\, e_{\underline{j}}$ on $\mathcal{B}_Q$ is found in the form $(D_{Q,n})^{-1}\, e_{\underline{j}} =$

$\left((\mathcal{Q}_n)^{-1}\cdot E_{Q,n}\right) e_{\underline{j}}$ and then it is shown that the constants $X_{\underline{j}}$ in $\mathcal{B}_Q$ under the Q-cocycle condition are expressed by

$$X_{\underline{j}} = \left(C_{Q,n}\cdot(\mathcal{Q}_{n-1})^{-1}\cdot E_{Q,n}\right) e_{\underline{j}} \tag{26}$$

$\underline{j}\in\widehat{Q}$, see Theorem 1 and Theorem 2 of [9], where an operator $C_{Q,n}$ on $\mathcal{B}_Q$ is given by (8), a diagonal operator $\mathcal{Q}_n$ on $\mathcal{B}_Q$ is given by

$$\mathcal{Q}_n\, e_{\underline{j}} = \left(id - Q_{\{1,2\}}\right)\cdot\left(id - Q_{\{1,2,3\}}\right)\cdots\left(id - Q_{\{1,2,\ldots,n\}}\right) e_{\underline{j}} \tag{27}$$

and similarly $\mathcal{Q}_{n-1}\, e_{\underline{j}} = \left(id - Q_{\{1,2\}}\right)\cdot\left(id - Q_{\{1,2,3\}}\right)\cdots\left(id - Q_{\{1,2,\ldots,n-1\}}\right) e_{\underline{j}}$, which can be written by applying (24) in the following form

$$\mathcal{Q}_{n-1}\, e_{j_1j_2\cdots j_n} = (1-\sigma_{j_1j_2})\cdot(1-\sigma_{j_1j_2j_3})\cdots(1-\sigma_{j_1j_2\cdots j_{n-1}})\, e_{j_1j_2\cdots j_n} \tag{28}$$

$\underline{j} = j_1j_2\ldots j_n\in\widehat{Q}$, see also (23). Moreover, we write here the operator $E_{Q,n}$ on $\mathcal{B}_Q$ in the following form

$$E_{Q,n}\, e_{j_1j_2\cdots j_n} = \sum_{g\in S_1\times S_{n-1}}\left(\prod_{i\in Des(g^{-1})}\sigma_{j_{g^{-1}(1)}j_{g^{-1}(2)}\cdots j_{g^{-1}(i)}}\prod_{(a,b)\in I(g^{-1})} q_{j_{g^{-1}(a)}j_{g^{-1}(b)}}\right) e_{j_{g^{-1}(1)}j_{g^{-1}(2)}\cdots j_{g^{-1}(n)}} \tag{29}$$

where

$$I(g^{-1}) = \{(a,b)\mid 1\le a<b\le n,\ g^{-1}(a)>g^{-1}(b)\}$$

denotes the set of all inversions (a,b) of the permutation $g^{-1}\in S_1\times S_{n-1}$ and

$$Des(g^{-1}) = \{1\le i\le n-1\mid g^{-1}(i)>g^{-1}(i+1)\}$$

denotes the descent set of the permutation $g^{-1}\in S_1\times S_{n-1}$, see also [4].

It is obvious that $g^{-1}\in S_1\times S_{n-1}$ is the inverse of the permutation $g\in S_1\times S_{n-1}$.

Remark 2.5. *We note that an operator $E_{Q,n}$ given by (29) is equal to the operator $E_{Q,n}$ from Theorem 1 of [9], which we repeat here*

$$E_{Q,n} = \sum_{g\in S_1\times S_{n-1}} W_n(g)\cdot g \tag{30}$$

where $W_n(g) = \prod_{i\in Des(g^{-1})} Q_{\{1,2,\ldots,i\}}$. Moreover, by applying (24), a diagonal operator $W_n(g)$ can be written in the following form $W_n(g)\, e_{j_1j_2\cdots j_n} = \prod_{i\in Des(g^{-1})}\sigma_{j_1j_2\cdots j_i}\, e_{j_1j_2\cdots j_n}$. We emphasize that there is a misprint in (30) (which is written in Theorem 1 and Theorem 3 of [9]), because on the right-hand side of (30) G should be written instead of g. In fact, we denote by $G = \varrho(g^*)$ a (twisted regular) representation on the subspace $\mathcal{B}_Q$ of an element $g^*\in\mathcal{A}(S_n)$ from a twisted group algebra $\mathcal{A}(S_n)$ of the symmetric group S_n, given by $g^* = \prod_{(a,b)\in I(g^{-1})} X_{a\,b}\, g$, see [8, Definition 2.1] where $I(g^{-1})$ denotes the set of inversions of $g^{-1}\in S_n$. Then we obtain that

$$G\, e_{j_1j_2\cdots j_n} = \varrho(g^*)\, e_{j_1j_2\cdots j_n} = \prod_{(a,b)\in I(g^{-1})} q_{j_{g^{-1}(a)}j_{g^{-1}(b)}}\, e_{j_{g^{-1}(1)}j_{g^{-1}(2)}\cdots j_{g^{-1}(n)}}. \tag{31}$$

We note that the right-hand side of (31) can be written as

$$\prod_{(b',a')\in I(g)} q_{j_{a'}j_{b'}}\, e_{j_{g^{-1}(1)}j_{g^{-1}(2)}\cdots j_{g^{-1}(n)}} = \prod_{(a',b')\in I(g)} q_{j_{b'}j_{a'}}\, e_{j_{g^{-1}(1)}j_{g^{-1}(2)}\cdots j_{g^{-1}(n)}},$$

see [7, Lemma 4.6], but we will not use this notation here because (31) is more convenient with the remaining notation of the operator $E_{Q,n}$ from (29). Briefly, if $(a,b)\in I(g^{-1})$ then it holds that $a<b$ and $g^{-1}(a)>g^{-1}(b)$. Let denote by $a'=g^{-1}(a)$ and $b'=g^{-1}(b)$, from which follows $g(a')=a$ and $g(b')=b$. Then $a<b$ and $g^{-1}(a)>g^{-1}(b)$ implies $g(a')<g(b')$ and $a'>b'$. We thus obtain $b'<a'$ and $g(b')>g(a')$, from which follows $(b',a')\in I(g)$.

Note that if $\underline{k}=k_1\dots k_n\in\widehat{Q}$ and $\underline{j}=j_1\dots j_n\in\widehat{Q}$ are in the relation with $k_p=j_{g^{-1}(p)}$ for all $1\le p\le n$, then the monomial $e_{\underline{k}}=e_{k_1k_2\dots k_n}$ in the monomial basis of $\mathcal{B}_Q$ is given by

$$e_{\underline{k}} = e_{j_{g^{-1}(1)}j_{g^{-1}(2)}\cdots j_{g^{-1}(n)}},$$

so that (29) can be written in the following shorter form

$$E_{Q,n}\, e_{\underline{j}} = \sum_{g\in S_1\times S_{n-1}} \left(\prod_{i\in Des(g^{-1})} \sigma_{j_{g^{-1}(1)}j_{g^{-1}(2)}\cdots j_{g^{-1}(i)}} \prod_{(a,b)\in I(g^{-1})} q_{j_{g^{-1}(a)}j_{g^{-1}(b)}} \right) e_{\underline{k}}.$$

Recall also that q_{ij} are complex numbers, so it is easy to see that it follows from the right-hand side of the formula (29) that $E_{Q,n}$ is a diagonal operator. Using the fact that the product of diagonal operators is commutative, we get $(D_{Q,n})^{-1} = (\mathcal{Q}_n)^{-1}\cdot E_{Q,n}\cdot = E_{Q,n}\cdot(\mathcal{Q}_n)^{-1}$ and then with (27) we get $(D_{Q,n})^{-1}\cdot\big(id - Q_{\{1,2,\dots,n\}}\big) = E_{Q,n}\cdot(\mathcal{Q}_n)^{-1}\cdot\big(id-Q_{\{1,2,\dots,n\}}\big) = E_{Q,n}\cdot(\mathcal{Q}_{n-1})^{-1}$, therefore from (25) follows (26).

Remark 2.6. *In the generic case there are $n!$ (nontrivial) vectors $X_{\underline{j}}\in\ker\partial^Q$ (c.f. (26)), but they are not linearly independent for each $\underline{j}\in\widehat{Q}$. In other words, they do not form a basis of $\ker\partial^Q$, see [9]. If we now use the abbreviations $V := \overleftarrow{\prod}_{2\le m\le n-1}(id - T_{m,1})$ and $W := \overleftarrow{\prod}_{2\le m\le n-1}\big(id - T_1^2\,T_{m,2}\big)$, then the operators $C_{Q,n}$ and $D_{Q,n}$, given by (8) and (9), can be written as $C_{Q,n}=(id-T_{n,1})\cdot V$ and $D_{Q,n}=\big(id-T_1^2\,T_{n,2}\big)\cdot W$, therefore we can rewrite (19) as*

$$X = (id - T_{n,1})\cdot V\cdot W^{-1}\cdot U_{\underline{j}} \tag{32}$$

where W is invertible under the Q-cocycle condition. Then $X\in\ker\partial^Q$ if $U_{\underline{j}}\in\ker\big(id-T_1^2\,T_{n,2}\big)$, so it turns out that

$$\dim\left(\ker(id - T_1^2\,T_{n,2})\right) = n\cdot(n-2)! \quad \text{and} \quad \dim\left(\ker(id-T_{n,1})\right) = (n-1)!.$$

Moreover, we obtain from (32) that $\dim(\ker\partial^Q) = \dim(\ker(id-T_1^2\,T_{n,2})) - \dim(\ker(id-T_{n,1})) = (n-2)!$, which leads to an alternative result of Frønsdal and Galindo [3, Theorem 4.1.2] that the space of constants in the generic case has dimension $(n-2)!$.

We can now conclude that if $Q=\{l_1,l_2,\dots,l_n\}$ is a set of cardinality $n\ge 2$, then under the Q-cocycle condition the number of vectors $X_{\underline{j}}\in\ker\partial^Q$ given by (26) can be reduced to $(n-2)!$ by

$$X_{l_1l_2j_3\dots j_n} = \left(C_{Q,n}\cdot(\mathcal{Q}_{n-1})^{-1}\cdot E_{Q,n}\right) e_{l_1l_2j_3\dots j_n} \tag{33}$$

for all $j_3\dots j_n\in\widehat{P}$, where $P=Q\backslash\{l_1,l_2\}=\{l_3,\dots,l_n\}$ is a set of cardinality $n-2$ obtained from the set Q by omitting the first two elements l_1 and l_2 of Q. Then $|\widehat{P}|=(n-2)!$. Thus, the indices of X on the left-hand side of the formula (33) have the form that the first two indices $l_1,l_2\in Q$ are fixed and the remaining $n-2$ indices $l_3,\ldots,l_n \in Q$ vary. Here we should note that the application of (29) leads us to the conclusion that the right-hand side of the formula (33) consists of $(n-1)!$ terms in which the indices of the monomial $e_{l_1 l_2 j_3 \ldots j_n}$ are such that its first index l_1 is fixed and the remaining $n-1$ indices $l_2 j_3 \ldots j_n$ vary. Moreover, we first introduce certain iterated $\boldsymbol{q}$-commutators $Y_{j_1\ldots j_p}$, which are recursively expressed by

$$Y_{j_1} = e_{j_1}, \qquad Y_{j_1\ldots j_p} = \left[Y_{j_1\ldots j_{p-1}}, e_{j_p}\right]_{q_{j_p j_1}\cdots q_{j_p j_{p-1}}}, \tag{34}$$

where $\left[Y_{j_1\ldots j_{p-1}}, e_{j_p}\right]_{q_{j_p j_1}\cdots q_{j_p j_{p-1}}} = Y_{j_1\ldots j_{p-1}} e_{j_p} - q_{j_p j_1}\cdots q_{j_p j_{p-1}} e_{j_p} Y_{j_1\ldots j_{p-1}}$, and we then obtain

$$Y_{j_1 j_2 \ldots j_n} = C_{Q,n}\, e_{j_1 j_2\ldots j_n} \tag{35}$$

$j_1 j_2 \ldots j_n \in \widehat{Q}$, see [9, Proposition 4]. On the other hand, we obtain with (28) that

$$(\mathcal{Q}_{n-1})^{-1} e_{j_1 j_2 \ldots j_n} = \frac{1}{(1-\sigma_{j_1 j_2})\cdot(1-\sigma_{j_1 j_2 j_3})\cdots(1-\sigma_{j_1 j_2\ldots j_{n-1}})}\, e_{j_1 j_2\ldots j_n}, \tag{36}$$

where $\sigma_{j_1 j_2 \ldots j_p}$ is given by (23) for each $2 \le p \le n-1$. Let $R = \{l_2,\ldots,l_n\} \subset Q$ be a set of cardinality $n-1$, $n \ge 2$, which we obtain from the set $Q = \{l_1, l_2, \ldots, l_n\}$ by omitting the first element l_1 of Q, then a monomial $e_{l_1 l_2 j_3 \ldots j_n}$ on the right-hand side of (33) can take the form $e_{l_1 j_2 j_3 \ldots j_n}$, where $j_2 j_3 \ldots j_n \in \widehat{R}$. Using (29), (35) and (36), we thus rewrite the right-hand side of (33) into the form

$$\sum_{g\in S_1\times S_{n-1}} \frac{\prod\limits_{(a,b)\in I(g^{-1})} q_{j_{g^{-1}(a)} j_{g^{-1}(b)}} \prod\limits_{i\in Des(g^{-1})} \sigma_{l_1 j_{g^{-1}(2)}\cdots j_{g^{-1}(i)}}}{\left(1-\sigma_{l_1 j_{g^{-1}(2)}}\right)\cdot\left(1-\sigma_{l_1 j_{g^{-1}(2)} j_{g^{-1}(3)}}\right)\cdots\left(1-\sigma_{l_1 j_{g^{-1}(2)}\cdots j_{g^{-1}(n-1)}}\right)} \cdot Y_{l_1 j_{g^{-1}(2)}\cdots j_{g^{-1}(n)}}, \tag{37}$$

where we used that the multiplication in the numerator of (37) is commutative, since all values are complex numbers. Thus, if $g = id$, then $g^{-1} = id$, so the sets $I(id)$ and $Des(id)$ are equal to an empty set, resulting in both products in the numerator of the fraction of (37) being equal to one. From the fact that $g \in S_1 \times S_{n-1}$ fixes the first index l_1 in Q, it follows that (37) consists of $(n-1)!$ terms in which the first index is fixed and the remaining $n-1$ indices vary. Thus, under the Q-cocycle condition $1-\sigma_Q = 0$ (c.f. (17)), each vector $X_{\underline{j}} \in \ker \partial^Q$ in (33) takes the form (37). This gives rise to the following theorem.

Theorem 2.7. *Let the generic weight subspace $\mathcal{B}_Q \subseteq \mathcal{B}$ correspond to a set $Q = \{l_1, l_2, \ldots, l_n\}$ of cardinality $n \ge 2$ and $P = \{l_3,\ldots,l_n\}$. If $1-\sigma_Q = 0$, then the space $\mathcal{C}_Q$ of all constants belonging to the subspace $\mathcal{B}_Q$ consists of $(n-2)!$ nontrivial basic constants which can be expressed in the form*

$$C_{l_1 l_2 j_3 \ldots j_n} = \tag{38}$$

$$= \sum_{g\in S_1\times S_{n-1}} \frac{\prod\limits_{(a,b)\in I(g^{-1})} q_{j_{g^{-1}(a)} j_{g^{-1}(b)}} \cdot \prod\limits_{i\in Des(g^{-1})} \sigma_{l_1 j_{g^{-1}(2)}\cdots j_{g^{-1}(i)}}}{\left(1-\sigma_{l_1 j_{g^{-1}(2)}}\right)\cdot\left(1-\sigma_{l_1 j_{g^{-1}(2)} j_{g^{-1}(3)}}\right)\cdots\left(1-\sigma_{l_1 j_{g^{-1}(2)}\cdots j_{g^{-1}(n-1)}}\right)} \cdot Y_{l_1 j_{g^{-1}(2)}\cdots j_{g^{-1}(n)}}$$

for every $j_3 \ldots j_n \in \widehat{P}$, where $g \in S_1 \times S_{n-1}$ fixes the first index.

Note that Theorem 2.7 gives the same result as [9, Theorem 3], where G should be written instead of g, see Remark 2.5. The nontrivial basic constants are described in more detail here using the formula (38), which is explained in the following examples, where we write the Q-cocycle condition (17) in the form $1-\sigma_Q = 0$.

Example 2.8. *Let $Q = \{l_1, l_2\}$ and $1-\sigma_{l_1 l_2} = 0$. Then it follows from $n = |Q| = 2$ that $(n-2)! = 0! = 1$ such that under the Q-cocycle condition $1-\sigma_{l_1 l_2} = 0$ the space $\mathcal{C}_Q$ of all constants belonging to the subspace $\mathcal{B}_Q$ consists of a nontrivial basic constant, consists of one term. Note that in the set $\widehat{Q} = \{l_1 l_2, l_2 l_1\}$*

there is only one permutation $g = l_1l_2 = id$ which fixes the first index $l_1 \in Q$, where $g^{-1} = l_1l_2 = id$ and $I(id) = Des(id) = \emptyset$. For $n = 2$ the numerator of the fraction of (38) multiplying the corresponding iterated $\boldsymbol{q}$-commutator $Y_{l_1 j_{g^{-1}(2)}} = Y_{l_1l_2}$ is equal to one. On the other hand, it follows directly that the denominator of the given fraction is also equal to one, so the corresponding fraction multiplied by $Y_{l_1l_2}$ is equal to one. Thus, under the Q-cocycle condition $1 - \sigma_{l_1l_2} = 0$ there is only one nontrivial basic constant

$$C_{l_1l_2} = Y_{l_1l_2},$$

where $Y_{l_1l_2} = [e_{l_1}, e_{l_2}]_{q_{l_2l_1}} = e_{l_1l_2} - q_{l_2l_1}e_{l_2l_1}$.

Example 2.9. *Let $Q = \{l_1, l_2, l_3\}$ be the set of cardinality 3 and let $1 - \sigma_{l_1l_2l_3} = 0$. Then the space $\mathcal{C}_Q$ of all constants belonging to the subspace $\mathcal{B}_Q$ consists of a nontrivial basic constant $C_{l_1l_2l_3}$ consisting of two terms as follows*

$$C_{l_1l_2l_3} = \frac{1}{1 - \sigma_{l_1 j_{g_1^{-1}(2)}}} \cdot Y_{l_1 j_{g_1^{-1}(2)} j_{g_1^{-1}(3)}} + \frac{q_{j_{g_2^{-1}(2)} j_{g_2^{-1}(3)}} \cdot \sigma_{l_1 j_{g_2^{-1}(2)}}}{1 - \sigma_{l_1 j_{g_2^{-1}(2)}}} \cdot Y_{l_1 j_{g_2^{-1}(2)} j_{g_2^{-1}(3)}}. \tag{39}$$

Note that the set $\widehat{Q}$ consists of six permutations, of which only the following two permutations $g_1 = l_1l_2l_3 = id$ and $g_2 = l_1l_3l_2$ are elements of $S_1 \times S_2$. Now it is easy to see that $g_1^{-1} = g_1 = id$, so that $I(id) = Des(id) = \emptyset$. So, by applying $j_{g_1^{-1}(2)} = l_2$ and $j_{g_1^{-1}(3)} = l_3$, the first term of the sum of the formula (39) is given by

$$\frac{1}{1 - \sigma_{l_1 j_{g_1^{-1}(2)}}} \cdot Y_{l_1 j_{g_1^{-1}(2)} j_{g_1^{-1}(3)}} = \frac{1}{1 - \sigma_{l_1l_2}} \cdot Y_{l_1l_2l_3}.$$

On the other hand, it follows that $g_2^{-1} = g_2 = l_1l_3l_2$, so $I(g_2^{-1}) = \{(2,3)\}$, $Des(g_2^{-1}) = \{2\}$. Then, by applying $j_{g_2^{-1}(2)} = l_3$ and $j_{g_2^{-1}(3)} = l_2$, the second term of the sum of the formula (39) is given by

$$\frac{q_{j_{g_2^{-1}(2)} j_{g_2^{-1}(3)}} \cdot \sigma_{l_1 j_{g_2^{-1}(2)}}}{1 - \sigma_{l_1 j_{g_2^{-1}(2)}}} \cdot Y_{l_1 j_{g_2^{-1}(2)} j_{g_2^{-1}(3)}} = \frac{q_{l_3l_2} \cdot \sigma_{l_1l_3}}{1 - \sigma_{l_1l_3}} \cdot Y_{l_1l_3l_2}.$$

Thus, under the Q-cocycle condition $1 - \sigma_{l_1l_2l_3} = 0$, the space $\mathcal{C}_Q$ consists of a nontrivial basic constant

$$C_{l_1l_2l_3} = \frac{1}{1 - \sigma_{l_1l_2}} \cdot Y_{l_1l_2l_3} + \frac{q_{l_3l_2}\sigma_{l_1l_3}}{1 - \sigma_{l_1l_3}} \cdot Y_{l_1l_3l_2}, \tag{40}$$

where $Y_{i_1i_2i_3} = [Y_{i_1i_2}, e_{i_3}]_{q_{i_3i_1}q_{i_3i_2}} = e_{i_1i_2i_3} - q_{i_2i_1}e_{i_2i_1i_3} - q_{i_3i_1}q_{i_3i_2}e_{i_3i_1i_2} + q_{i_2i_1}q_{i_3i_1}q_{i_3i_2}e_{i_3i_2i_1}$.

Example 2.10. *Let $Q = \{l_1, l_2, l_3, l_4\}$ be the set of cardinality 4 and let $1 - \sigma_{l_1l_2l_3l_4} = 0$. Then the space $\mathcal{C}_Q$ consists of two nontrivial basic constants $C_{l_1l_2l_3l_4}$ and $C_{l_1l_2l_4l_3}$, each consisting of six terms. Note that the set $\widehat{Q}$ consists of 24 permutations, of which only the next six permutations $g_1 = l_1l_2l_3l_4 = id$, $g_2 = l_1l_2l_4l_3$, $g_3 = l_1l_3l_2l_4$, $g_4 = l_1l_3l_4l_2$, $g_5 = l_1l_4l_2l_3$, $g_6 = l_1l_4l_3l_2$ are elements of $S_1 \times S_3$. Then we obtain $g_1^{-1} = l_1l_2l_3l_4 = id$, $I(id) = Des(id) = \emptyset$, $g_2^{-1} = l_1l_2l_4l_3$, $I(g_2^{-1}) = \{(3,4)\}$, $Des(g_2^{-1}) = \{3\}$, $g_3^{-1} = l_1l_3l_2l_4$, $I(g_3^{-1}) = \{(2,3)\}$, $Des(g_3^{-1}) = \{2\}$, $g_4^{-1} = l_1l_4l_2l_3$, $I(g_4^{-1}) = \{(2,3),(2,4)\}$, $Des(g_4^{-1}) = \{2\}$, $g_5^{-1} = l_1l_3l_4l_2$, $I(g_5^{-1}) = \{(2,4),(3,4)\}$, $Des(g_5^{-1}) = \{3\}$, $g_6 = l_1l_4l_3l_2$, $I(g_6^{-1}) = \{(2,3),(2,4),(3,4)\}$, $Des(g_6^{-1}) = \{2,3\}$, from which we deduce, that under the Q-cocycle condition $1 - \sigma_{l_1l_2l_3l_4} = 0$, the space $\mathcal{C}_Q$ consists of the following nontrivial basic constants*

$$\begin{aligned}
C_{l_1l_2l_3l_4} &= \frac{1}{(1-\sigma_{l_1l_2})(1-\sigma_{l_1l_2l_3})}\cdot Y_{l_1l_2l_3l_4} + \frac{q_{l_4l_3}\sigma_{l_1l_2l_4}}{(1-\sigma_{l_1l_2})(1-\sigma_{l_1l_2l_4})}\cdot Y_{l_1l_2l_4l_3}\\
&+ \frac{q_{l_3l_2}\sigma_{l_1l_3}}{(1-\sigma_{l_1l_3})(1-\sigma_{l_1l_2l_3})}\cdot Y_{l_1l_3l_2l_4} + \frac{q_{l_4l_2}q_{l_4l_3}\sigma_{l_1l_4}}{(1-\sigma_{l_1l_4})(1-\sigma_{l_1l_2l_4})}\cdot Y_{l_1l_4l_2l_3}\\
&+ \frac{q_{l_3l_2}q_{l_4l_2}\sigma_{l_1l_3l_4}}{(1-\sigma_{l_1l_3})(1-\sigma_{l_1l_3l_4})}\cdot Y_{l_1l_3l_4l_2} + \frac{q_{l_3l_2}q_{l_4l_2}q_{l_4l_3}\sigma_{l_1l_4}\sigma_{l_1l_3l_4}}{(1-\sigma_{l_1l_4})(1-\sigma_{l_1l_3l_4})}\cdot Y_{l_1l_4l_3l_2}\\
C_{l_1l_2l_4l_3} &= \frac{1}{(1-\sigma_{l_1l_2})(1-\sigma_{l_1l_2l_4})}\cdot Y_{l_1l_2l_4l_3} + \frac{q_{l_3l_4}\sigma_{l_1l_2l_3}}{(1-\sigma_{l_1l_2})(1-\sigma_{l_1l_2l_3})}\cdot Y_{l_1l_2l_3l_4}\\
&+ \frac{q_{l_4l_2}\sigma_{l_1l_4}}{(1-\sigma_{l_1l_4})(1-\sigma_{l_1l_2l_4})}\cdot Y_{l_1l_4l_2l_3} + \frac{q_{l_3l_2}q_{l_3l_4}\sigma_{l_1l_3}}{(1-\sigma_{l_1l_3})(1-\sigma_{l_1l_2l_3})}\cdot Y_{l_1l_3l_2l_4}\\
&+ \frac{q_{l_3l_2}q_{l_4l_2}\sigma_{l_1l_3l_4}}{(1-\sigma_{l_1l_4})(1-\sigma_{l_1l_3l_4})}\cdot Y_{l_1l_4l_3l_2} + \frac{q_{l_3l_2}q_{l_3l_4}q_{l_4l_2}\sigma_{l_1l_3}\sigma_{l_1l_3l_4}}{(1-\sigma_{l_1l_3})(1-\sigma_{l_1l_3l_4})}\cdot Y_{l_1l_3l_4l_2} \qquad (41)
\end{aligned}$$

The expressions of the nontrivial basic constants in $\mathcal{B}_{l_1l_2l_3l_4l_5}$ can be found in [9, Example 4], where we have used the abbreviations $x^* := \frac{1}{1-x}$, $x^+ := \frac{x}{1-x}$. In this case, there are six nontrivial basic constants, each consisting of 24 terms in lexicographic order.

In agreement with the notation introduced earlier, the problem of finding the explicit formula describing the constants in the generic subspaces of the algebra $\mathcal{B}$ is solved by the formula (38) of Theorem 2.7, in which under the Q-cocycle condition (17) the constants in $\mathcal{B}_Q$ are expressed by certain iterated $\boldsymbol{q}$-commutators. In doing so, we have shown that vectors play a crucial role in the kernel of the operator $(id - T_1^2 T_{n,2})$. Similarly, we can compute constants in the degenerate subspaces of the algebra $\mathcal{B}$. However, the problem of finding the explicit formula describing the constants in the degenerate subspaces of the algebra $\mathcal{B}$ is not so easy to solve, because for each multiset Q the polynomial $\det \mathbf{B}_Q$ has a different factorization with the factors β_T, $T \subseteq Q$, so it is much more difficult to express the corresponding determinant (11) of the matrix $\mathbf{B}_Q$ by an explicit formula for each multiset Q. Thus, the factor β_Q in the Q-cocycle condition (14) takes a different form depending on the given multiset Q. In accordance with the above procedure, we briefly describe below the corresponding determinant (11) of the matrix $\mathbf{B}_Q$, the Q-cocycle condition (14), and nontrivial basic constant in the degenerate subspace $\mathcal{B}_Q$ corresponding to the multiset Q, which first takes the form $Q = \{k^n\}$ and then $Q = \{k_1^{n-1}, k_2\}$, see also [6], where the basic constants are also given under the $(Q;T)$-*cocycle condition* for the fixed $T \subset Q$, which is not considered here.

1. Let $Q = \{k^n\}$ be a multiset of cardinality $n \geq 2$. Then in this case we obtain that the determinant (11) is given by $\det \mathbf{B}_Q = [n]_{q_{kk}}$ (see (4)), so that the Q-cocycle condition (14) has the form $c_Q = \{[n]_{q_{kk}} = 0\}$, which we can rewrite into the form

$$1 + q_{kk} + q_{kk}^2 + \cdots + q_{kk}^{n-1} = 0. \qquad (42)$$

If the Q-cocycle condition (42) is satisfied, then the space $\mathcal{C}_Q = \{C_{k^n}\}$ of all constants belonging to the degenerate subspace $\mathcal{B}_Q$ consists of a nontrivial basic constant of the following form

$$C_{k^n} = e_{k^n}. \qquad (43)$$

2. Let $Q = \{k_1^{n-1}, k_2\}$ be a multiset of cardinality $n \geq 3$. Then the determinant (11) is given by

$$\det \mathrm{B}_{k_1^{n-1}k_2} = [n-1]_{q_{k_1k_1}}! \cdot \prod_{i=0}^{n-2} \left(1 - q_{k_1k_1}^i \sigma_{k_1k_2}\right)$$

(c.f. (13) for $n = 3$), where $[n-1]_{q_{k_1k_1}}!$ denotes the factorial, i.e., the product of all polynomials $[p]_{q_{k_1k_1}}$ with the form given in (4) for all $2 \leq p \leq n-1$. Then the Q-cocycle condition

(14) has the form $c_Q = \left\{1 - q_{k_1k_1}^{n-2}\sigma_{k_1k_2} = 0,\ 1 - q_{k_1k_1}^{i}\sigma_{k_1k_2} \neq 0,\ [p]_{q_{k_1k_1}} \neq 0\right\}$ for all $0 \leq i \leq n-3$, $2 \leq p \leq n-1$, which we rewrite in the form

$$1 - q_{k_1k_1}^{n-2}\sigma_{k_1k_2} = 0. \tag{44}$$

If the Q-cocycle condition (44) is satisfied, then the space $\mathcal{C}_Q = \{C_{k_1^{n-1}k_2}\}$ of all constants belonging to the degenerate subspace $\mathcal{B}_Q$ consists of a nontrivial basic constant of the form

$$C_{k_1^{n-1}k_2} = Y_{k_2k_1^{n-1}}, \tag{45}$$

where $Y_{k_2k_1^{n-1}}$ denotes an iterated $\boldsymbol{q}$-commutator, see (34). For more details, see [6, Degenerate cases] and [2, Appendix. Examples].

3. The relation of the constants in degenerate subspaces $\mathcal{B}_Q$ of the algebra $\mathcal{B}$ to the corresponding constants in the generic case

Considering that it is much more difficult to formulate an explicit formula describing the constants in the degenerate subspaces $\mathcal{B}_Q$ of the multiparametric algebra $\mathcal{B}$, the following questions naturally arise for arbitrary multiset Q: is it possible to determine the corresponding Q-cocycle condition (14) from (17) and is it then possible to determine constants in degenerate subspaces of the algebra $\mathcal{B}$ using the formula (38)?

Following the works of [2, 5] and also [6], we first consider that any multiset of cardinality n can be viewed as the set of the same cardinality n in which some of its elements are repeated. Suppose, then, that $Q = \{l_1, \ldots, l_n\} = \left\{k_1^{n_1}, \ldots, k_m^{n_m}, \ldots, k_p^{n_p}\right\}$ is a multiset of cardinality $n = n_1 + \cdots + n_p$, where $k_i \neq k_j$ for each $1 \leq i < j \leq p$ and there is at least one n_m such that $n_m \neq 1$. We recall that n_m is considered as the repetition frequency of the element k_m in the multiset Q. Then we define the submultiset Q_{k_m}, $1 \leq m \leq p$, by removing one copy of k_m from the multiset Q, i.e., $Q_{k_m} = Q\backslash\{k_m\} = \left\{k_1^{n_1}, \ldots, k_m^{n_m-1}, \ldots, k_p^{n_p}\right\}$. In agreement with the introduced notation $\widehat{Q}$ for the set of all unique permutations of the multiset Q, we denote by $\widehat{Q}_{k_m}$ the set of all unique permutations of the multiset Q_{k_m}. Then we define the functions $a\colon \widehat{Q} \to \mathbb{C}\backslash\{0\}$ and $b_{k_m}\colon \widehat{Q}_{k_m} \to \mathbb{C}\backslash\{0\}$, $1 \leq m \leq p$ by

$$a\left(j_1 \ldots j_n\right) = q_{j_nj_1} \cdots q_{j_nj_{n-1}}, \qquad j_1 \ldots j_n \in \widehat{Q}, \tag{46}$$

$$b_{k_m}\left(j_1 \ldots \widehat{k_m} \ldots j_n\right) = q_{k_mj_n} \cdot a\left(k_m\, j_1 \ldots \widehat{k_m} \ldots j_n\right) \qquad j_1 \ldots \widehat{k_m} \ldots j_n \in \widehat{Q}_{k_m} \tag{47}$$

which are called commutation factors (c.f. [2]). We note here that we define b_{k_m} only for distinct elements k_m of the multiset Q, hence we write $1 \leq m \leq p$. In other words, by $k_m^{n_m}$ we mean that the element k_m of Q is repeated n_m times, so for all $k_m^{n_m}$ we get only one multiset Q_{k_m} and also one function b_{k_m}. Thus, in (47), one k_m of $k_m^{n_m}$ is deleted (it does not matter which k_m of $k_m^{n_m}$). Note that we can write (47) as follows

$$b_{k_m}\left(j_1 \ldots \widehat{k_m} \ldots j_n\right) = \sigma_{k_mj_n} q_{j_nj_1} \cdots \widehat{q_{j_nk_m}} \cdots q_{j_nj_{n-1}}, \tag{48}$$

where we have used (46) and the identity $\sigma_{ij} = q_{ij}q_{ji}$. We emphasize, the factors $a\left(j_1 \ldots j_m\right)$ and $b_{j_1}\left(j_2 \ldots j_m\right)$ occur in the expressions $q_{j_mj_1} \cdots q_{j_mj_{m-1}}$ and $\sigma_{j_1j_m} q_{j_mj_2} \cdots q_{j_mj_{m-1}}$ of the action of the simpler operators $T_{m,1}$ and $T_1^2\, T_{m,2}$, $2 \leq m \leq n$ on $\mathcal{B}_Q$, given by (6) and (10). Similar to [6], in the following we will briefly discuss the $\langle t_{m,1}\rangle$-orbit and the $\langle t_1^2\, t_{m,2}\rangle$-orbit $(2 \leq m \leq n)$ on $\mathcal{B}_Q$ generated by $e_{\underline{j}}$ for all $\underline{j} = j_1 \ldots j_n \in \widehat{Q}$ with the motivation to explain their connection with the Q-cocyclic conditions (14) and (17) in the degenerate and generic cases. Note that the term used here for the $\langle t_1^2\, t_{m,2}\rangle$-orbit on $\mathcal{B}_Q$ is equal to the term $\langle t_{2,1}^2 t_{i,2}\rangle$-orbit on $\mathcal{B}_Q$ in [6] for the same orbit on $\mathcal{B}_Q$ when $m = i$, where

$t_1^2\, t_{m,2} = t_{2,1}^2 t_{m,2}$. It should be noted that among the orbits we distinguish the long and the short singular orbits, which we call by the common name singular orbits.

Let us denote by $\mathcal{B}_Q^{(j_1 j_2 \dots j_m) j_{m+1} \dots j_n} := span_{\mathbb{C}} \left\{ e_{t_{m,1}^{\alpha} \cdot \underline{j}} \mid 0 \leq \alpha \leq m-1 \right\}$ the $\langle t_{m,1} \rangle$-orbit on $\mathcal{B}_Q$ generated by $e_{\underline{j}}$, where

$$e_{t_{m,1} \cdot \underline{j}} = e_{j_{t_{1,m}(1)} \cdots j_{t_{1,m}(n)}}$$

for each $\underline{j} = j_1 \dots j_n \in \widehat{Q}$ (c.f. [5]); $t_{1,m}$ denotes the inverse of $t_{m,1}$. These orbits are in one-to-one correspondence with cyclic $t_{m,1}$-equivalence classes $(j_1 j_2 \dots j_m) j_{m+1} \dots j_n$ of sequences $\underline{j} \in \widehat{Q}$, see [6]. Here we considered that $\langle t_{m,1} \rangle = \{id, t_{m,1}, (t_{m,1})^2, \dots, (t_{m,1})^{m-1}\}$ is the cyclic subgroup of the symmetric group S_n generated by the cycle $t_{m,1} = (1\,2 \dots m) \in S_n$. Then for each $1 \leq m \leq n$ we obtain that $T_{m,1} \left(e_{t_{m,1}^{\alpha} \cdot \underline{j}} \right) = c_\alpha \, e_{t_{m,1}^{\alpha+1} \cdot \underline{j}}$, $0 \leq \alpha \leq m-1$, where $c_0 = a\,(j_1 \dots j_m)$, $c_1 = a\,(j_m \dots j_{m-1})$, $c_2 = a\,(j_{m-1} \dots j_{m-2}), \dots, c_{m-2} = a\,(j_3 \dots j_2)$, $c_{m-1} = a\,(j_2 \dots j_1)$, see (46). Therefore, we obtain that $T_{m,1} | \mathcal{B}_Q^{(j_1 j_2 \dots j_m) j_{m+1} \dots j_n}$ is a cyclic operator such that

$$\det \left(\mathbf{I} - \mathbf{T}_{m,1} | \mathcal{B}_Q^{(j_1 j_2 \dots j_m) j_{m+1} \dots j_n} \right) = 1 - \prod_{0 \leq \alpha \leq m-1} c_\alpha .$$

Here we denote by $\mathbf{T}_{m,1}$ the corresponding matrix of the operator $T_{m,1}$ in the monomial basis of the subspace $\mathcal{B}_Q$ and $\mathbf{I}$ is the unit matrix corresponding to the operator $T_{1,1} = id$. A $\langle t_{m,1} \rangle$-orbit on $\mathcal{B}_Q$, $|Q| = n$ is thus singular if

$$1 - \prod_{0 \leq \alpha \leq m-1} c_\alpha = 0 \tag{49}$$

and it is long singular if $m = n$, where (49) reduces to the form

$$1 - \prod_{1 \leq a \neq b \leq n} q_{l_a l_b} = 0, \tag{50}$$

where the product runs over all $n \cdot (n-1)$ pairs $l_a l_b$ of elements from the given multiset Q. The identity (49) represents the Q-cocycle condition (14). On the other hand, the identity (50) represents the corresponding Q-cocycle condition (17) in the generic case (Q is a set of cardinality n), since in these cases all orbits are long.

Now consider $\mathcal{B}_Q^{j_1 (j_2 j_3 \dots j_m) j_{m+1} \dots j_n} = span_{\mathbb{C}} \left\{ e_{t_{m,2}^{\beta} \cdot \underline{j}} \mid 0 \leq \beta \leq m-2 \right\}$ the $\langle t_1^2\, t_{m,2} \rangle$-orbit on $\mathcal{B}_Q$, which are in one-to-one correspondence with cyclic $t_{m,2}$-equivalence classes $j_1 (j_2 \dots j_m) j_{m+1} \dots j_n$ of sequences $\underline{j} \in \widehat{Q}$, where $\langle t_{m,2} \rangle$, $2 \leq m \leq n$ is the cyclic subgroup of $S_1 \times S_{n-1}$ generated by the cycle $t_{m,2} = (2\,3 \dots m) \in S_1 \times S_{n-1}$. Then for each $2 \leq m \leq n$ we obtain that $T_1^2\, T_{m,2} \left(e_{t_{m,2}^{\beta} \cdot \underline{j}} \right) = d_\beta \, e_{t_{m,2}^{\beta+1} \cdot \underline{j}}$, $0 \leq \beta \leq m-2$, where $d_0 = b_{j_1} \left(\widehat{j_1} j_2 \dots j_m \right)$, $d_1 = b_{j_1} \left(\widehat{j_1} j_m \dots j_{m-1} \right)$, $d_2 = b_{j_1} \left(\widehat{j_1} j_{m-1} \dots j_{m-2} \right)$, $\dots, d_{m-3} = b_{j_1} \left(\widehat{j_1} j_4 \dots j_3 \right)$, $d_{m-2} = b_{j_1} \left(\widehat{j_1} j_3 \dots j_2 \right)$, see (47) and also (48). Then we obtain that $T_1^2\, T_{m,2} | \mathcal{B}_Q^{j_1 (j_2 j_3 \dots j_m) j_{m+1} \dots j_n}$ is a cyclic operator such that

$$\det \left(\mathbf{I} - \mathbf{T}_1^2\, \mathbf{T}_{m,2} | \mathcal{B}_Q^{j_1 (j_2 j_3 \dots j_m) j_{m+1} \dots j_n} \right) = 1 - \prod_{0 \leq \beta \leq m-2} d_\beta ,$$

where $\mathbf{T}_1^2\, \mathbf{T}_{m,2}$, $2 \leq m \leq n$ denotes the corresponding matrix of the operator $T_1^2\, T_{m,2}$ in the monomial basis of the subspace $\mathcal{B}_Q$. Thus, a $\langle t_1^2\, t_{m,2} \rangle$-orbit on $\mathcal{B}_Q$ is singular if

$$1 - \prod_{0 \leq \beta \leq m-2} d_\beta = 0 \tag{51}$$

and it is long singular if (51) reduces to (50). Then we conclude that a $\langle t_{m,1}\rangle$-orbit or a $\langle t_1^2\, t_{m,2}\rangle$-orbit on $\mathcal{B}_Q$ is short singular if the left-hand side of (49) or the left-hand side of (51) is a nontrivial divisor of the left-hand side of (50). Considering the introduced notation of the corresponding matrices of the given operators with respect to the monomial basis of a subspace $\mathcal{B}_Q$ of the algebra $\mathcal{B}$, see Remark 2.2, we note that under the Q-cocycle condition it is sufficient to consider only the matrices $(\mathbf{I}-\mathbf{T}_{n,1})$ and $\big(\mathbf{I}-\mathbf{T}_1^2\,\mathbf{T}_{n,2}\big)$. If these matrices are transformed into block diagonal matrices, then the number of blocks in a block diagonal matrix is equal to the number of distinct singular orbits on $\mathcal{B}_Q$. The difference between the number of distinct singular $\langle t_1^2\, t_{n,2}\rangle$-orbits and $\langle t_{n,1}\rangle$-orbits on $\mathcal{B}_Q$ is equal to the dimension of $\mathcal{C}_Q$, the space of all constants belonging to the subspace $\mathcal{B}_Q$ of the algebra $\mathcal{B}$, see [6]. In the generic case this difference is equal to $(n-2)!$, see Remark 2.6. It follows that there is a relation between the Q-cocycle conditions (14) and (17) in the corresponding degenerate and generic subspaces of the algebra $\mathcal{B}$, which leads us to conclude that we can also establish a relation between constants in the corresponding degenerate and generic subspaces of the algebra $\mathcal{B}$.

In the interest of clearer notation and a more sophisticated notation of multiset and set, in what follows we denote by Q' a multiset and by Q a set, assuming that Q and Q' have the same cardinality. We recall that any multiset Q' of cardinality n can be obtained from the set of the same cardinality n by specializing the elements of the set Q such that some of them are repeated. In this case, with the given specialization, we can obtain from the set $\widehat{Q}$ of all unique permutations of the set Q the set $\widehat{Q'}$ of all unique permutations of the multiset Q' by removing the elements that are repeated, which is explained in the following example.

Example 3.1. *Considering the set $Q=\{l_1,l_2,l_3\}$ of cardinality 3, we obtain, that the set of all unique permutations of the set Q is given by $\widehat{Q}=\{j_1j_2j_3, j_1j_3j_2, j_3j_1j_2, j_3j_2j_1, j_2j_3j_1, j_2j_1j_3\}$. On the other hand, if we specialize the elements of the set Q such that $k_1=l_1=l_2$ and $k_2=l_3$, then we obtain the multiset $Q'=\{k_1^2,k_2\}$ of cardinality 3. Moreover, if we apply the given specialization to the elements of the set $\widehat{Q}$, we obtain the set $\widehat{Q'}=\{i_1i_1i_2, i_1i_2i_1, i_2i_1i_1, i_2i_1i_1, i_1i_2i_1, i_1i_1i_2\}$, from which we get $\widehat{Q'}=\{i_1i_1i_2, i_1i_2i_1, i_2i_1i_1\}$, where we used $i_1=j_1=j_2$, $i_2=j_3$. Thus we can realize the monomial basis $\mathtt{B}_{Q'}=\{e_{i_1i_1i_2}, e_{i_1i_2i_1}, e_{i_2i_1i_1}\}$ of a subspace $\mathcal{B}_{Q'}$ from the monomial basis $\mathtt{B}_Q=\{e_{j_1j_2j_3}, e_{j_1j_3j_2}, e_{j_3j_1j_2}, e_{j_3j_2j_1}, e_{j_2j_3j_1}, e_{j_2j_1j_3}\}$ of a subspace $\mathcal{B}_Q$. Considering the obtained matrices $\mathbf{B}_Q$ and $\mathbf{B}_{Q'}$ from Example 2.3, we note that with the given specialization $k_1=l_1=l_2$, $k_2=l_3$ the matrix $\mathbf{B}_{Q'}$ is reduced from the matrix $\mathbf{B}_Q$. At the same time, we note that some elements of the matrix $\mathbf{B}_{Q'}$ are polynomials obtained by adding the corresponding elements (monomials) of the matrix $\mathbf{B}_Q$ to the same reduced elements of the monomial basis. In particular, it is easy to see that from*

$$\mathbf{B}_Q\, e_{j_1j_2j_3}=(\mathbf{T}_{3,1}+\mathbf{T}_{2,1}+\mathbf{I})\, e_{j_1j_2j_3}=q_{j_3j_2}q_{j_3j_2}\, e_{j_3j_1j_2}+q_{j_2j_1}\, e_{j_2j_1j_3}+e_{j_1j_2j_3}$$

by the given specialization $k_1=l_1=l_2$, $k_2=l_3$, which is equivalent to $i_1=j_1=j_2$, $i_2=j_3$, we obtain

$$\begin{aligned}\mathbf{B}_{Q'}\, e_{i_1i_1i_2}&=(\mathbf{T}_{3,1}+\mathbf{T}_{2,1}+\mathbf{I})\, e_{i_1i_1i_2}=q_{i_2i_1}q_{i_2i_1}\, e_{i_2i_1i_1}+q_{i_1i_1}\, e_{i_1i_1i_2}+e_{i_1i_1i_2}\\&=q_{i_2i_1}^2\, e_{i_2i_1i_1}+(1+q_{i_1i_1})\, e_{i_1i_1i_2}.\end{aligned}$$

Then it follows from (12), given the specialization, that

$$\begin{aligned}\det\mathbf{B}_{Q'}&=(1-\sigma_{k_1k_1})\cdot(1-\sigma_{k_1k_2})\cdot(1-\sigma_{k_1k_2})\cdot(1-\sigma_{k_1k_1k_2})\\&=\big(1-q_{k_1k_1}^2\big)\cdot(1-\sigma_{k_1k_2})^2\cdot\big(1-q_{k_1k_1}^2\sigma_{k_1k_2}^2\big)\\&=(1-q_{k_1k_1})\cdot(1+q_{k_1k_1})\cdot(1-\sigma_{k_1k_2})^2\cdot(1-q_{k_1k_1}\sigma_{k_1k_2})\cdot(1+q_{k_1k_1}\sigma_{k_1k_2})\end{aligned}$$

is a multiple of $\det\mathbf{B}_{Q'}$ from (13). In other words, $\det\mathbf{B}_{Q'}$ from (13) is a nontrivial divisor of $\det\mathbf{B}_{Q'}$, which we have obtained here by the given specialization. We emphasize that $\sigma_{ij}=q_{ij}q_{ji}$ implies $\sigma_{k_1k_1}=q_{k_1k_1}^2$, and from (24) and (23) it follows $\sigma_{k_1k_1k_2}=\sigma_{k_1k_1}\sigma_{k_1k_2}\sigma_{k_1k_2}=q_{k_1k_1}^2\sigma_{k_1k_2}^2$.

Similar to Example 3.1, we can obtain the Q'-cocycle conditions in degenerate cases from an appropriate "generic" Q-cocycle condition by using a certain specialization procedure in which it turns out that the Q'-cocycle condition is a nontrivial divisor of the corresponding Q'-cocycle condition obtained by the certain specialization. In this way, any constant in degenerate subspaces $\mathcal{B}_Q$ of the multiparametric algebra $\mathcal{B}$ can be constructed from those in the generic case by a certain specialization procedure, which we will discuss in more detail below. Therefore, by applying a certain specialization procedure, we explain the determination of the corresponding Q'-cycle condition (14) from the Q-cycle condition (17) and then the determination of constants in degenerate subspaces $\mathcal{B}_{Q'}$ of $\mathcal{B}$ using the formula (38), where starting from the set $Q = \{l_1, \dots, l_n\}$ of cardinality n by certain specializations of its elements, first a multiset $Q' = \{k^n\}$ and then a multiset $Q' = \{k_1^{n-1}, k_2\}$ of the same cardinality is obtained.

3.1. Multiset $Q' = \{k^n\},\ n \geq 2$

Proposition 3.2. *Let $Q' = \{k^n\}$ be a multiset of cardinality $n \geq 2$. The nontrivial basic constant (43) in the degenerate subspace $\mathcal{B}_{Q'}$ and also the Q'-cocycle condition (42) can be constructed from the generic case by a certain specialization procedure.*

Proof. Let $Q = \{l_1, \dots, l_n\}$ be a set of cardinality $n \geq 2$. Then the Q-cocycle condition is given by (17) and the nontrivial basic constants by (38). Specializing the elements of the set Q such that all l_j are equal to k, one can consider the set $Q = \{l_1, \dots, l_n\}$ as a multiset $Q' = \{k^n\}$ in which the element $k \in \mathcal{N}$ is repeated n times. In this case, the corresponding Q-cocycle condition (17), given by $1 - \sigma_Q = 0$ and $1 - \sigma_T \neq 0$ for all $T \subset Q$, here has the form $1 - \sigma_{k^n} = 0$, $1 - \sigma_{k^i} \neq 0$ for all $2 \leq i \leq n-1$, or shorter $1 - \sigma_{k^n} = 0$. Moreover, by applying (24) and (23), we obtain $\sigma_{k^m} = (q_{kk}^2)^{\binom{m}{2}} = (q_{kk})^{m\cdot(m-1)} = q_{kk}^{m^2-m}$ for all $2 \leq m \leq n$, from which it follows that

$$
\begin{aligned}
1 - \sigma_{k^n} = 1 - q_{kk}^{n^2-n} &= (1 - q_{kk}) \cdot \left(1 + q_{kk} + q_{kk}^2 + \cdots + q_{kk}^{n^2-n-1}\right) \\
&= (1 - q_{kk}) \cdot (1 + q_{kk} + \cdots + q_{kk}^{n-1}) \cdot \left(1 + q_{kk}^{n} + q_{kk}^{2n} + \cdots + q_{kk}^{(n-2)\cdot n}\right),
\end{aligned}
$$

which we can write in the following form

$$
1 - \sigma_{k^n} = (1 - q_{kk}) \cdot [n]_{q_{kk}} \cdot \left(1 + \sum_{j=1}^{n-2} q_{kk}^{j\cdot n}\right). \tag{52}
$$

Then $1 - \sigma_{k^n} = 0$ if and only if $1 - q_{kk} = 0$ or $[n]_{q_{kk}} = 0$ or $1 + \sum_{j=1}^{n-2} q_{kk}^{j\cdot n} = 0$, so it is easy to see that the left-hand side of (42) is a nontrivial divisor of (52), see also (4). So in the following we will show that the Q'-cocycle condition $1 - \sigma_{k^n} = 0$ can be reduced to $[n]_{q_{kk}} = 0$ with $1 - q_{kk} \neq 0$ and $1 + \sum_{j=1}^{n-2} q_{kk}^{j\cdot n} \neq 0$. In particular, for $n = 2$ see Remark 3.3 below. On the other hand, if all $l_j \in Q$ are equal to k, then all $(n-2)!$ nontrivial basic constants from (38) reduce to a constant C_{k^n}, where all $(n-1)!$ iterated $\boldsymbol{q}$-commutators are equal to the iterated $\boldsymbol{q}$-commutator Y_{k^n} (c.f. (34)) given by $Y_{k^n} = \left(\prod_{s=1}^{n-1}(1 - q_{kk}^s)\right) e_{k^n}$, $n \geq 2$, which by applying the property $1 - q_{kk}^s = (1 - q_{kk}) \cdot (1 + q_{kk} + \cdots + q_{kk}^{s-1}) = (1 - q_{kk}) \cdot [s]_{q_{kk}}$, can be written as follows

$$
Y_{k^n} = \left(\prod_{s=1}^{n-1} (1 - q_{kk}) \cdot [s]_{q_{kk}}\right) e_{k^n}. \tag{53}
$$

Here we consider that $e_{k^n} = e_k^n$ and that $n \geq 2$. We emphasize that (4) implies $[1]_{q_{kk}} = 1$. Moreover, from (38), considering the given specialization, we obtain that the denominators of all fractions of the obtained constant C_{k^n} are equal to the product $\prod_{i=2}^{n-1}(1 - \sigma_{k^i})$, so we express C_{k^n} in terms of a fraction, whose numerator is equal to the product of the factors $\left(1 + \sum_{j=1}^{m-2} q_{kk}^{j\cdot m}\right)$ for all $3 \leq m \leq n$. In other words, we obtain

$$C_{k^n} = \frac{\prod_{m=3}^{n}\left(1+\sum_{j=1}^{m-2} q_{kk}^{j\cdot m}\right)}{\prod_{i=2}^{n-1}(1-\sigma_{k^i})} \cdot Y_{k^n}, \tag{54}$$

where by applying (52) for $n = i$ and the identity (53) we further obtain

$$\begin{aligned} C_{k^n} &= \frac{\prod_{m=3}^{n}\left(1+\sum_{j=1}^{m-2} q_{kk}^{j\cdot m}\right)\cdot \prod_{s=1}^{n-1}(1-q_{kk})\cdot [s]_{q_{kk}}}{\prod_{i=2}^{n-1}(1-q_{kk})\cdot [i]_{q_{kk}}\cdot\left(1+\sum_{j=1}^{i-2} q_{kk}^{j\cdot i}\right)}\, e_{k^n} \\ &= \left(\frac{\left(1+\sum_{j=1}^{n-2} q_{kk}^{j\cdot n}\right)\cdot(1-q_{kk})\cdot(1-q_{kk})\cdot[2]_{q_{kk}}}{(1-q_{kk})\cdot[2]_{q_{kk}}}\cdot \prod_{m=3}^{n-1}\frac{\left(1+\sum_{j=1}^{m-2} q_{kk}^{j\cdot m}\right)\cdot(1-q_{kk})\cdot[m]_{q_{kk}}}{(1-q_{kk})\cdot[m]_{q_{kk}}\cdot\left(1+\sum_{j=1}^{m-2} q_{kk}^{j\cdot m}\right)}\right) e_{k^n} \end{aligned}$$

from which it follows directly

$$C_{k^n} = (1-q_{kk})\cdot\left(1+q_{kk}^{n}+q_{kk}^{2n}+\cdots+q_{kk}^{(n-2)\cdot n}\right) e_{k^n}. \tag{55}$$

Here we have considered that $[1]_{q_{kk}} = 1$ and that the sum $1+\sum_{j=1}^{i-2} q_{kk}^{j\cdot i}$ is equal to one if $i = 2$. If we compare (52) with (55), it is easy to see that $1-\sigma_{k^n}$ and C_{k^n} consist of the same factors $1-q_{kk}$ and $1+q_{kk}^{n}+q_{kk}^{2n}+\cdots+q_{kk}^{(n-2)\cdot n}$. First, we exploit the fact that from the Q-cocycle condition $1-\sigma_Q = 0$ (c.f. (17)) with (52) it follows the Q'-cocycle condition, given by $1-\sigma_{k^n} = 0$, and then there is a constant (c.f. (55)) in the degenerate subspace $\mathcal{B}_{k^n}$ if and only if the Q'-cocycle condition $1-\sigma_{k^n} = 0$ is satisfied, then we obtain

1. if $1-q_{kk} = 0$ or $1+q_{kk}^{n}+q_{kk}^{2n}+\cdots+q_{kk}^{(n-2)\cdot n} = 0$, then the Q'-cocycle condition is satisfied and the constant C_{k^n} is zero (i.e., a trivial constant);

2. if $1-q_{kk} \neq 0$ and $1+q_{kk}^{n}+q_{kk}^{2n}+\cdots+q_{kk}^{(n-2)\cdot n} \neq 0$, then the Q'-cocycle condition is given by $[n]_{q_{kk}} = 0$ (i.e., $1+q_{kk}+\cdots+q_{kk}^{n-1} = 0$, see (52) and (4)) and in this case C_{k^n} is a nontrivial constant;

3. if $1-q_{kk} \neq 0$, $1+q_{kk}^{n}+q_{kk}^{2n}+\cdots+q_{kk}^{(n-2)\cdot n} \neq 0$ and $[n]_{q_{kk}} \neq 0$, then the Q'-cocycle condition is not satisfied, so there are no constants in $\mathcal{B}_{k^n}$.

From this we can conclude that the obtained Q'-cocycle condition $1-\sigma_{k^n} = 0$ (c.f. (52)) can be reduced to $[n]_{q_{kk}} = 0$, compare with (42). On the other hand, we emphasize that q_{ij}'s are complex numbers, which means that the constant C_{k^n} given in (55) is a multiple of e_{k^n} (c.f. (43)). Thus, under the Q'-cocycle condition $[n]_{q_{kk}} = 0$, it follows from the constant C_{k^n} that e_{k^n} is a nontrivial basic constant in the space $\mathcal{C}_{k^n}$ of all constants belonging to the degenerate subspace $\mathcal{B}_{k^n}$. In this way, we proved that for the multiset $Q' = \{k^n\}$, $n \geq 2$ the Q'-cocycle condition as well as the nontrivial basic constant in the degenerate subspace $\mathcal{B}_{Q'}$ can be constructed from the generic case by a certain specialization procedure. □

Remark 3.3. *For $n = 2$ it follows from (52) that the Q'-cocycle condition $1 - \sigma_{k^2} = 0$ is of the form $(1 - q_{kk}) \cdot [2]_{q_{kk}} = 0$, that is, $(1 - q_{kk}) \cdot (1 + q_{kk}) = 0$ when we apply (4), and it is a multiple of the corresponding Q'-cocycle condition $1 + q_{kk} = 0$ from (42) for $n = 2$. We note here that in this case the sum $1 + q_{kk}^n + \cdots + q_{kk}^{(n-2)\cdot n}$ from (52) is equal to one. In addition, both the numerator and the denominator of the fraction of (54) are equal to one, so the constant in the degenerate subspace $\mathcal{B}_{k^2}$ under the Q'-cocycle condition is given by $C_{k^2} = Y_{k^2} = (1 - q_{kk})\, e_{k^2}$, which is also consistent with (55), where for $n = 2$ the sum $1 + q_{kk}^n + \cdots + q_{kk}^{(n-2)\cdot n}$ equals one. Similar to the above, the Q'-cocycle condition $1 - \sigma_{k^2} = 0$ is satisfied if and only if $1 - q_{kk} = 0$ or $1 + q_{kk} = 0$, where the obtained constant C_{k^2} is trivial if $1 - q_{kk} = 0$ and nontrivial if $1 - q_{kk} \neq 0$ and $1 + q_{kk} = 0$ and in this case e_{k^2} is a nontrivial basic constant in $\mathcal{B}_{k^2}$.*

Example 3.4. *Let us take $n = 3$. Then by applying the following specialization $k = l_1 = l_2 = l_3$ we can consider the set $Q = \{l_1, l_2, l_3\}$ as a multiset $Q' = \{k^3\}$. Then the Q-cocycle condition $1 - \sigma_{l_1 l_2 l_3} = 0$ reduces to the Q'-cocycle condition $1 - \sigma_{k^3} = 0$, which has the form*

$$(1 - q_{kk}) \cdot [3]_{q_{kk}} \cdot (1 + q_{kk}^3) = 0,$$

see (52). On the other hand, with the introduced specialization, a nontrivial basic constant in the generic subspace $\mathcal{B}_Q$ of the algebra $\mathcal{B}$, given by (40) from Example 2.9, reduces to the form

$$C_{k^3} = \frac{1 + q_{kk}^3}{1 - \sigma_{k^2}} \cdot Y_{k^3} = (1 - q_{kk}) \cdot (1 + q_{kk}^3)\, e_{k^3},$$

see (55), where we applied $1 - \sigma_{k^2} = (1 - q_{kk}) \cdot [2]_{q_{kk}}$ and $Y_{k^3} = (1 - q_{kk})^2 \cdot [2]_{q_{kk}}$. Considering the obtained Q'-cocycle condition and the obtained constant C_{k^3}, we conclude that a constant C_{k^3} is trivial (zero) in the degenerate subspace $\mathcal{B}_{k^3}$ if $1 - q_{kk} = 0$ or $1 + q_{kk}^3 = 0$. On the other hand, if $1 - q_{kk} \neq 0$ and $1 + q_{kk}^3 \neq 0$, but $[3]_{q_{kk}} = 0$, then C_{k^3} is a nontrivial constant in $\mathcal{B}_{k^3}$. Thus, in this case e_{k^3} is a nontrivial basic constant in $\mathcal{B}_{k^3}$, where the Q'-cocycle condition $1 - \sigma_{k^3} = 0$ is reduced to $[3]_{q_{kk}} = 0$.

Example 3.5. *Let us now take $n = 4$. Then, similarly to Example 3.4, by applying the specialization $k = l_1 = l_2 = l_3 = l_4$, the set $Q = \{l_1, l_2, l_3, l_4\}$ corresponds to a multiset $Q' = \{k^4\}$. Then the Q-cocycle condition $1 - \sigma_{l_1 l_2 l_3 l_4} = 0$ reduces to the Q'-cocycle condition*

$$(1 - q_{kk}) \cdot [4]_{q_{kk}} \cdot (1 + q_{kk}^4 + q_{kk}^8) = 0$$

c.f. (52). We note that in the generic case, under the Q-cocycle condition $1 - \sigma_{l_1 l_2 l_3 l_4} = 0$, there are two nontrivial basic constants (see Example 2.10), which reduce to the same constant with the introduced specialization

$$C_{k^4} = \frac{(1 + q_{kk}^3) \cdot (1 + q_{kk}^4 + q_{kk}^8)}{(1 - \sigma_{k^2}) \cdot (1 - \sigma_{k^3})} \cdot Y_{k^4} = (1 - q_{kk}) \cdot (1 + q_{kk}^4 + q_{kk}^8)\, e_{k^4},$$

c.f. (55), where we have used $1 - \sigma_{k^2} = (1 - q_{kk}) \cdot [2]_{q_{kk}}$ and $1 - \sigma_{k^3} = (1 - q_{kk}) \cdot [3]_{q_{kk}} \cdot (1 + q_{kk}^3)$ and also $Y_{k^4} = (1 - q_{kk}) \cdot (1 - q_{kk}) \cdot [2]_{q_{kk}} \cdot (1 - q_{kk}) \cdot [3]_{q_{kk}}$. Then we obtain that C_{k^4} is zero (a trivial constant) in the degenerate subspace $\mathcal{B}_{k^4}$, if $1 - q_{kk} = 0$ or $1 + q_{kk}^4 + q_{kk}^8 = 0$ and that it is a nontrivial constant if $1 - q_{kk} \neq 0$ and $1 + q_{kk}^4 + q_{kk}^8 \neq 0$ but $[4]_{q_{kk}} = 0$. Thus, the Q'-cocycle condition $1 - \sigma_{k^4} = 0$ reduces to $[4]_{q_{kk}} = 0$ under which e_{k^4} is a nontrivial basic constant in $\mathcal{B}_{k^4}$.

3.2. Multiset $Q' = \{k_1^{n-1}, k_2\}$, $n \geq 3$

Proposition 3.6. *Let $Q' = \{k_1^{n-1}, k_2\}$ be a multiset of cardinality $n \geq 3$. The nontrivial basic constant (45) in the degenerate subspace $\mathcal{B}_{Q'}$ and also the Q'-cocycle condition (44) can be constructed from the generic case by a certain specialization procedure.*

Proof. Let $Q = \{l_1, \ldots, l_n\}$ be a set of cardinality $n \geq 3$. By specializing the elements of the set Q such that one element is equal to k_2 and all remaining $n - 1$ elements are equal to k_1, we obtain from the set $Q = \{l_1, \ldots, l_n\}$ the multiset $Q' = \{k_1^{n-1}, k_2\}$, $k_1 \neq k_2$, in which the element $k_1 \in \mathcal{N}$ is repeated $n-1$ times. The corresponding Q-cocycle condition $1 - \sigma_Q = 0$, $1 - \sigma_T \neq 0$, for all $T \subset Q$ here has the form $1 - \sigma_{k_1^{n-1}k_2} = 0$, $1 - \sigma_{k_1^m k_2} \neq 0$, $1 - \sigma_{k^i} \neq 0$ for all $1 \leq m \leq n-2$, $2 \leq i \leq n-1$, where of particular interest is the factor $1 - \sigma_{k_1^{n-1}k_2} = 0$, which we usually call the Q'-cocycle condition. So, if we consider

$$\begin{aligned} 1 - \sigma_{k_1^{n-1}k_2} &= 1 - q_{k_1k_1}^{(n-1)\cdot(n-2)} \sigma_{k_1k_2}^{n-1} = 1 - (q_{k_1k_1}^{n-2}\sigma_{k_1k_2})^{n-1} \\ &= \left(1 - q_{k_1k_1}^{n-2}\sigma_{k_1k_2}\right) \cdot \left(1 + q_{k_1k_1}^{n-2}\sigma_{k_1k_2} + \cdots + (q_{k_1k_1}^{n-2}\sigma_{k_1k_2})^{n-2}\right) \end{aligned}$$

for $n \geq 3$, which we write in the following form

$$1 - \sigma_{k_1^{n-1}k_2} = \left(1 - q_{k_1k_1}^{n-2}\sigma_{k_1k_2}\right) \cdot \left(1 + \sum_{j=1}^{n-2} \left(q_{k_1k_1}^{n-2}\sigma_{k_1k_2}\right)^j\right). \tag{56}$$

If we compare (56) with (44), we can easily see that (56) is a multiple of the left-hand side of (44), which leads us to conclude that the obtained Q'-cocycle condition $1 - \sigma_{k_1^{n-1}k_2} = 0$ (c.f. (56)) can be reduced to $1 - q_{k_1k_1}^{n-2}\sigma_{k_1k_2} = 0$ with $1 + \sum_{j=1}^{n-2} \left(q_{k_1k_1}^{n-2}\sigma_{k_1k_2}\right)^j \neq 0$, see (44). Taking into account the introduced specialization of the elements of the set Q, we obtain that all $(n-2)!$ nontrivial basic constants $(n \geq 3)$ from (38) are reduced to a constant $C_{k_1^{n-1}k_2}$, which after further calculations we can write in the form

$$C_{k_1^{n-1}k_2} = -\frac{q_{k_2k_1}^{n-1}}{[n-1]_{q_{k_1k_1}}} \cdot \prod_{m=1}^{n-2} \frac{\left(1 + \sum_{j=1}^{m} \left(q_{k_1k_1}^{m}\sigma_{k_1k_2}\right)^j\right)}{\left(1 - \sigma_{k_1^m k_2}\right)} \cdot Y_{k_2k_1^{n-1}}, \tag{57}$$

where the denominators of the given fractions on the right-hand side of the formula (57) are nonzero, since the Q'-cocycle condition implies that $1 - \sigma_{k_1^{n-1}k_2} = 0$ but $1 - \sigma_{k_1^m k_2} \neq 0$ for all $1 \leq m \leq n-2$ and also $1 - \sigma_{k^i} \neq 0$ for all $2 \leq i \leq n-1$. Note that by using (52) from $1 - \sigma_{k^{n-1}} \neq 0$ it follows $[n-1]_{q_{k_1k_1}} \neq 0$. If we now apply (56) for each $1 \leq m \leq n-2$, we obtain that the given constant $C_{k_1^{n-1}k_2}$ can be written as

$$\begin{aligned} C_{k_1^{n-1}k_2} &= -\frac{q_{k_2k_1}^{n-1}}{[n-1]_{q_{k_1k_1}}} \cdot \prod_{m=1}^{n-2} \frac{\left(1 + \sum_{j=1}^{m} \left(q_{k_1k_1}^{m}\sigma_{k_1k_2}\right)^j\right)}{\left(1 - q_{k_1k_1}^{m-1}\sigma_{k_1k_2}\right) \cdot \left(1 + \sum_{j=1}^{m-1} \left(q_{k_1k_1}^{m-1}\sigma_{k_1k_2}\right)^j\right)} \cdot Y_{k_2k_1^{n-1}} \\ &= -\frac{q_{k_2k_1}^{n-1}}{[n-1]_{q_{k_1k_1}}} \cdot \frac{\left(1 + \sum_{j=1}^{n-2} \left(q_{k_1k_1}^{n-2}\sigma_{k_1k_2}\right)^j\right)}{\prod_{m=1}^{n-2} \left(1 - q_{k_1k_1}^{m-1}\sigma_{k_1k_2}\right)} \cdot Y_{k_2k_1^{n-1}} \end{aligned}$$

which we further rewrite into the following form

$$C_{k_1^{n-1}k_2} = -\frac{q_{k_2k_1}^{n-1} \cdot \left(1 + \sum_{j=1}^{n-2} \left(q_{k_1k_1}^{n-2}\sigma_{k_1k_2}\right)^j\right)}{[n-1]_{q_{k_1k_1}} \cdot \prod_{m=0}^{n-3} \left(1 - q_{k_1k_1}^{m}\sigma_{k_1k_2}\right)} \cdot Y_{k_2k_1^{n-1}}. \tag{58}$$

We note that it follows from (56), the corresponding Q'-cocycle condition $1-\sigma_{k_1^{n-1}k_2}=0$ is given by $1-q_{k_1k_1}^{n-2}\sigma_{k_1k_2}=0$ and $1+\sum_{j=1}^{n-2}\left(q_{k_1k_1}^{n-2}\sigma_{k_1k_2}\right)^j=0$ and we also recall that there is a constant in the degenerate subspace $\mathcal{B}_{k_1^{n-1}k_2}$ if and only if the Q'-cocycle condition $1-\sigma_{k_1^{n-1}k_2}=0$ is satisfied. Thus, if $1+\sum_{j=1}^{n-2}\left(q_{k_1k_1}^{n-2}\sigma_{k_1k_2}\right)^j=0$, then the constant $C_{k_1^{n-1}k_2}$ (c.f. (58)) is a trivial constant and it is a nontrivial constant if

$$1-q_{k_1k_1}^{n-2}\sigma_{k_1k_2}=0, \qquad \sum_{j=0}^{n-2}\left(q_{k_1k_1}^{n-2}\sigma_{k_1k_2}\right)^j\neq 0, \tag{59}$$

which is equal to (44). From this we can conclude that the corresponding Q'-cocycle condition $1-\sigma_{k_1^{n-1}k_2}=0$ reduces to (59), under which the space $\mathcal{C}_{k_1^{n-1}k_2}$ of all constants belonging to the degenerate subspace $\mathcal{B}_{k_1^{n-1}k_2}$ consists of a constant $C_{k_1^{n-1}k_2}$ given by (58). We recall that q_{ij}'s are complex numbers, so the constant given by (58) is a multiple of an iterated $\boldsymbol{q}$-commutator $Y_{k_2k_1^{n-1}}$, see (45). Therefore, under the Q'-cocycle condition (59), it follows from (58) that an iterated $\boldsymbol{q}$-commutator $Y_{k_2k_1^{n-1}}$ is a nontrivial basic constant in space $\mathcal{C}_{k_1^{n-1}k_2}$. Thus we have proved that for the multiset $Q'=\{k_1^{n-1},k_2\}$, $n\geq 3$ the Q'-cocycle condition as well as the nontrivial basic constant in the degenerate subspace $\mathcal{B}_{Q'}$ can be constructed from the generic case by a certain specialization procedure. □

Example 3.7. *Let us consider the set $Q=\{l_1,l_2,l_3\}$ as a multiset $Q'=\{k_1^2,k_2\}$ using the specialization $k_1=l_1=l_2,\ k_2=l_3$. Then the Q-cocycle condition $1-\sigma_{l_1l_2l_3}=0$ reduces to the Q'-cocycle condition $1-\sigma_{k_1^2k_2}=0$, from which follows*

$$(1-q_{k_1k_1}\sigma_{k_1k_2})\cdot(1+q_{k_1k_1}\sigma_{k_1k_2})=0 \tag{60}$$

see also Example 3.1. In agreement with the given specialization, a nontrivial basic constant (40) of the generic $\mathcal{B}_Q$ has the form

$$C_{k_1^2k_2}=\frac{1}{1-\sigma_{k_1^2}}\cdot Y_{k_1^2k_2}+\frac{q_{k_2k_1}\sigma_{k_1k_2}}{1-\sigma_{k_1k_2}}\cdot Y_{k_1k_2k_1},$$

see Example 2.9, where by applying (34) we find that the iterated $\boldsymbol{q}$-commutators $Y_{k_1^2k_2}$, $Y_{k_1k_2k_1}$ are given by

$$\begin{aligned}Y_{k_1^2k_2}&=(1-q_{k_1k_1})\cdot(e_{k_1^2k_2}-q_{k_2k_1}^2e_{k_2k_1^2})\\ Y_{k_1k_2k_1}&=(1+q_{k_1k_1}\sigma_{k_1k_2})\,e_{k_1k_2k_1}-q_{k_2k_1}e_{k_2k_1^2}-q_{k_1k_1}q_{k_1k_2}e_{k_1^2k_2}.\end{aligned}$$

After further calculations we get

$$C_{k_1^2k_2}=-\frac{q_{k_2k_1}^2\cdot(1+q_{k_1k_1}\sigma_{k_1k_2})}{(1+q_{k_1k_1})\cdot(1-\sigma_{k_1k_2})}\cdot Y_{k_2k_1^2}. \tag{61}$$

It is easy to verify that $Y_{k_2k_1^2}=e_{k_2k_1^2}-q_{k_1k_2}(1+q_{k_1k_1})\,e_{k_1k_2k_1}+q_{k_1k_1}q_{k_1k_2}^2e_{k_1^2k_2}$ is obtained by applying (34). Then it follows directly from the obtained Q'-cocycle condition (60) and the obtained constant $C_{k_1^2k_2}$ (c.f. (61)) that a constant $C_{k_1^2k_2}$ is zero, if $1+q_{k_1k_1}\sigma_{k_1k_2}=0$ and is a nontrivial constant if $1+q_{k_1k_1}\sigma_{k_1k_2}\neq 0$, but $1-q_{k_1k_1}\sigma_{k_1k_2}=0$. In this case a constant $C_{k_1^2k_2}$ is a multiple of an iterated $\boldsymbol{q}$-commutator $Y_{k_2k_1^2}$. We conclude that the obtained Q'-cocycle condition reduces to $1-q_{k_1k_1}\sigma_{k_1k_2}=0$ under which an iterated $\boldsymbol{q}$-commutator $Y_{k_2k_1^2}$ is a nontrivial basic constant in $\mathcal{B}_{k_1^2k_2}$.

In the way described, by specializing some elements of a set Q and considering it as a corresponding multiset, we conclude that (14) can be determined from (17) and that under the obtained Q'-cocycle condition, the constants in the degenerate subspaces of the algebra $\mathcal{B}$ can be obtained from the constants in the corresponding generic subspace of the algebra $\mathcal{B}$ by applying the formula (38). Note, however, that it is therefore more difficult to execute the general formulas for determining the constants in all degenerate subspaces of the algebra $\mathcal{B}$, since for each different specialization of some elements of a set Q of cardinality n, different degenerate cases arise from the corresponding generic case. In this way, there are different multisets Q' of cardinality n associated with different Q'-cocycle conditions under which there are suitable constants.

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