J. Algebra Comb. Discrete Appl.
11(3) • 151–164

Received: 11 February 2023
Accepted: 2 April 2024

Journal of Algebra Combinatorics Discrete Structures and Applications

Local and 2-local automorphisms of null-filiform and filiform associative algebras *

Research Article

Farhodjon Arzikulov, Iqboljon Karimjanov, Sardorbek Umrzaqov

Abstract: In the present paper automorphisms, local and 2-local automorphisms of n-dimensional null-filiform and filiform associative algebras are studied. Namely, a common form of the matrix of automorphisms and local automorphisms of these algebras is clarified. It turns out that the common form of the matrix of an automorphism on these algebras does not coincide with the local automorphism's matrices common form on these algebras. Therefore, these associative algebras have local automorphisms that are not automorphisms. Also, that each 2-local automorphism of null-filiform algebra is an automorphism and some associative filiform algebras admit 2-local automorphisms which are not automorphisms are proved.

2020 MSC: 16S50, 16W20

Keywords: Associative algebra, Nilpotent algebra, Null-filiform algebra, Filiform algebra, Automorphisms, Local automorphisms, 2-Local automorphisms

Introduction

The Gleason-Kahane-Żelazko theorem [13, 16], which is a fundamental contribution to the theory of Banach algebras, asserts that every unital linear local homomorphism from an unital complex Banach algebra A into $\mathbb{C}$ is multiplicative. We recall that a linear map T from a Banach algebra A into a Banach algebra B is said to be a local homomorphism if for every a in A there exists a homomorphism $\Phi_a : A \to B$, depending on a, such that $T(a) = \Phi_a(a)$.

* *This work was partially supported by RSF 22-71-10001.*
Farhodjon Arzikulov (Corresponding Author), Iqboljon Karimjanov, Sardorbek Umrzaqov; Andijan State University, 129, Universitet Street, Andijan, 170100, Uzbekistan (email: arzikulovfn@gmail.com, iqboli@gmail.com, sardor.umrzaqov1986@gmail.com).
Farhodjon Arzikulov, Iqboljon Karimjanov; V. I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Univesity Street, 9, Olmazor district, Tashkent, 100174, Uzbekistan.
Iqboljon Karimjanov; Saint-Petersburg State University, 7-9 Universitetskaya Embankment, St Petersburg, 199034, Russia.

https://doi.org/10.13069/jacodesmath.v11i3.279 ISSN 2148-838X

Later, in [15], R. Kadison introduces the concept of local derivations and proves that each continuous local derivation from a von Neumann algebra into its dual Banach bemodule is a derivation. B. Jonson [14] extends the above result by proving that every local derivation from a C*-algebra into its Banach bimodule is a derivation. In particular, Johnson gives an automatic continuity result by proving that local derivations of a C*-algebra A into a Banach A-bimodule X are continuous even if not assumed a priori to be so (cf. [14, Theorem 7.5]). Based on these results, many authors have studied local derivations on operator algebras.

A similar notion, which characterizes non-linear generalizations of automorphisms, was introduced by Šemrl in [21] as 2-local automorphisms. He described such maps on the algebra $B(H)$ of all bounded linear operators on an infinite dimensional separable Hilbert space H.

The first results concerning local derivations and automorphisms on finite-dimensional Lie algebras were obtained in [4]. Namely, in [4] the authors have proved that every local derivation on semi-simple Lie algebras is a derivation and gave examples of nilpotent finite-dimensional Lie algebras with local derivations which are not derivations. Sh.A.Ayupov, K.K.Kudaybergenov, B.A.Omirov proved similar results concerning local derivations and automorphisms on simple Leibniz algebras in their recent paper [7]. Local automorphisms of certain finite-dimensional simple Lie and Leibniz algebras are investigated in [5]. Concerning local automorphism, T.Becker, J.Escobar, C.Salas, and R.Turdibaev in [9] established that the set of local automorphisms $LAut(sl_2)$ coincides with the group $Aut^{\pm}(sl_2)$ of all automorphisms and anti-automorphisms. Later in [11] M.Costantini proved that a linear map on a simple Lie algebra is a local automorphism if and only if it is either an automorphism or an anti-automorphism. The local derivation of semisimple Leibniz algebras investigated in [19]. Similar results concerning local derivations and automorphisms on Lie superalgebras were obtained in [10, 22] and [23].

In the paper [6], local derivations of solvable Lie algebras are studied, and it is proved that in the class of solvable Lie algebras, there exist algebras that admit local derivations which are not derivation. Also, algebras, every local derivation of which is a derivation, are found. Moreover, every local derivation on a finite-dimensional solvable Lie algebra with model nilradical and the maximal dimension of complementary space is a derivation. Sh.A.Ayupov, A.Kh.Khudoyberdiyev, and B.B.Yusupov proved similar results concerning local derivations on solvable Leibniz algebras in their recent papers [8, 24]. F.N.Arzikulov, I.A.Karimjanov, and S.M.Umrzaqov established that every local and 2-local automorphisms on the solvable Leibniz algebras with null-filiform and naturally graded non-Lie filiform nilradicals, whose dimension of complementary space is maximal, is an automorphism [2]. Recently, local derivations and automorphisms of Cayley algebras, local derivations on the simple Malcev algebra and local and 2-local derivations of simple $n-$ary algebras considered in [1, 12, 19]

In the paper [18], I.A.Karimjanov, S.M.Umrzaqov, and B.Yusupov describe automorphisms, local and 2-local automorphisms of solvable Leibniz algebras with a model or abelian null-radicals. They show that any local automorphisms on solvable Leibniz algebras with a model nilradical, the dimension of the complementary space of which is maximal, is an automorphism. But solvable Leibniz algebras with an abelian nilradical with a 1-dimensional complementary space admit local automorphisms which are not automorphisms.

In the present paper automorphisms, local and 2-local automorphisms of n-dimensional filiform and null-filiform associative algebras are studied. Namely, a common form of the matrix of automorphisms and local automorphisms of these algebras is clarified. It turns out that the common form of the matrix of an automorphism on these algebras does not coincide with the local automorphism's matrix's common form on these algebras. Therefore, these associative algebras have local automorphisms that are not automorphisms. Also, that each 2-local automorphism of null-filiform algebra is an automorphism and some associative filiform algebras admit 2-local automorphisms which are not automorphisms are proved.

1. Preliminaries

Null-filiform and filiform associative algebras. For an algebra $\mathcal{A}$ of an arbitrary variety, we consider the series

$$\mathcal{A}^1 = \mathcal{A}, \qquad \mathcal{A}^{i+1} = \sum_{k=1}^{i} \mathcal{A}^k \mathcal{A}^{i-k+1}, \qquad i \geq 1.$$

We say that an algebra $\mathcal{A}$ is nilpotent if $\mathcal{A}^i = 0$ for some $i \in \mathbb{N}$. The smallest integer satisfying $\mathcal{A}^i = 0$ is called the index of nilpotency or nilindex of $\mathcal{A}$.

Definition 1.1. *An n-dimensional associative algebra $\mathcal{A}$ is called null-filiform if $dim\mathcal{A}^i = (n+1)-i, 1 \leq i \leq n+1$.*

Theorem 1.2 ([20])**.** *An arbitrary n-dimensional null-filiform associative algebra is isomorphic to the following algebra:*

$$\mu_0 : \quad e_i e_j = e_{i+j}, 2 \leq i+j \leq n,$$

where $\{e_1, e_2, \ldots, e_n\}$ is a basis of the algebra $\mathcal{A}$ and the omitted products vanish.

Definition 1.3. *An n-dimensional associative algebra $\mathcal{A}$ is called filiform if $dim(\mathcal{A}^i) = n-i$, $2 \leq i \leq n$.*

Theorem 1.4 ([17])**.** *For $n > 3$ every n-dimensional filiform associative algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero is isomorphic to one of the following pairwise non - isomorphic algebras with a basis $\{e_1, e_2, \ldots, e_n\}$:*

$$\begin{aligned} \mu_{1,1} :& \quad e_i e_j = e_{i+j} \\ \mu_{1,2} :& \quad e_i e_j = e_{i+j}, e_n e_n = e_{n-1} \\ \mu_{1,3} :& \quad e_i e_j = e_{i+j}, e_1 e_n = e_{n-1} \\ \mu_{1,4} :& \quad e_i e_j = e_{i+j}, e_1 e_n = e_n e_n = e_{n-1}, \end{aligned}$$

where $2 \leq i+j \leq n-1$.

2. Description of automorphisms of finite-dimensional null-filiform and filiform associative algebras

Here we describe automorphisms of the associative algebras from Theorems 1.2 and 1.4.

Theorem 2.1. *A linear map $\varphi : \mu_0 \to \mu_0$ is an automorphism if and only if it has the following form:*

$$\varphi(e_1) = \sum_{i=1}^{n} a_i e_i,$$

$$\varphi(e_i) = \sum_{j=i}^{n} (\sum_{k_1+k_2+\ldots+k_i=j} a_{k_1} \cdot a_{k_2} \cdot \ldots \cdot a_{k_i}) e_j, \quad 2 \leq i \leq n, \tag{1}$$

where $a_1 \neq 0$.

Proof. Let

$$\varphi(e_1) = \sum_{i=1}^{n} a_i e_i.$$

Then

$$\varphi(e_2) = \varphi(e_1 e_1) = \left(\sum_{i=1}^{n} a_i e_i\right)\left(\sum_{i=1}^{n} a_i e_i\right)$$

$$= \sum_{j=2}^{n}\left(\sum_{k_1+k_2=j} a_{k_1}\cdot a_{k_2}\right) e_j.$$

Also we have

$$\varphi(e_3) = \varphi(e_1 e_2) = \varphi(e_1)\varphi(e_2) = \sum_{j=3}^{n}\left(\sum_{k_1+k_2+k_3=j} a_{k_1}\cdot a_{k_2}\cdot a_{k_3}\right) e_j.$$

Similarly, for any $i = 2, 3, \ldots n$, we have

$$\varphi(e_i) = \varphi(e_1 e_{i-1}) = \varphi(e_1)\varphi(e_{i-1}) = \left(\sum_{i=1}^{n} a_i e_i\right)\left(\sum_{i=1}^{n} a_i e_i\right)^{i-1}$$

$$= \left(\sum_{i=1}^{n} a_i e_i\right)^{i} = \sum_{j=i}^{n}\left(\sum_{k_1+k_2+\ldots+k_i=j} a_{k_1}\cdot a_{k_2}\cdot\ldots\cdot a_{k_i}\right) e_j.$$

The proof is complete. □

Theorem 2.2. *A linear map $\varphi : \mu_{1,1} \to \mu_{1,1}$ is an automorphism if and only if it has the following form:*

$$\varphi(e_1) = \sum_{i=1}^{n} a_i e_i,$$

$$\varphi(e_i) = \sum_{j=i}^{n-1}\Big(\sum_{k_1+k_2+\ldots+k_i=j} a_{k_1}\cdot a_{k_2}\cdot\ldots\cdot a_{k_i}\Big)e_j, \quad 2 \le i \le n-1$$

$$\varphi(e_n) = b_{n-1}e_{n-1} + b_n e_n$$

where $a_1 \neq 0$.

Proof. Let

$$\varphi(e_1) = \sum_{i=1}^{n} a_i e_i.$$

Then similar to the proof of Theorem 2.1 we have

$$\varphi(e_i) = \sum_{j=i}^{n}\Big(\sum_{k_1+k_2+\ldots+k_i=j} a_{k_1}\cdot a_{k_2}\cdot\ldots\cdot a_{k_i}\Big)e_j, \quad 2 \le i \le n-1,$$

where $a_1 \neq 0$.

Now, let

$$\varphi(e_n) = \sum_{i=1}^{n} b_i e_i.$$

Then, by the table of multiplication of the algebra $\mu_{1,1}$ and equality

$$\varphi(e_1 e_n) = 0,$$

we have

$$0 = \varphi(e_1 e_n) = \varphi(e_1)\varphi(e_n) = \left(\sum_{i=1}^{n} a_i e_i\right)\left(\sum_{i=1}^{n} b_i e_i\right) =$$

$$= \sum_{j=1}^{n-2}\sum_{i=j+1}^{n-1} a_j b_{i-1} e_i.$$

From this it follows that

$$b_i = 0, \quad 1 \leq i \leq n-2, \quad b_{n-1} \neq 0, b_n \neq 0.$$

So,

$$\varphi(e_n) = b_{n-1}e_{n-1} + b_n e_n.$$

The proof is complete. □

We can similarly prove the following theorems.

Theorem 2.3. *A linear map $\varphi : \mu_{1,2} \to \mu_{1,2}$ is an automorphism if and only if it has the following form:*

$$\varphi(e_1) = \sum_{i=1}^{n} a_i e_i,$$

$$\varphi(e_2) = \sum_{j=2}^{n-1}(\sum_{k_1+k_2=j} a_{k_1} \cdot a_{k_2})e_j + a_n^2 e_{n-1}$$

$$\varphi(e_i) = \sum_{j=i}^{n-1}(\sum_{k_1+k_2+...+k_i=j} a_{k_1} \cdot a_{k_2} \cdot ... \cdot a_{k_i})e_j, \quad 3 \leq i \leq n-1$$

$$\varphi(e_n) = -a_n\sqrt{a_1^{n-3}}e_{n-2} + b_{n-1}e_{n-1} + \sqrt{a_1^{n-1}}e_n$$

where $a_1 \neq 0$.

Theorem 2.4. *A linear map $\varphi : \mu_{1,3} \to \mu_{1,3}$ is an automorphism if and only if the map φ has the following form:*

$$\varphi(e_1) = \sum_{i=1}^{n} a_i e_i,$$

$$\varphi(e_2) = \sum_{j=2}^{n-1}(\sum_{k_1+k_2=j} a_{k_1} \cdot a_{k_2})e_j + a_1 a_n e_{n-1}$$

$$\varphi(e_i) = \sum_{j=i}^{n-1}(\sum_{k_1+k_2+\ldots+k_i=j} a_{k_1} \cdot a_{k_2} \cdot \ldots \cdot a_{k_i})e_j, \quad 3 \leq i \leq n-1$$

$$\varphi(e_n) = b_{n-1}e_{n-1} + a_1^{n-2}e_n$$

where $a_1 \neq 0$.

Theorem 2.5. *A linear map* $\varphi : \mu_{1,4} \to \mu_{1,4}$ *is an automorphism if and only if the map* φ *has the following form:*

$$\varphi(e_1) = \sum_{i=1}^{n} a_i e_i,$$

$$\varphi(e_2) = \sum_{j=2}^{n-1}(\sum_{k_1+k_2=j} a_{k_1} \cdot a_{k_2})e_j + (a_1 a_n + a_n^2)e_{n-1},$$

$$\varphi(e_i) = \sum_{j=i}^{n-1}(\sum_{k_1+k_2+\ldots+k_i=j} a_{k_1} \cdot a_{k_2} \cdot \ldots \cdot a_{k_i})e_j, \quad 3 \leq i \leq n-1,$$

$$\varphi(e_n) = -a_n e_{n-2} + b_{n-1}e_{n-1} + e_n$$

where $a_1 \neq 0$.

3. Description of local automorphisms of finite-dimensional null-filiform and filiform associative algebras

Now we describe local automorphisms of the associative algebras from Theorems 1.2 and 1.4.

Definition 3.1. *Let* A *be an algebra. A linear map* $\Phi : A \to A$ *is called a local automorphism, if for any element* $x \in A$ *there exists an automorphism* $\varphi_x : A \to A$ *such that* $\Phi(x) = \varphi_x(x)$.

Theorem 3.2. *A linear map* Φ *is a local automorphism of* μ_0 *if and only if the matrix of* Φ *has the following lower triangular form*

$$\begin{pmatrix} b_{1,1} & 0 & 0 & \ldots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & \ldots & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \ldots & b_{n-1,n-1} & 0 \\ b_{n,1} & b_{n,2} & b_{n,3} & \ldots & b_{n,n-1} & b_{n,n} \end{pmatrix}$$

Proof. Let Φ be an arbitrary local automorphism on μ_0. By the definition of local derivation, for any element $x \in \mu_0$, there exists an automorphism φ_x on μ_0 such that

$$\Phi(x) = \varphi_x(x).$$

By Theorem 2.1, the matrix of the automorphism φ_x has the following form:

$$A_x = \begin{pmatrix} a_1^x & 0 & \dots & 0 & 0 \\ a_2^x & (a_1^x)^2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1}^x & \sum\limits_{k_1+k_2=n-1} a_{k_1}^x a_{k_2}^x & \dots & (a_1^x)^{n-1} & 0 \\ a_n^x & \sum\limits_{k_1+k_2=n} a_{k_1}^x a_{k_2}^x & \dots & \sum\limits_{k_1+k_2+\dots+k_{n-1}=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_{n-1}}^x & (a_1^x)^n \end{pmatrix}.$$

Let A be the matrix of Φ and

$$A = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \dots & b_{1,n-1} & b_{1,n} \\ b_{2,1} & b_{2,2} & b_{2,3} & \dots & b_{2,n-1} & b_{2,n} \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & b_{3,n-1} & b_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,1} & b_{n,2} & b_{n,3} & \dots & b_{n,n-1} & b_{n,n} \end{pmatrix}.$$

Then, by choosing subsequently $x = e_1$, $x = e_2$, $\dots, x = e_n$ and using $\Phi(x) = \varphi_x(x)$, i.e. $A\bar{x} = A_x\bar{x}$, where $\bar{x} = (x_1, x_2, \dots x_n)^T$ is the vector corresponding to $x = x_1e_1 + \dots + x_ne_n$, we have $b_{i,j} = 0$, $i < j$, and $b_{k,k} \neq 0$, $1 \leq k \leq n$, which implies

$$A = \begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & \dots & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & 0 \\ b_{n,1} & b_{n,2} & b_{n,3} & \dots & b_{n,n-1} & b_{n,n} \end{pmatrix}$$

Now we prove that the linear operator, defined by the matrix A is a local automorphism. If, for each element $x \in \mu_0$, there exists a matrix A_x of the form in Theorem 2.1 such that

$$A\bar{x} = A_x\bar{x}, \tag{2}$$

then the linear operator, defined by the matrix A is a local automorphism. In other words, if, for each element $x \in \mu_0$, the system of equations

$$\begin{cases} b_{1,1}x_1 = a_1^x x_1, \\ \sum\limits_{j=1}^{i} b_{i,j}x_j = a_i^x x_1 + \sum\limits_{j=2}^{i} \sum\limits_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x x_j, \quad 2 \leq i \leq n, \end{cases} \tag{3}$$

obtained from (2), has a solution with respect to the variables

$$a_1^x,\ a_2^x,\ \dots a_n^x,$$

then the linear operator, defined by the matrix A, is a local automorphism.

Let us consider the following cases

- If $x \neq 0$ then $a_1^x = b_{1,1}$,
  $a_i^x = b_{i,1} + \frac{1}{x_1} \sum\limits_{j=2}^{i} (b_{i,j} - \sum\limits_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j$, $2 \leq i \leq n$, $(a_1^x \neq 0)$.
- If $x_1 = x_2 = \dots = x_{m-1} = 0$ and $x_m \neq 0$ then $(a_1^x)^m = b_{m,m}$,
  $a_{i-m+1}^x = \frac{1}{m(a_1^x)^{m-1}} \Big(b_{i,k} - \sum\limits_{k_1+k_2+\dots+k_m=i}^{k_{l_1}+k_{l_2}+\dots+k_{l_{m-1}} \neq m-1} a_{k_1} a_{k_2} \dots a_{k_m} +$

$$+\frac{1}{x_k}\sum_{j=k+1}^{i}\Big(b_{i,j}-\sum_{k_1+k_2+\cdots+k_j=i}a^x_{k_1}a^x_{k_2}\dots a^x_{k_j})x_j\Big),$$
$$m+1\le i\le n,\ l_1,l_2,\dots,l_{m-1}\in\{1,2,\dots,m\},\ (a^x_1\neq 0).$$

Hence, the system of equation (3) always has a solution. Therefore, the linear operator, defined by the matrix A is a local automorphism. This completes the proof.

□

Theorem 3.3. *A linear map Φ is a local automorphism of $\mu_{1,1}$ if and only if the matrix of Φ has the following lower triangular form*

$$\begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 & 0\\ b_{2,1} & b_{2,2} & 0 & \dots & 0 & 0\\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n}\\ b_{n,1} & 0 & 0 & \dots & 0 & b_{n,n} \end{pmatrix}$$

Proof. Let Φ be an arbitrary local automorphism on $\mu_{1,1}$. Then, by the definition, for any element $x\in\mu_{1,1}$, there exists an automorphism φ_x on $\mu_{1,1}$ such that

$$\Phi(x)=\varphi_x(x).$$

By Theorem 2.2, the matrix of the automorphism φ_x has the following form:

$$A_x=\begin{pmatrix} a^x_1 & 0 & 0 & \dots & 0 & 0\\ a^x_2 & (a^x_1)^2 & 0 & \dots & 0 & 0\\ a^x_3 & \sum\limits_{k_1+k_2=3}a^x_{k_1}a^x_{k_2} & (a^x_1)^3 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ a^x_{n-1} & \sum\limits_{k_1+k_2=n-1}a^x_{k_1}a^x_{k_2} & \sum\limits_{k_1+k_2+k_3=n-1}a^x_{k_1}a^x_{k_2}a^x_{k_3} & \dots & (a^x_1)^{n-1} & b^x_{n-1}\\ a^x_n & 0 & 0 & \dots & 0 & b^x_n \end{pmatrix}.$$

Let A be the matrix of Φ and, let

$$A=\begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \dots & b_{1,n-1} & b_{1,n}\\ b_{2,1} & b_{2,2} & b_{2,3} & \dots & b_{2,n-1} & b_{2,n}\\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & b_{3,n-1} & b_{3,n}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n}\\ b_{n,1} & b_{n,2} & b_{n,3} & \dots & b_{n,n-1} & b_{n,n} \end{pmatrix}.$$

Then, by choosing subsequently $x=e_1$, $x=e_2$, $\dots$, $x=e_n$ and using $\Phi(x)=\varphi_x(x)$, i.e. $A\bar{x}=A_x\bar{x}$, where $\bar{x}=(x_1,x_2,\dots x_n)^T$ is the vector corresponding to $x=x_1e_1+\cdots+x_ne_n$, we have $b_{i,j}=0$, $i<j$, $i\neq n-1$, and $b_{k,k}\neq 0$, $1\le k\le n$, which implies

$$A=\begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 & 0\\ b_{2,1} & b_{2,2} & 0 & \dots & 0 & 0\\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n}\\ b_{n,1} & 0 & 0 & \dots & 0 & b_{n,n} \end{pmatrix}$$

Similar to the proof of Theorem 3.2 we prove that, for each element $x \in \mu_{1,1}$, the system of equations

$$\begin{cases} b_{1,1}x_1 = a_1^x x_1, \\ \sum_{j=1}^{i} b_{i,j}x_j = a_i^x x_1 + \sum_{j=2}^{i} \sum_{k_1+k_2+\cdots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x x_j, \quad 2 \le i \le n-2, \\ \sum_{j=1}^{n} b_{n-1,j}x_j = a_{n-1}^x x_1 + \sum_{j=2}^{n-1} \sum_{k_1+k_2+\cdots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x x_j + b_{n-1}^x x_n, \\ b_{n,1}x_1 + b_{n,n}x_n = a_n^x x_1 + b_n^x x_n, \end{cases} \tag{4}$$

obtained from the equality $A\bar{x} = A_x\bar{x}$, has a solution with respect to the variables

$$a_1^x,\ a_2^x,\ \dots a_n^x, b_{n-1}^x, b_n^x.$$

Let us consider the following cases

- If $x_1 \neq 0$ then $a_1^x = b_{1,1}$,
  $a_i^x = b_{i,1} + \frac{1}{x_1}\sum_{j=2}^{i}(b_{i,j} - \sum_{k_1+k_2+\cdots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x)x_j$, $2 \le i \le n-2$,
  $a_{n-1}^x = b_{n-1,n} + \frac{1}{x_1}\sum_{j=2}^{n-1}(b_{n-1,j} - \sum_{k_1+k_2+\cdots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x)x_j + \frac{1}{x_1}(b_{n,n} - b_n^x)x_n$,
  $a_n^x = b_{n,1} + \frac{1}{x_1}(b_{n,n} - b_n^x)x_n$,
  where b_{n-1}^x and b_n^x are defined arbitrarily.
- If $x_1 = x_2 = \cdots = x_{m-1} = 0$ and $x_m \neq 0$ then $(a_1^x)^m = b_{m,m}$,
  $a_{i-m+1}^x = \frac{1}{m(a_1^x)^{m-1}}\Big(b_{i,m} - \sum_{k_1+k_2+\cdots+k_m=i}^{k_{l_1}+k_{l_2}+\cdots+k_{l_{m-1}}\neq m-1} a_{k_1}^x a_{k_2}^x \dots a_{k_m}^x +$
  $+\frac{1}{x_m}\sum_{j=m+1}^{i}\big(b_{i,j} - \sum_{k_1+k_2+\cdots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x)x_j\Big)$, $m+1 \le i \le n-2, l_1, l_2, \dots, l_{m-1} \in \{1, 2, \dots, m\}$,
  $(a_1^x \neq 0)$,
  $a_{n-m}^x = \frac{1}{m(a_1^x)^{m-1}}\Big(b_{n-1,k} - \sum_{k_1+k_2+\cdots+k_m=n-1}^{k_{l_1}+k_{l_2}+\cdots+k_{l_{m-1}}\neq m-1} a_{k_1}^x a_{k_2}^x \dots a_{k_m}^x +$
  $+\frac{1}{x_m}\sum_{j=m+1}^{n-1}\big(b_{n-1,j} - \sum_{k_1+k_2+\cdots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x)x_j + \frac{1}{x_m}(b_{n-1,n} - b_{n-1}^x)x_n\Big)$,
  $l_1, l_2, \dots, l_{m-1} \in \{1, 2, \dots, m\}$, $(a_1^x \neq 0)$.
  In this case, by the last equation of System (4), we have $b_n^x = b_{n,n}$ if $x_n \neq 0$, otherwise b_n^x is defined arbitrarily. In this case, a_n^x, a_{n-1}^x, …, a_{n-m+1}^x and b_{n-1}^x are defined arbitrarily.
- If $x_1 = x_2 = \cdots = x_{n-1} = 0$ and $x_n \neq 0$, then all equation of System (4) except the last two equations are zero. So, in this case, a_n^x, a_{n-1}^x, …, a_1^x are defined arbitrarily. By the last two equations of System (4), $b_{n-1}^x = b_{n-1,n}$ and $b_n^x = b_{n,n}$.

Hence, the system of equation (4) always has a solution. Therefore, the linear operator, defined by the matrix A is a local automorphism. The proof is complete. □

Theorem 3.4. *A linear map Φ is a local automorphism of $\mu_{1,2}$ if and only if the matrix of Φ has the following lower triangular form*

$$\begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & \dots & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,1} & 0 & 0 & \dots & 0 & b_{n,n} \end{pmatrix}$$

Proof. Let Φ be an arbitrary local automorphism on $\mu_{1,2}$. Then, by the definition, for any element $x \in \mu_{1,2}$, there exists an automorphism φ_x on $\mu_{1,2}$ such that

$$\Phi(x) = \varphi_x(x).$$

By Theorem 2.3, the matrix of automorphism φ_x has the following form:

$$A_x = \begin{pmatrix} a_1^x & 0 & 0 & \dots & 0 & 0 \\ a_2^x & (a_1^x)^2 & 0 & \dots & 0 & 0 \\ a_3^x & \sum\limits_{k_1+k_2=3} a_{k_1}^x a_{k_2}^x & (a_1^x)^3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1}^x & \sum\limits_{k_1+k_2=n-1} a_{k_1}^x a_{k_2}^x + (a_n^x)^2 & \sum\limits_{k_1+k_2+k_3=n-1} a_{k_1}^x a_{k_2}^x a_{k_3}^x & \dots & (a_1^x)^{n-1} & b_{n-1}^x \\ a_n^x & 0 & 0 & \dots & 0 & \sqrt{(a_1^x)^{n-1}} \end{pmatrix}.$$

Let A be the matrix of Φ, then from $\Phi(x) = \varphi_x(x)$ for $x = e_i$, we have that

$$A = \begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & \dots & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,1} & 0 & 0 & \dots & 0 & b_{n,n} \end{pmatrix}$$

Similar to the proof of Theorem 3.2 we prove that, for each element $x \in \mu_{1,2}$, the system of equations

$$\begin{cases} b_{1,1}x_1 = a_1^x x_1, \\ \sum\limits_{j=1}^{i} b_{i,j}x_j = a_i^x x_1 + \sum\limits_{j=2}^{i} \sum\limits_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x x_j, \quad 2 \le i \le n-3, \\ \sum\limits_{j=1}^{n-2} b_{n-2,j}x_j + b_{n-2,n}x_n = \\ = a_{n-2}^x x_1 + \sum\limits_{j=2}^{n-2} \sum\limits_{k_1+k_2+\dots+k_j=n-2} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x x_j - a_n^x \sqrt{(a_1^x)^{n-3}} x_n, \\ \sum\limits_{j=1}^{n} b_{n-1,j}x_j = a_{n-1}^x x_1 + \sum\limits_{j=2}^{n-1} \sum\limits_{k_1+k_2+\dots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x x_j + (a_n^x)^2 x_2 + b_{n-1}^x x_n, \\ b_{n,1}x_1 + b_{n,n}x_n = a_n^x x_1 + \sqrt{(a_1^x)^{n-1}} x_n, \end{cases} \tag{5}$$

obtained from the equality $A\bar{x} = A_x\bar{x}$, has a solution with respect to the variables

$$a_1^x, \ a_2^x, \ \dots a_n^x, b_{n-1}^x.$$

Let us consider the following cases

- If $x_1 \neq 0$ then $a_1^x = b_{1,1}$,
  $a_i^x = b_{i,1} + \frac{1}{x_1} \sum\limits_{j=2}^{i} (b_{i,j} - \sum\limits_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j$, $2 \le i \le n-3$,
  $a_{n-2}^x = b_{n-2,1} + \frac{1}{x_1} \sum\limits_{j=2}^{n-2} (b_{n-2,j} - \sum\limits_{k_1+k_2+\dots+k_j=n-2} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j + \frac{1}{x_1}(b_{n-2,n} - a_n^x \sqrt{(a_1^x)^{n-3}}) x_n$,where a_n^x is defined arbitrarily,
  $a_{n-1}^x = b_{n-1,1} + \frac{1}{x_1} \sum\limits_{j=2}^{n-1} (b_{n-1,j} - \sum\limits_{k_1+k_2+\dots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j + \frac{1}{x_1}((b_{n-1,n} - b_{n-1}^x) x_n - (a_n^x)^2 x_2)$,where a_n^x, b_{n-1}^x are defined arbitrarily,
  $a_n^x = b_{n,1} + \frac{1}{x_1}(b_{n,n} - \sqrt{(a_1^x)^{n-1}}) x_n$.

- If $x_1 = x_2 = \cdots = x_{m-1} = 0$ and $x_m \neq 0$ then $(a_1^x)^m = b_{m,m}$,
  $a_{i-m+1}^x = \frac{1}{m(a_1^x)^{m-1}}\Big(b_{i,m} - \sum_{k_1+k_2+\cdots+k_m=i}^{k_{l_1}+k_{l_2}+\cdots+k_{l_{m-1}}\neq m-1} a_{k_1}^x a_{k_2}^x \dots a_{k_m}^x +$
  $+ \frac{1}{x_m}\sum_{j=m+1}^{i}(b_{i,j} - \sum_{k_1+k_2+\cdots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x)x_j\Big),$
  $m+1 \leq i \leq n-3,\ l_1, l_2, \dots, l_{m-1} \in \{1, 2, \dots, m\},\ (a_1^x \neq 0),$
  $a_{n-m-1}^x = \frac{1}{m(a_1^x)^{m-1}}\Big(b_{n-2,m} - \sum_{k_1+k_2+\cdots+k_m=n-2}^{k_{l_1}+k_{l_2}+\cdots+k_{l_{m-1}}\neq m-1} a_{k_1}^x a_{k_2}^x \dots a_{k_m}^x +$
  $+ \frac{1}{x_m}((\sum_{j=m+1}^{n-2}(b_{n-2,j} - \sum_{k_1+k_2+\cdots+k_j=n-2} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x)x_j + (b_{n-2,n}+$
  $a_n^x\sqrt{(a_1^x)^{n-3}})x_n)\Big)$, where a_n^x is defined arbitrarily, $l_1, l_2, \dots, l_{m-1} \in \{1, 2, \dots, m\}$, $(a_1^x \neq 0)$.

  $a_{n-m}^x = \frac{1}{m(a_1^x)^{m-1}}\Big(b_{n-1,m} - \sum_{k_1+k_2+\cdots+k_m=n-1}^{k_{l_1}+k_{l_2}+\cdots+k_{l_{m-1}}\neq m-1} a_{k_1}^x a_{k_2}^x \dots a_{k_m}^x +$
  $+ \frac{1}{x_m}((\sum_{j=m+1}^{n-1}(b_{n-1,j} - \sum_{k_1+k_2+\cdots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x)x_j + (b_{n-1,n} - b_{n-1}^x)x_n)\Big)$, $b_n^x = b_{n,n}$, where a_n^x, a_{n-1}^x, $\dots$, a_{n-m+1}^x and b_{n-1}^x are defined arbitrarily.

Hence, the system of equation (5) always has a solution. Therefore, the linear operator, defined by the matrix A is a local automorphism. This ends the proof. □

We can similarly prove the following Theorems using Theorems 2.4 and 2.5.

Theorem 3.5. *A linear map Φ is a local automorphism of the algebras $\mu_{1,2}$ and $\mu_{1,3}$ if and only if the matrix of Φ has the following lower triangular form*

$$\begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & \dots & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,1} & 0 & 0 & \dots & 0 & b_{n,n} \end{pmatrix}$$

Theorem 3.6. *A linear map Φ is a local automorphism of $\mu_{1,4}$ if and only if the matrix of Φ has the following lower triangular form*

$$\begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & \dots & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,1} & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Remark 3.7. *Note that the common form of the matrix of a local automorphism on an algebra includes the common form of the matrix of an automorphism on this algebra. The coincidence of these common forms denotes that every local automorphism of the considering algebra is an automorphism. But the common form of the matrix of an automorphism on the associative algebras μ_0, $\mu_{1,1}$, $\mu_{1,2}$, $\mu_{1,3}$ and $\mu_{1,4}$ does not coincide with the common form of the matrix of a local automorphism on these algebras by theorems 3.2, 3.3, 3.5 and 3.6. Therefore, the associative algebras μ_0, $\mu_{1,1}$, $\mu_{1,2}$, $\mu_{1,3}$ and $\mu_{1,4}$ have local automorphisms that are not automorphisms.*

Also, note that local automorphisms of an arbitrary low-dimension algebra can be similarly described using a common form of the matrix of automorphisms on this algebra.

4. Description of 2-local automorphisms of finite-dimensional null-filiform and filiform associative algebras

Theorem 4.1. *Each 2-local automorphism of μ_0 is an automorphism.*

Proof. Let ϕ be an arbitrary 2 -local automorphism of μ_0. Then, by the definition, for every element $x \in \mu_0$ and e_1, there exist a matrix A_{x,e_1}

$$A_{x,e_1} = \begin{pmatrix} a_1^{x,e_1} & 0 & \dots & 0 & 0 \\ a_2^{x,e_1} & (a_1^{x,e_1})^2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1}^{x,e_1} & \sum\limits_{k_1+k_2=n-1} a_{k_1}^{x,e_1} a_{k_2}^{x,e_1} & \dots & (a_1^{x,e_1})^{n-1} & 0 \\ a_n^{x,e_1} & \sum\limits_{k_1+k_2=n} a_{k_1}^{x,e_1} a_{k_2}^{x,e_1} & \dots & \sum\limits_{k_1+k_2+\dots+k_{n-1}=n-1} a_{k_1}^{x,e_1} a_{k_2}^{x,e_1} \dots a_{k_{n-1}}^{x,e_1} & (a_1^{x,e_1})^n \end{pmatrix},$$

such that $\phi(x) = \widehat{A_{x,e_1}\bar{x}}$ and $\phi(e_1) = \widehat{A_{x,e_1}\bar{e_1}}$, where $\bar{x} = (x_1, x_2, \dots, x_n)^T$ is the vector corresponding to x and $\widehat{\bar{x}}$ is an operation on $\bar{x}$ such that $\widehat{\bar{x}} = x$. Then

$$\phi(e_1) = \widehat{A_{x,e_1}\overline{e_1}} = (a_1^{x,e_1}, a_2^{x,e_1}, \widehat{a_3^{x,e_1}}, \dots, a_1^{x,e_1})^T.$$

Since $\phi(e_1) = \varphi_{x,e_1}(e_1) = \varphi_{y,e_1}(e_1)$, we have

$$\phi(e_1) = (a_1^{x,e_1}, a_2^{x,e_1}, \widehat{a_3^{x,e_1}}, \dots, a_n^{x,e_1})^T =$$

$$= (a_1^{y,e_1}, a_2^{y,e_1}, \widehat{a_3^{y,e_1}}, \dots, a_n^{y,e_1})^T$$

for each pair, x, y of elements in μ_0. Hence, $a_k^{x,e_1} = a_k^{y,e_1}$, $k = 1, 2, \dots n$. Therefore

$$\phi(x) = \widehat{A_{y,e_1}\bar{x}}$$

for any $x \in \mu_0$, and the matrix of $\phi(x)$ does not depend on x. Hence ϕ is a linear operator, and the matrix of φ_{y,e_1} is the matrix of ϕ. Thus, by Proposition 2.1, ϕ is an automorphism. $\square$

Theorem 4.2. *The associative algebras $\mu_{1,1}$, $\mu_{1,2}$, $\mu_{1,3}$ and $\mu_{1,4}$ admit 2-local automorphisms which are not automorphisms.*

Proof. We prove the theorem for the algebra $\mu_{1,1}$; for the algebras $\mu_{1,2}$, $\mu_{1,3}$, $\mu_{1,4}$ the proofs are similar. Let us define a homogeneous non additive function f on $\mathbb{C}^2$ as follows

$$f(z_1, z_n) = \begin{cases} \frac{z_1^2}{z_n}, & \text{if} \quad z_n \neq 0; \\ 0, & \text{if} \quad z_n = 0. \end{cases}$$

where $(z_1, z_n) \in \mathbb{C}^2$. Consider the map $\Delta : \mu_{1,1} \to \mu_{1,1}$ defined by the rule

$$\Delta(x) = x + f(x_1, x_n)e_{n-1}, \text{ where } x = \sum_{i=1}^{n} x_i e_i \in \mu_{1,1}.$$

Since f is not additive, we have Δ is not an automorphism. Let us show that Δ is a 2-local automorphism. For the elements

$$x = \sum_{i=1}^{n} x_i e_i, y = \sum_{i=1}^{n} y_i e_i,$$

we search an automorphism Φ with the matrix of the form (by Theorem 2.2 this matrix defines an automorphism of $\mu_{1,1}$):

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & 0 & 0 & \dots & 1 & b_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

such that $\Delta(x) = \Phi(x)$ and $\Delta(y) = \Phi(y)$. Then we obtain the following system of equations for a_{n-1} and b_{n-1}:

$$\begin{cases} x_1 a_{n-1} + x_{n-1} + x_n b_{n-1} = x_{n-1} + f(x_1, x_n), \\ y_1 a_{n-1} + y_{n-1} + y_n b_{n-1} = y_{n-1} + f(y_1, y_n), \end{cases}$$

i.e.

$$\begin{cases} x_1 a_{n-1} + x_n b_{n-1} = f(x_1, x_n), \\ y_1 a_{n-1} + y_n b_{n-1} = f(y_1, y_n). \end{cases}$$

Case 1. Let $x_1 y_n - x_n y_1 = 0$, then the system has infinitely many solutions, because of the right-hand side of this system is homogeneous.

Case 2. Let $x_1 y_n - x_n y_1 \neq 0$, then the system has a unique solution. The proof is complete. The proof is complete. □

Acknowledgment: We are very grateful to the referee for the careful study of our article and significant comments.

References

[1] F. N. Arzikulov, I. A. Karimjanov, A criterion of local derivations on the seven-dimensional simple Malcev algebras, Operators and Matrices, 16(2) (2022) 495–511.
[2] F. N. Arzikulov, I. A. Karimjanov, S. M. Umrzaqov, Local and 2-local automorphisms of some solvable Leibniz algebras, Journal of Geometry and Physics, 178 (2022) 104573.
[3] Sh. A. Ayupov, A. Elduque, K. K. Kudaybergenov, Local derivations and automorphisms of Cayley algebras, Journal of Pure and Applied Algebra, 227(5) (2023) 107277.
[4] Sh. A. Ayupov, K. K. Kudaybergenov, Local derivations on finite-dimensional Lie algebras, Linear Algebra and its Applications, 493 (2016) 381–398.
[5] Sh. A. Ayupov, K. K. Kudaybergenov, Local automorphisms on finite-dimensional Lie and Leibniz algebras, Algebra, Complex Analysis and Pluripotential Theory, Springer Proceedings in Mathematics and Statistics, 264 (2017) 31–44.
[6] Sh. A. Ayupov, A. Kh. Khudoyberdiyev, Local derivations on solvable Lie algebras, Linear and Multilinear Algebra, 69 (2021) 1286–1301.
[7] Sh. A. Ayupov, K. K. Kudaybergenov, B. A. Omirov, Local and 2-local derivations and automorphisms on simple Leibniz algebras, Bulletin of the Malaysian Mathematical Sciences Society, 43 (2020) 2199–2234.
[8] Sh. A. Ayupov, A. Kh. Khudoyberdiyev, B. B. Yusupov, Local and 2-local derivations of solvable Leibniz algebras, International Journal of Algebra and Computation, 30(6) (2020) 1185–1197.
[9] T. Becker, J. Escobar, C. Salas, R. Turdibaev, On local automorphisms of sl_2, Uzbek Mathematical Journal, 2 (2019) 27–34.

[10] H. Chen, Y. Wang, Local superderivation on Lie superalgebra $q(n)$, Czechoslovak Mathematical Journal, 68(143) (2018) 661–675.
[11] M. Costantini, Local automorphisms of finite dimensional simple Lie algebras, Linear Algebra and its Applications, 562 (2019) 123–134.
[12] B. Ferreira, I. Kaygorodov, K. Kudaybergenov, Local and 2-local derivations of simple n-ary algebras, Ricerche di Matematica, 73 (2024) 341–350.
[13] A. Gleason, A characterization of maximal ideals, Journal d'Analyse Mathématique, 19 (1967) 171–172.
[14] B. Johnson, Local derivations on C*-algebras are derivations, Transactions of the American Mathematical Society, 353 (2001) 313–325.
[15] R. Kadison, Local derivations, Journal of Algebra, 130 (1990) 494–509.
[16] J. Kahane, W. Żelazko, A characterization of maximal ideals in commutative Banach algebras, Studia Mathematica, 29 (1968) 339–343.
[17] I. A. Karimjanov, M. Ladra, Some classes of nilpotent associative algebras, Mediterranean Journal of Mathematics, 17(2) (2020) 1–21.
[18] I. A. Karimjanov, S. M. Umrzaqov, B. B. Yusupov, Local and 2-local automorphisms of solvable Leibniz algebras with abelian and model nilradicals, Quaestiones Mathematicae, 47(5) (2023) 1039–1054.
[19] I. Kaygorodov, K. Kudaybergenov, I. Yuldashev, Local derivations of semisimple Leibniz algebras, Communications in Mathematics, 30(2) (2022) 1–12.
[20] K. K. Masutova, B. A. Omirov, On some zero-filiform algebras, Ukrainian Mathematical Journal, 66(4) (2014) 541–492.
[21] P. Šemrl, Local automorphisms and derivations on $B(H)$, Proceedings of the American Mathematical Society, 125 (1997) 2677–2680.
[22] Y. Wang, H. Chen, J. Nan, 2-Local superderivations on basic classical Lie superalgebras, Journal of Mathematical Research with Applications, 37 (2017) 527–534.
[23] Y. Wang, H. Chen, J. Nan, 2-Local automorphisms on basic classical Lie superalgebras, Acta Mathematica Sinica, English Series, 35(3) (2019) 427–437.
[24] Yusupov B. B., Local and 2-local derivations of some solvable Leibniz algebras, Uzbek Mathematical Journal, 2 (2019) 154–166.