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Valuation overrings of polynomial rings and group of divisibility*

Research Article

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Abstract: In this work, we discuss the types of valuation overrings of $K[x_1, x_2, ..., x_n]$ based on the rank and rational rank of value groups. Also, we describe the group of divisibility of a finite intersection of valuation overrings of $K[x_1, x_2, ..., x_n]$. In particular, we focus on the case for $n > 3$.

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1. Introduction

Let R be an integral domain with quotient field K, and let $K^* = K - \{0\}$. The *group of divisibility* $G(R)$ of R is defined as the multiplicative group of nonzero principal fractional ideals of R. The ring R is the identity element of $G(R)$. We define a partial order on $G(R)$ by setting $xR \leq yR$ if and only if $yR \subseteq xR$. For background on the group of divisibility see [7].

A partially ordered group G is called a *lattice-ordered group (ℓ-group)* if for any two elements a and b of G, $\sup\{a, b\}$ and $\inf\{a, b\}$ exist in G. If any two elements of the group G are comparable, then G is called a *totally ordered group*. We recall that a *valuation* on K is a mapping ν of K onto a totally ordered group $G \cup \{\infty\}$, where ∞ is a symbol such that $g + \infty = \infty + g = \infty + \infty = \infty$ and $g < \infty$ for all $g \in G$, for which the following conditions are satisfied.

(i) $\nu(K^*) = G$, $\nu(0) = \infty$.

(ii) $\nu(xy) = \nu(x) + \nu(y)$ for all $x, y \in K$.

(ii) $\nu(x + y) \geq \inf\{\nu(x), \nu(y)\}$ for all x, $y \in K$.

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Then $R_\nu = \{x \in K^* : \nu(x) \geq 0\} \cup \{0\}$ is a subring of K. Moreover, $G(R_\nu) \cong G$ [2, page 103] and the domain R_ν is called a *valuation domain.* If $G \cong \mathbb{Z} \times_\ell \mathbb{Z} \times_\ell \cdots \times_\ell \mathbb{Z}$, then G is called a *discrete value group*, where the product $\times_\ell$ is a *lexicographic product.* We define the *lexicographic order* on $\mathbb{Z} \times_l \mathbb{Z} \times_l ... \times_l \mathbb{Z}$ as follows: $(\alpha_1, \alpha_2, ..., \alpha_n) \geq (\beta_1, \beta_2, ..., \beta_n)$ if $\alpha_1 > \beta_1$ or if for some $k > 1, \alpha_i = \beta_i$ for $i = 1, 2, ..., k-1$ and $\alpha_k > \beta_k$. Two valuation rings V_1 and V_2 of K are said to be *independent* if K is the only common overring of both V_1 and V_2. Otherwise, V_1 and V_2 are *dependent.* A family of valuation rings $\{V_i\}_{i \in I}$ of a field K is said to be a *dependent family* if there exists a valuation ring $V \subsetneq K$ such that $V_i \subseteq V$ for each $i \in I$. Any valuation ring of Krull dimension one is independent with other incomparable valuation rings.

The group of divisibility encodes much information about the divisibility theory and ideal structure of a domain. For example, Krull proved that there exists an order-reversing correspondence between the prime ideals of a valuation domain and the convex subgroups of its group of divisibility [7]. A result of Krull for valuation domains is generalized by Yakabe to a Bézout domain and its group of divisibility by proving a correspondence between the prime ideals of a Bézout domain and the prime filters of its group of divisibility [7].

Let G_1 and G_2 be two ℓ-groups. The group $G_1 \times_c G_2$ is called the *cardinal product* of G_1 and G_2. We define the *pointwise ordering* on $G_1 \times_c G_2$ as follows: $(a, b) \geq (c, d)$ if $a \geq c$ and $b \geq d$. Let $\{G_i : i \in I\}$ denote a collection of ℓ-groups. The group $\prod_{i \in I} G_i$ with pointwise ordering is an ℓ-group called the *cardinal product* of the G_i. The group $\bigoplus_{i \in I} G_i$ with pointwise ordering is an ℓ-group called the *cardinal sum* of the G_i.

Let K be a field. Let $R = \bigcap_{i=1}^n V_i$, where for each $i = 1, 2, ..., n$, V_i is a valuation ring of K. If the valuation rings are independent, then $G(R) \cong \bigoplus_{i=1}^n G(V_i)$ [6, Exercise 6, page 285]. Otherwise, if the valuation rings are dependent, then the map $\phi : G(R) \to \bigoplus_{i=1}^n G(V_i)$ defined by $\phi(xR) = (xV_1, xV_2, ..., xV_n)$ is not surjective [3, page 711], and hence finding the group of divisibility is more complicated. Doering and Lequain in 1999 introduced a weak approximation theorem for dependent valuation rings [3, Theorem 4]. They showed that if each of the valuation rings in the intersection has a finitely generated value group then the group of divisibility of their intersection can be calculated explicitly.

For an ℓ-group G the *rational rank* of G is the dimension of $\mathbb{Q} \otimes_{\mathbb{Z}} G$ as a vector space over $\mathbb{Q}$ and is denoted by rat.rank(G). The *rank* of a totally ordered group G is the order type of the set of proper convex subgroups of G. The Krull dimension of a valuation domain is equal to the rank of its value group [11, Corollary, page 5]. We have $\text{rank}(G_\nu) \leq \text{rat.rank}(G_\nu)$ [11, page 9].

By [5, Theorem 2.1.4], all the valuation overrings of $k[x]$ are obtained by localizing $k[x]$ at some prime ideal P and each of them has both rank and rational rank one. Abhyankar in [1] showed that there are only three types of valuation overrings of $k[x_1, x_2]$ based on the nature of the value group. In [10], we characterized the valuation overrings of $k[x_1, x_2, x_3, ..., x_n]$ for $n = 3$. Also, in [10], we described the group of divisibility of a finite intersection of valuation overrings $k[x_1, x_2, x_3, ..., x_n]$ for $n \leq 3$. In this paper, we discuss the types of valuation overrings of $k[x_1, x_2, x_3, ..., x_n]$ based on rank and rational rank of their value groups. Also, we describe the group of divisibility of a finite intersection of valuation overrings of $k[x_1, x_2, x_3, ..., x_n]$. In Proposition 2.5, we show the types of valuation overrings of $k[x_1, x_2, x_3, ..., x_n]$ based on rank and rational rank of value groups. In Proposition 4.2, we discuss the minimum number of dependent families of a finite collection of valuations overrings of $k[x_1, x_2, x_3, ..., x_n]$, where the group of divisibility of the intersection of valuation overrings of each dependent family cannot be determined explicitly (the group of divisibility expresses in terms of non-splitting lex-exact sequences). In Proposition 4.4 and 4.5, we describe the group of divisibility of a finite intersection of valuation overrings of $k[x_1, x_2, x_3, ..., x_n]$ for $n = 4$ and $n = 5$ respectively using the Weak Approximation Theorem for Valuations [3, Theorem 4]. Finally, in Theorem 4.6, we describe the group of divisibility of a finite intersection of valuation overrings of $k[x_1, x_2, x_3, ..., x_n]$ using the Weak Approximation Theorem for Valuations [3, Theorem 4].

2. Types of valuation overrings of $k[x_1, x_2, x_3, ..., x_n]$

Let V be a valuation overring of $k[x_1, x_2, x_3, ..., x_n]$. Then both rank and rational rank of $G(V)$ is less or equal to n [1, Theorem 1] and hence dimension of V is less or equal to n. If rat.rank $G(V) = n$, then $G(V)$ is finitely generated [11, Page 494].

The following proposition shows that there exist three types of valuation overrings of $k[x_1, x_2]$ based on rank and rational rank of value groups.

Proposition 2.1. *([1, Theorem 1]) Each valuation overring of $k[x_1, x_2]$ belongs to one of the following three sets.*

a) Valuation rings with rational value group; i.e., the value group is isomorphic to a subgroup of $\mathbb{Q}$.

b) Valuation rings with finitely generated value group of rank 1 and rational rank 2.

c) Valuation rings with discrete value group of rank two.

A short exact sequence of partially ordered groups

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is called *lexicographically exact* if

$$B_+ = \{b \in B : b \geq 0\} = \{b \in B : \beta(b) > 0\} \cup \{\alpha(a) : a \in A, a \geq 0\}.$$

In this case, the group B is called a *lexicographic extension (or lex-extension)* of A by C and if the sequence splits, then $B \cong C \times_\ell A$ [3, page 714].

The proposition below shows types of valuation overrings of $k[x_1, x_2, x_3]$ and the proof follows from [10, Lemma 3.5].

Proposition 2.2. *There exist six types of valuation overrings of $k[x_1, x_2, x_3]$.*

(1) Valuation rings with value group of rank one and rational rank is either one, two or three (three types of valuation rings). Then the value group is isomorphic to a subgroup of $\mathbb{Q}$, subgroup of $\mathbb{R}$ with rational rank two or subgroup of $\mathbb{R}$ with rational rank three respectively.

(2) Valuation rings with value group of rank two and rational rank is two or three (two types of valuation rings). Then the value group is isomorphic to a group of the form (rational rank two) lex-extension of $\mathbb{Z}$ by H, where H is a subgroup of $\mathbb{Q}$ or a subgroup of $\mathbb{R}$ with rational rank two of the form $\mathbb{Z}+\alpha_1\mathbb{Z}$, where α_1 is an irrational number.

(3) Valuation rings with value group of rank three and rational rank three. Then the value group will be in the form $\mathbb{Z} \times_\ell \mathbb{Z} \times_\ell \mathbb{Z}$.

Proposition 2.3. *([11, Corollary, Page 485]) Let V be a valuation overring of $k[x_1, x_2, x_3, ..., x_n]$. If rank of $G(V) = i$, $\quad 1 \leq i \leq n$, then rational rank of $G(V) = j$, $\quad i \leq j \leq n$.*

Proposition 2.4. *There exist at most ten types of valuation overrings of $k[x_1, x_2, x_3, x_4]$ based on rank and rational rank of value group.*

Proof. For $n = 4$, let V be a valuation overring of $k[x_1, x_2, x_3, x_4]$. Then we have the following cases using the Proposition 2.3.

1. If rank of $G(V) = 1$, the rational rank of $G(V)$ will be either $1, 2, 3$ or 4.
2. If rank of $G(V) = 2$, the rational rank of $G(V)$ will be $2, 3$ or 4.

3. If rank of $G(V) = 3$, the rational rank of $G(V)$ will be 3 or 4.
4. If rank of $G(V) = 4$, the rational rank of $G(V)$ will be 4.

$\square$

Proposition 2.5. *Based on rank and rational rank of value groups, there exist at most $\frac{n(n+1)}{2}$ types of valuation overrings of $k[x_1, x_2, x_3, ..., x_n]$.*

Proof. Let V be a valuation overring of $k[x_1, x_2, x_3, ..., x_n]$. Since rank $G(V) \leq$ rat.rank $G(V)$ [11, page 9], so if rank of $G(V) = 1$, then the rational rank of $G(V)$ will be less or equal to n. This means there exist n number of valuation overrings of $k[x_1, x_2, x_3, ..., x_n]$ with rank 1 and rational rank is less or equal to n. Again, if rank of $G(V) = 2$, then the rational rank of $G(V)$ will be less or equal to n. This means there exist $n-1$ number of valuation overrings of $k[x_1, x_2, x_3, ..., x_n]$ with rank 2 and rational rank is greater or equal to 2 and less or equal to n. Continuing this way, if rank of $G(V) = n-1$, then the rational rank of $G(V)$ will be less or equal to n. This means there exist 2 number of valuation overrings of $k[x_1, x_2, x_3, ..., x_n]$ with rank $n-1$ and rational rank is either $n-1$ or n. Finally, if rank of $G(V) = n$, then there exists one valuation overring of $k[x_1, x_2, x_3, ..., x_n]$ with rational rank n. Thus the total number of different types of valuation overrings of $k[x_1, x_2, x_3, ..., x_n]$ is

$$1 + 2 + 3 + ... + (n-1) + n = \frac{n(n+1)}{2}.$$

$\square$

3. Group of divisibility of a valuation overring of $k[x_1, x_2, ..., x_n]$

In this section, we describe the group of divisibility of a valuation overring of $k[x_1, x_2, ..., x_n]$ in terms of lex-exact sequences.

Proposition 3.1. *Let V be a valuation overring of $k[x_1, x_2, x_3, ..., x_n]$ with rank of $V = m \leq n$. Then $G(V)$ is order isomorphic to a group of the form*

lex-extension of $(\ldots($ *lex-extension of* $($ *lex-extension of* A_1 *by* $A_2)$ *by* $A_3)\ldots)$ *by* A_m*, where each* A_j*,* $j = 1, 2, ..., m$ *is order isomorphic to a group of the form* $A = B_1 + \alpha_1 B_2 + \alpha_2 B_3 + ... + \alpha_{t-1} B_t$*, where* $\alpha_1, \alpha_2, ..., \alpha_{t-1}$ *are either zero or an irrational number, for each* $i = 1, 2, ..., t, B_i$ *is a subgroup of* $\mathbb{Q}$ *(may or may or not be finitely generated) and* $t \leq n$*.*

Proof. Since rank of $V = m$, dimension of $V = m \leq n$. Then there exists a sequence of prime ideals of V such that $0 \subset p_1 \subseteq p_2 \subseteq \cdots \subseteq p_m$ and $V = V_{p_m} \subseteq V_{p_{m-1}} \cdots \subseteq V_{p_2} \subseteq V_{p_1}$. Also, by [5, Lemma 2.3.1], there exists a sequence of convex subgroups $G(V) \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{m-1} \supseteq 0$ of $G(V)$ such that H_{j-1}/H_j is a totally order group with rank one and is order isomorphic to a subgroup of $\mathbb{R}$ for $j = 1, 2, \cdots, m$ [5, Proposition 2.1.1], where for each $j = 1, 2, ..., m, H_j$ is corresponding to p_j. By using the Weak Approximation Theorem for valuation [3, Theorem 4], $G(V)$ is order isomorphic to a group of the form

$$\text{lex-extension of } \Big(\ldots\Big(\text{lex-extension of } (\text{ lex-extension of } H_{V_{p_{m-1}}, V} \text{ by } H_{V_{p_{m-2}}, V_{p_{m-1}}}) \text{ by } H_{V_{p_{m-3}}, V_{p_{m-2}}}\Big)\ldots\Big) \text{ by } H_{k(x_1, x_2, x_3, ..., x_n), V_{p_1}},$$

where $H_{V_{p_{j-1}}, V_{p_j}} = \ker(G(V_{p_j}) \to G(V_{p_{j-1}}))$ for $j = 1, 2, ..., m$ with $V_0 = k(x_1, x_2, x_3, ..., x_n)$.

Since H_j is a convex subgroup of $G(V)$ corresponding to the prime ideal p_j of V, then

$H_{V_{p_{j-1}},V_{p_j}} = \ker(G(V_{p_j}) \to G(V_{p_{j-1}})) = H_{j-1}/H_j$ for $j = 1, 2, ..., m$.

Let $A_j = H_{j-1}/H_j$ for $j = 1, 2, ..., m$. Then $G(V)$ is order isomorphic to a group of the form

lex-extension of (... (lex-extension of (lex-extension of A_1 by A_2) by A_3) ...) by A_m.

□

Remark 3.2. The sequence in proposition 3.1 splits and $G(V) \cong A_m \times_\ell A_{m-1} \times_\ell ... \times_\ell A_2 \times_\ell A_1$ if $G(V)$ satisfies one of the following conditions:

1. $G(V)$ is finitely generated. In this case, each of the groups appearing in the sequence will be finitely generated and the sequence will be split [3, Corollary 5].
2. $G(V)$ is divisible. In this case each of the factor groups and subgroups of $G(V)$ will be divisible and divisible groups are injective, then the sequence splits [4].
3. Each $A_i,\ i = 2, 3, ..., m$ is finitely generated, the sequence splits [3, Corollary 5].
4. Each $A_i,\ i = 1, 2, 3, ..., m-1$ is divisible, the sequence splits [4].

Remark 3.3. Let V be a valuation overring of $k[x_1, x_2, ..., x_n]$. The value groups of valuation overrings of $k[x_1, x_2, ..., x_n]$ with rank 1, 2, 3, 4, 5, ..., $n-1$ and n are given below based on Proposition 3.1. If rat.rank$(G(V)) = n$, then $G(V)$ is finitely generated [11, Page 494].

1. Valuation rings of rank one with rational rank is either $1, 2, ...,$ or n. If rat.rank$(G(V)) = 1$, then $G(V) \subseteq \mathbb{Q}$, and for other cases $G(V) \subset \mathbb{R}$.
2. Valuation rings of rank two with rational rank is either $2, 3, ...,$ or n. Then $G(V)$ is order isomorphic to a group of the form
lex-extension of C_1 by C_2, where C_1 and C_2 are subgroup of group of rational numbers or group of real numbers.
3. Valuation rings of rank three with rational rank is either $3, 4, ...,$ or n. Then $G(V)$ is order isomorphic to a group of the form
lex-extension of (lex-extension of C_1 by C_2) by C_3, where C_1, C_2 and C_3 are subgroup of group of rational numbers or group of real numbers.
4. Valuation rings of rank four with rational rank is either $4, 5, ...,$or n. Then $G(V)$ is order isomorphic to a group of the form
lex-extension of (lex-extension of (lex-extension of C_1 by C_2) by C_3) by C_4, where C_1, C_2, C_3 and C_4 are subgroup of group of rational numbers or group of real numbers.
5. Valuation rings of rank five with rational rank is either $5, 6, ...,$ or n. Then $G(V)$ is order isomorphic to a group of the form
lex-extension of (lex-extension of [lex-extension of (lex-extension of C_1 by C_2) by C_3] by C_4) by C_5, where C_1, C_2, C_3, C_4 and C_5 are subgroups of group of rational numbers or group of real numbers.
6. Valuation rings of rank $n-1$ with rational rank is $n-1$ or n. If rational rank is $n-1$, then $G(V)$ is in the form of the group in Proposition 3.1 with $m = n-1$.
7. Valuation rings of rank n with rational rank is n. Then $G(V) \cong \mathbb{Z} \times_\ell \mathbb{Z} \times_\ell ... \times_\ell \mathbb{Z} \times_\ell \mathbb{Z}$ [1, Theorem 1].

An ℓ-group G is called *realizable over* a polynomial ring $k[x_1, x_2, ..., x_n]$ if there exists a Bézout overring R of $k[x_1, x_2, ..., x_n]$ such that $G(R) \cong G$ as ℓ-groups. Let G be an ℓ-group with finite rational rank such that G is order isomorphic to a group of the form lex-extension of (A) by B, where A is an

ℓ-group and B is a totally ordered group. If A is divisible or B is finitely generated, then the sequence splits and if G has finite rational rank, then the group G is realizable over a polynomial ring in finitely many variables [9, Corollary 4.5], [9, Corollary 4.8], and [9, Theorem 4.11].

Definition 3.4. *[3, page 723] Let G be an ordered group. A decomposition of the type*

$$\prod_{\sigma_1\in\Lambda(\sigma_0)}\Big(\textit{ lex- extension of }_1\Big[\prod_{\sigma_2\in\Lambda(\sigma_1)}\Big(\textit{ lex-extension of }_2\Big[\prod_{\sigma_3\in\Lambda(\sigma_2)}$$
$$\Big(\cdots\prod_{\sigma_d\in\Lambda(\sigma_{d-1})}\Big(\textit{ lex-extension of }_d\Big[\prod_{\sigma_{d+1}\in\Lambda(\sigma_d)}H_{\sigma_d,\sigma_{d+1}}\Big]_d\textit{ by }H_{\sigma_{d-1},\sigma_d}\Big)\cdots\Big)\Big]_2$$
$$\textit{by }H_{\sigma_1,\sigma_2}\Big)\Big]_1\textit{ by }H_{\sigma_0,\sigma_1}\Big)$$

where

- *d is an integer ≥ 0,*
- *$\Lambda(\sigma_0)$ is a non-empty finite set, and*
- *for every $i \geq 1$ and all $\sigma_1 \in \Lambda(\sigma_0), \sigma_2 \in \Lambda(\sigma_1), ..., \sigma_i \in \Lambda(\sigma_{i-1})$ such that $H_{\sigma_{i-1},\sigma_i}$ is a totally ordered nonzero group and $\Lambda(\sigma_i)$ is a finite set, we have $\Lambda(\sigma_i) = \varnothing$, or $|\Lambda(\sigma_i)| \geq 2$,*

will be called a lexico-cardinal decomposition of G.

Having such a decomposition, set

$$\Lambda := \Lambda(\sigma_0) \cup \left(\bigcup_{\sigma_1\in\Lambda(\sigma_0)}\Lambda(\sigma_1)\right) \cup \cdots \cup \left(\bigcup_{\sigma_1\in\Lambda(\sigma_0)}\cdots\bigcup_{\sigma_d\in\Lambda(\sigma_{d-1})}\Lambda(\sigma_d)\right),$$

where the unions are disjoint. The elements of the disjoint union $\{\sigma_0\} \cup \Lambda$ will be called *indices* of the decomposition. If $\sigma \in \Lambda$ and $\Lambda(\sigma) = \varnothing$, we shall say that σ is a *final index*. If $\sigma \in \Lambda$, we shall define the *line of predecessors* of σ to be the unique sequence of indices $\sigma_0, \sigma_1, ..., \sigma_m$ such that $\sigma_m = \sigma, \sigma_m \in \Lambda(\sigma_{m-1}), ..., \sigma_1 \in \Lambda(\sigma_0)$.

Let G be a semilocal ℓ-group with finite rational rank. Then G admits a lexico-cardinal decomposition form [9, Remark 3.7]. If each of the lex-exact sequence appears in the lexicocardinal decomposition form of G splits, then the group G is realizable over a polynomial ring in finitely many variables since the group of divisibility can be determined uniquely. Any ℓ-group with finite rational rank which is either finitely generated or divisible or cardinal sum of totally ordered groups, then the ℓ-group is realizable over a polynomial ring in finitely many variables by [9, Corollary 4.8], [9, Corollary 4.5] and [8, Theorem 5.8] respectively.

An ℓ-group G is called *weakly realizable over* $k[x_1, x_2, ..., x_n]$ if there exists a Bézout overring R of $k[x_1, x_2, ..., x_n]$ such that G and $G(R)$ admit a lexico-cardinal decomposition of the same form. If G is order isomorphic to a group of the form

lex-extension of A by B,

then to be weakly realizable means, the group $G(R)$ is order isomorphic to a group of the form

lex-extension of A by B,

and the sequence does not split and the group A may or may not be realizable over $k[x_1, x_2, ..., x_n]$.

Theorem 3.5. *[9, Theorem 3.1] Each ℓ-group G having finite rational rank can be weakly realized over $k[x_1, x_2, ..., x_n]$, where k is a field and $n \geq rat.rank(G)$. If R is the corresponding semilocal Bézout domain, then each of the valuation overrings $R_M, M \in Max(R)$, has residue field k.*

Corollary 3.6. *Any totally ordered group G is weakly realizable over $k[x_1, x_2, ..., x_n]$, where k is a field and $n \geq$ rat.rank(G).*

Proof. Let G be a totally ordered group with $n \geq \text{rat.rank}(G)$. Then G is realizable over $k[x_1, x_2, ..., x_n]$ [8, Lemma 4.3]. Since any realizable ℓ-group is weakly realizable, so G is weakly realizable over $k[x_1, x_2, ..., x_n]$. □

4. Group of divisibility of a finite intersection of valuation overrings of $k[x_1, x_2, ..., x_n]$

Let $\{V_1, V_2, ..., V_m\}$ be a collection of valuation overrings of $k[x_1, x_2, ..., x_n]$. Let $R = \bigcap_{j=1}^{m} V_j$. The group of divisibility $G(R)$ is determined in terms of value groups in [10] for $n \leq 3$. Here, we discuss the case for $n > 3$. To determine $G(R)$, we use Weak Approximation Theorem for Valuations [3, Theorem 4] and we express the $G(R)$ in lexico-cardinal decomposition form (the form in Definition 3.4).

Remark 4.1. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two dependent family such that each family has a finite collection of valuation overrings of $k[x_1, x_2, ..., x_n]$. Let $R = \bigcap_{V \in \mathfrak{F}_1} V \cap \bigcap_{V \in \mathfrak{F}_2} V$. Then using the Weak Approximation Theorem for valuations [3, Theorem 4],*

$$G(R) = G(\bigcap_{V \in \mathfrak{F}_1} V) \oplus G(\bigcap_{V \in \mathfrak{F}_2} V) = G_1 \oplus G_2,$$

where $G_1 = G(\bigcap_{V \in \mathfrak{F}_1} V)$ and $G_2 = G(\bigcap_{V \in \mathfrak{F}_2} V)$.

The groups G_1 and G_2 in Remark 4.1 can be expressed in terms of lexico-cardinal decomposition form. If each of the lex-exact sequence appearing in lexico-cardinal decomposition form of G_1 and G_2 splits, then the groups G_1 and G_2 can be realizable over $k[x_1, x_2, ..., x_n]$ [9] and if any of the lex-exact sequence does not split, then the group G_1 and G_2 cannot be realizable over $k[x_1, x_2, ..., x_n]$. Thus to describe the group of divisibility of a finite intersection of valuation overrings of $k[x_1, x_2, ..., x_n]$, we focus on the dependent families of valuation overrings of $k[x_1, x_2, ..., x_n]$ in which the group of divisibility of their intersection cannot be determined explicitly.

Let V be a valuation overring of $k[x_1, x_2, ..., x_n]$ with rank$(G(V)) = 1$. Let $R = \bigcap_{i=1}^{m} V_i$ and $V_i \subseteq V$ for each $i = 1, 2, ..., m$. For $n = 3$, when $G(V) = H$, a subgroup of $\mathbb{Q}$ which is not finitely generated, then we cannot determine the group of divisibility $G(R)$ explicitly [10]. Below, we describe the case for $n > 3$.

For $n = 4$, by Proposition 3.1, the group of divisibility of any valuation overring of $k[x_1, x_2, x_3, x_4]$ with rank greater than one can be in the form

lex-extension of (lex-extension of (lex-extension of A_1 by A_2) by A_3) by A_4. Then based on value group, there are three dependent families of valuations whose group of divisibility cannot be determined explicitly.

1. When $G(V) = H$ is a subgroup of $\mathbb{Q}$ which is not finitely generated. We can assume either A_2 or A_3 is order isomorphic to $G(V) = H$ and $A_4 = 0$. In this case, the above lex-exact sequence does not split. Then the group $G(R)$ is order isomorphic to a group of the form lex-extension of T_1 by H, where the group T_1 is not a divisible group and the group T_1 is related with the valuations $V_i, i = 1, 2, ..., m$ [3, Theorem 4].
2. When $G(V) = \mathbb{Z}$. We can assume either A_2 or A_3 is order isomorphic to $G(V) = \mathbb{Z}$ and $A_4 = 0$. In this case, the above lex-exact sequence does not split (here we can have situation, where one of the group A_2 or A_3 different from $G(V) = \mathbb{Z}$ is neither finitely generated nor divisible). Then the group $G(R)$ is order isomorphic to a group of the form lex-extension of (lex-extension of T_2 by H) by Z, where the group T_2 is not divisible and the group T_2 is related with the valuations $V_i, i = 1, 2, ..., m$ [3, Theorem 4], and H is a subgroup of $\mathbb{Q}$ which is not finitely generated.

3. When $G(V) = \mathbb{Z}+\alpha H$, where H is a subgroup of $\mathbb{Q}$ and is not finitely generated and α is an irrational number. Then both A_3 and A_4 are zero group and A_2 is order isomorphic to $G(V) = \mathbb{Z} + \alpha H$. In this case, the above lex-exact sequence does not split. Then the group $G(R)$ is order isomorphic to a group of the form lex-extension of T_3 by $\mathbb{Z} + \alpha H$, where the group T_3 is not divisible and the group T_3 is related with the valuations $V_i, i = 1, 2, ..., m$ [3, Theorem 4].

For $n = 5$, based on value groups, there are five dependent families of valuation overrings whose group of divisibility cannot be determined explicitly. We describe the nature of the value group $G(V)$ and the group $G(R)$ will have the similar form as in the case $n = 4$. We cannot determine the group of divisibility $G(R)$ explicitly if $G(V)$ is one of the following five cases:

1. $G(V)$ is a subgroup of $\mathbb{Q}$ which is not finitely generated.
2. $G(V) = \mathbb{Z}$. .
3. $G(V) = \mathbb{Z} + \alpha\mathbb{Z}$.
4. $G(V) = \mathbb{Z} + \alpha_1 H_1$, where H_1 is a subgroup of $\mathbb{Q}$ which is not finitely generated.
5. $G(V) = \mathbb{Z} + \alpha_1 H_1 + \alpha_2 H_2$, where H_1 and H_2 are subgroups of $\mathbb{Q}$, and at least one of them is not finitely generated.

For $n = 6$, based on value groups, there are seven dependent families of valuation overrings whose group of divisibility cannot be determined explicitly if $G(V)$ is one of the following cases:

1. $G(V) = H$ is a subgroup of $\mathbb{Q}$ which is not finitely generated.
2. $G(V) = \mathbb{Z}$.
3. $G(V) = \mathbb{Z} + \alpha\mathbb{Z}$.
4. $G(V) = \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$.
5. $G(V) = \mathbb{Z} + \alpha_1 H_1$, where H_1 is a subgroup of $\mathbb{Q}$ which is not finitely generated.
6. $G(V) = \mathbb{Z} + \alpha_1 H_1 + \alpha_2 H_2$, where H_1 and H_2 are subgroups of $\mathbb{Q}$, and at least one of them is not finitely generated.
7. $G(V) = \mathbb{Z} + \alpha_1 H_1 + \alpha_2 H_2 + \alpha_2 H_3$, where H_1, H_2 and H_3 are subgroups of $\mathbb{Q}$, and at least one of them is not finitely generated.

In the proposition below, we describe the minimum number of dependent family of valuation overrings of $k[x_1, x_2, ..., x_n]$ in which the group of divisibility of their intersection cannot be determined explicitly.

Proposition 4.2. *For $n \geq 3$, there are $2n - 5$ types of dependent families of valuation overrings of $k[x_1, x_2, ..., x_n]$ based on value groups such that the group of divisibility of their intersection cannot be determined explicitly.*

Proof. Let V be a valuation overring of $k[x_1, x_2, ..., x_n]$. Then by Proposition 3.1, $G(V)$ is order isomorphic to a group of the form

lex-extension of (...(lex-extension of (lex-extension of A_1 by A_2) by A_3)...)
by A_m, where each A_j, $j = 1, 2, ..., m-1$ if nonzero is order isomorphic to a totally ordered group with rational rank one and and $A_m \neq 0$ is order isomorphic to a totally ordered group with rational rank one.

Whether the above sequence may or may not split completely depends on the groups A_j, $j = 1, 2, ..., m$. One of the inner sequences may split due to the nature of the group A_j and the outer sequence may not split due to the nature of the group A_{j+1}. On the other hand, the outer sequence may split due to the nature of the group A_{j+1} and the inner sequence may not split due to the nature of the group A_j. Thus the above lex-exact sequence does not split if A_m is one of the form $\mathbb{Z}$, $\mathbb{Z}+\alpha_1\mathbb{Z}$, $\mathbb{Z}+\alpha_1\mathbb{Z}+\alpha_2\mathbb{Z}, ..., \mathbb{Z}+\alpha_1\mathbb{Z}+\alpha_2\mathbb{Z}+...+\alpha_{n-4}\mathbb{Z}$ (the total number is $n-3$) or one of the form of the groups but not finitely generated $H_1, H_1+\alpha_1 H_2, H_1+\alpha_1 H_2+\alpha_2 H_3, ..., H_1+\alpha_1 H_2+\alpha_2 H_3+...+\alpha_{n-3}H_{n-2}$ (the total number is $n-2$), where $H_i, i=1,2,...,n-2$ are subgroups of $\mathbb{Q}$. Thus we have $(n-3)+(n-2)=2n-5$ number of dependent families of valuation domains exist over $k[x_1,x_2,...,x_n]$, and the group of divisibility of their intersection cannot be determined explicitly. □

Proposition 4.3. *Let V be a valuation overring of $k[x_1,x_2,x_3,x_4]$ with rank of $G(V)=1$, then $G(V)$ is order isomorphic to one of the following subgroups of $\mathbb{R}$.*

1. *$H \subseteq \mathbb{Q}$*
2. *$H_1+\alpha H_2$, where $H_1, H_2 \subseteq \mathbb{Q}$*
3. *$H_1+\alpha_1 H_2+\alpha_2 H_3$, where $H_1, H_2, H_3 \subseteq \mathbb{Q}$*
4. *$\mathbb{Z}+\alpha_1\mathbb{Z}+\alpha_2\mathbb{Z}+\alpha_3\mathbb{Z}$, where α_1, α_2 and α_3 are irrational numbers.*

Proof. From the Proposition 2.4, the rational rank of $G(V)$ is either $1,2,3,$ or 4.

1. If rat.rank$(G(V))=1$, then $G(V)\subseteq\mathbb{Q}$.
2. If rat.rank$(G(V))=2$, then $G(V)\cong H_1+\alpha H_2$.
3. If rat.rank$(G(V))=3$, then $G(V)\cong H_1+\alpha_1 H_2+\alpha_2 H_3$,
4. If rat.rank$(G(V))=4$, then $G(V)\cong \mathbb{Z}+\alpha_1\mathbb{Z}+\alpha_2\mathbb{Z}+\alpha_3\mathbb{Z}$.

□

Let G_1 be an ℓ-group isomorphic to a group of the form $\mathbb{Z}\times_\ell(A_3^{'})$, where $A_3^{'}$ is weakly realizable over $k[x_1,x_2,x_3]$, let G_2 be an ℓ-group isomorphic to a group of the form lex-extension of $(A_2^{'})$ by H, where $A_2^{'}$ is realizable over $k[x_1,x_2]$ but $A_2^{'}$ is not a divisible group, and let G_3 be an ℓ-group isomorphic to a group of the form lex-extension of $(A_1^{'})$ by $\mathbb{Z}+\alpha H$, where $A_1^{'}$ is realizable over $k[x_1]$. The three groups G_1, G_2 and G_3 will be corresponding to three dependent families of valuation overrings of $k[x_1,x_2,x_3,x_4]$ since each of the group G_1, G_2 and G_3 is indecomposable [9]. By using Proposition 4.2, for $n=4, 2n-5=3$ is the numbers of dependent families of valuation overrings of $k[x_1,x_2,x_3,x_4]$. Let G_4 be an ℓ-group isomorphic to a group which is a cardinal product of groups of the form $\mathbb{Z}\times_\ell(A_3)$, $(\mathbb{Z}+\alpha\mathbb{Z})\times_\ell A_2$, $(\mathbb{Z}+\alpha_1\mathbb{Z}+\alpha_2\mathbb{Z})\times_\ell A_1$ and $\mathbb{Q}\times_\ell\mathbb{Q}$, where A_3 is realizable over $k[x_1,x_2,x_3]$, A_2 is realizable over $k[x_1,x_2]$ and A_1 is realizable over $k[x_1]$ respectively. The group G_4 is realizable over $k[x_1,x_2,x_3,x_4]$ since G_4 is the cardinal product of realizable groups over $k[x_1,x_2,x_3,x_4]$. The following proposition describes the group of divisibility of a finite intersection of valuation overrings of $k[x_1,x_2,x_3,x_4]$.

Proposition 4.4. *Let $\{V_1,V_2,...,V_m\}$ be a collection of valuation overrings of $k[x_1,x_2,x_3,x_4]$ with $m>3$ and let $R=\bigcap_{j=1}^m V_j$. Then $G(R)$ is order isomorphic to a group of the form $G_1\times_c G_2\times_c G_3\times_c G_4\times_c G_5$, where G_1, G_2, G_3 and G_4 if nonzero are the groups described in above paragraph and if nonzero G_5 is the cardinal product of the group of divisibility of the rank one valuation overrings of $k[x_1,x_2,x_3,x_4]$. (In other words, the group $G(R)$ can be written as $G_1\times_c G_2\times_c G_3\times_c G_4\times_c G_5=($ Weakly realizable group over $k[x_1,x_2,x_3,x_4])\times_c($ Realizable group over $k[x_1,x_2,x_3,x_4])$, where G_1,G_2,G_3 weakly realizable over $k[x_1,x_2,x_3,x_4]$ and G_4,G_5 realizable over $k[x_1,x_2,x_3,x_4]$ respectively).*

Proof. Let $\{V_1,V_2,...,V_m\}$ be a collection of valuation overrings of $k[x_1,x_2,x_3,x_4]$ and let $R=\bigcap_{j=1}^m V_j$. For each dependent family of valuations, there exists a rank one valuation overring of $k[x_1,x_2,x_3,x_4]$ which contains each member of the family. Then by using the Proposition 4.3, the value group of the valuation is either $\mathbb{Z}$ or H or $\mathbb{Z}+\alpha\mathbb{Z}$ or $\mathbb{Z}+\alpha_1\mathbb{Z}+\alpha_2\mathbb{Z}$ or $\mathbb{Z}+\alpha H$. By using the Weak Approximation Theorem for Valuations [3, Theorem 4], $G(R)$ is order isomorphic to a group of the form $G_1\times_c G_2\times_c G_3\times_c G_4\times_c G_5$. □

Let G_1 be an ℓ-group isomorphic to a group of the form $\mathbb{Z} \times_\ell (A_4^{'})$, where $A_4^{'}$ is weakly realizable over $k[x_1, x_2, x_3, x_4]$, let G_2 be an ℓ-group isomorphic to a group of the form lex-extension of $(A_3^{'})$ by H, where $A_3^{'}$ is weakly realizable over $k[x_1, x_2, x_3]$, let G_3 be an ℓ-group isomorphic to a group of the form lex-extension of $(A_2^{''})$ by $\mathbb{Z} + \alpha Z$, where $A_2^{''}$ is weakly realizable over $k[x_1, x_2, x_3]$, let G_4 be an ℓ-group isomorphic to a group of the form lex-extension of $A_2^{'}$ by $\mathbb{Z} + \alpha H$, where $A_2^{'}$ is realizable over $k[x_1, x_2]$ and let G_5 be an ℓ-group isomorphic to a group of the form lex-extension of $A_1^{'}$ by $\mathbb{Z} + \alpha_1 H + \alpha_2 H$, where $A_1^{'}$ is realizable over $k[x_1]$. Let G_6 be an ℓ-group isomorphic to a group which is a cardinal product of groups of the form $\mathbb{Z} \times_\ell (A_4), (\mathbb{Z} + \alpha\mathbb{Z}) \times_\ell A_3, (\mathbb{Z} + \alpha_1\mathbb{Z} + \alpha_2\mathbb{Z}) \times_\ell A_2, (\mathbb{Z} + \alpha_1\mathbb{Z} + \alpha_2\mathbb{Z} + \alpha_3\mathbb{Z}) \times_\ell A_1, (\mathbb{Z} + \alpha\mathbb{Q}) \times_\ell A_2, \mathbb{Q} \times_\ell \mathbb{Q}, \mathbb{Q} \times_\ell (\mathbb{Q} + \alpha\mathbb{Q})$, and $(\mathbb{Q} + \alpha\mathbb{Q}) \times_\ell \mathbb{Q}$, where A_4 is realizable over $k[x_1, x_2, x_3, x_4]$, A_3 is realizable over $k[x_1, x_2, x_3]$, A_2 is realizable over $k[x_1, x_2]$ and A_1 is realizable over $k[x_1]$ respectively. For $n = 5$, the proof follows similarly to the case for $n = 4$.

Proposition 4.5. *Let $\{V_1, V_2, ..., V_m\}$ be a collection of valuation overrings of $k[x_1, x_2, x_3, x_4, x_5]$ with $m > 5$ and let $R = \bigcap_{j=1}^m V_j$. Then $G(R)$ is order isomorphic to a group of the form $\prod_{i=1}^7 G_i$, where G_1, G_2, G_3, G_4, G_5 and G_6 are described as in above and G_7 is the cardinal product of the group of divisibility of the rank one valuation overrings of $k[x_1, x_2, x_3, x_4, x_5]$.*

In the theorem below, we describe the group of divisibility of a finite intersection of valuation overrings of $k[x_1, x_2, ..., x_n]$ for $n \geq 3$.

Theorem 4.6. *Let $\mathfrak{F} = \{V_1, V_2, ..., V_m\}$ be a collection of valuation overrings of $k[x_1, x_2, ..., x_n]$ and let $R = \bigcap_{j=1}^m V_j$. Let $\mathfrak{F}_1, \mathfrak{F}_2, ..., \mathfrak{F}_{2n-5}$ be a collection of dependent families of valuation rings in $\mathfrak{F}$ in which the group of divisibility of the intersection of valuation rings of each family cannot be determined explicitly. Then $G(R)$ is order isomorphic to a group of the form*

$$(\prod_{i=1}^{2n-5} G(R_i)) \times_c G,$$

where $G(R_i)$ is the group of divisibility of the intersection of valuation rings corresponding to $\mathfrak{F}_i$ for $i = 1, 2, ..., 2n-5$ and G is an ℓ-group which can be realized over $k[x_1, x_2, ..., x_n]$.

Proof. Let $\{V_1, V_2, ..., V_m\}$ be a collection of valuation overrings of $k[x_1, x_2, ..., x_n]$. Assume that there are t number of dependent families of valuations $\mathfrak{F}_1, \mathfrak{F}_2, ..., \mathfrak{F}_t$. Then we can write $R = \bigcap_{j=1}^m V_j = \bigcap_{q=1}^t R_q$, where $R_q = \bigcap_{V \in \mathfrak{F}_q} V$. Now, by using the Weak Approximation Theorem for valuations [3, Theorem 4], $G(R) \cong \prod_{q=1}^t G(R_q) = (\prod_{q=1}^{2n-5} G(R_i)) \times_c (\prod_{q=2n-4}^t G(R_q)) = (\prod_{q=1}^{2n-5} G(R_i)) \times_c G$, where the group $G = \prod_{q=2n-4}^t G(R_q)$ and is realizable over $k[x_1, x_2, ..., x_n]$. □

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