

A note on binary graph labelings and linear preservers

Research Article

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Abstract: An FM labeling of the vertices of an undirected graph requires that half the neighbors of each vertex are labeled zero and the other half labeled one. Variations of this type of labeling are presented and examples of the smallest and largest of graphs having one of these FM labelings are given. It is also shown that if T is a linear operator on the set of all undirected graphs on n vertices that strongly preserves sets of graphs that are labelable by one of the various FM type labelings, then T is a vertex permutation.

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1. Introduction

Binary labeling of the vertices of graphs has recently been a focus of some research interest. There have been a few binary labelings of graphs introduced recently, see [2, 4, 7]. We continue those investigations.

A $(0, 1)$ -labeling of the vertices of a graph is cordial if the zeros and ones are evenly distributed over the vertices and if the edges are also evenly labeled by zeros and ones, where that labeling is the absolute difference of the vertex labelings of the incident vertices. This concept was introduced by Cahit [6] in 1987. Since then there have been a couple of further binary vertex labelings of graphs defined, see [2, 7]. In this article we investigate the linear operators that preserve them.

Let $\mathcal{G}_n$ denote the set of simple (loopless) undirected graphs on the vertex set $V = \{v_1, v_2, \dots, v_n\}$. In this article we need to specify the concept of a friendly labeling.

Definition 1.1. Let $\mathcal{Z}$ be a set and label the set with entries from the set $\mathcal{A}$. Let f be the mapping $f : \mathcal{Z} \rightarrow \mathcal{A}$ that performs this labeling. This labeling of a set is called friendly if $-1 \leq |f^{-1}(i)| - |f^{-1}(j)| \leq 1$ for any $i, j \in \mathcal{A}$. To specify that the indexing set is $\mathcal{A}$ we say the labeling f is $\mathcal{A}$ -friendly. The labeling is strongly friendly if $|f^{-1}(i)| = |f^{-1}(j)|$ for any $i, j \in \mathcal{A}$.

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Definition 1.2. Let $\mathcal{A}$ be a set and G be an undirected graph with vertex set V and edge set E . Let $f : V \rightarrow \mathcal{A}$ be an $\mathcal{A}$ -friendly labeling of the non-isolated vertices of V and let $g : E \rightarrow \mathcal{A}$ be an induced labeling of E , that is given an edge uv , $g(uv) = \hat{g}(f(u), f(v))$ where $\hat{g} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$. Note that $\hat{g}$ must be symmetric to be well defined. The graph G with the $\mathcal{A}$ -friendly labeling f of the non-isolated vertices of G and induced edge labeling g is said to be $(\mathcal{A}, f, g)$ -cordial if g is also an $\mathcal{A}$ -friendly mapping. Also, given the labeling f , and the induced labeling, g , we say that an $(\mathcal{A}, f, g)$ -cordial graph G is k -cordial if $\mathcal{A}$ is $\mathbb{Z}_k$, and we say G is cordial if $\mathcal{A}$ is $\mathbb{Z}_2$.

If the induced mapping $g : E \rightarrow \mathcal{A}$ is defined to be $g(uv) = f(u) * f(v)$ where $*$ is a binary operation on $\mathcal{A}$ and G is $\mathcal{A}$ -cordial with the friendly vertex label f , and induced labeling g , we say G is $(\mathcal{A}, *)$ -cordial.

Remark 1.3. It should be noted that $(\mathbb{Z}_2, +)$ -cordial and $(\mathbb{Z}_2, -)$ -cordial are the same and in fact is the usual definition of a cordial graph labeling where f is a $(0, 1)$ labeling and $g(u, v) = |f(u) - f(v)|$ using real arithmetic. Further, the restriction that f is a friendly labeling of the non-isolated vertices, not necessarily all vertices, is required to avoid ambiguity because, for example, if all vertices were labeled, the graph $2K_2$ would be $(\mathbb{Z}_2, +)$ -cordial as a graph on five or more vertices but not on four vertices.

For digraphs we have a similar definition, however, as digraphs are not usually symmetric, we do not require the induced mapping g to be symmetric:

Definition 1.4. Let $\mathcal{A}$ and $\mathcal{B}$ be sets and let $G = (V, A)$ be a directed graph with vertex set V and arc set A . Let $f : V \rightarrow \mathcal{A}$ be an $\mathcal{A}$ -friendly labeling of the non-isolated vertices of V and let $g : A \rightarrow \mathcal{B}$ be an induced labeling of A , that is, given an arc $\vec{uv}$, $g(\vec{uv}) = \hat{g}(f(u), f(v))$ where $\hat{g} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{B}$. The graph G with the $\mathcal{A}$ -friendly labeling f of the non-isolated vertices and induced edge labeling g is said to be $(\mathcal{A}, \mathcal{B})$ -cordial if g is a $\mathcal{B}$ -friendly mapping. When $\mathcal{A} = \mathbb{Z}_k$ and $\mathcal{B} = \mathbb{Z}_\ell$ we say that G is (k, ℓ) -cordial. In particular we say that G is $(2, 3)$ -cordial if $\mathcal{A} = \mathbb{Z}_2$, $\mathcal{B} = \mathbb{Z}_3$ and $g(\vec{uv}) = f(v) - f(u)$. Note that in this case g is anti-symmetric.

Note that in the above definitions, if $\hat{g}$ is the binary mapping corresponding to one of the binary operations on $\mathcal{B}$ we indicate that by placing that operator in the notation. For example a digraph is $(\mathcal{A}, \mathcal{B}, -)$ -cordial indicates that the induced labeling g is $g(\vec{uv}) = f(v) - f(u)$. So a digraph is $(2, 3, -)$ -cordial means that the arc labelings are $f(v) - f(u)$ for the arc $\vec{uv}$. In this specific case we usually drop the minus sign and write $(2, 3)$ -cordial. A graph with an $\mathcal{A}$ -friendly vertex labeling is *product cordial* if it is $(\mathcal{A}, \times)$ -cordial.

Definition 1.5. A graph $G \in \mathcal{G}_n$ is said to be $(2, 3)$ -orientable if some orientation of the edges of G yields a digraph that is $(2, 3)$ -cordial.

Let $\mathcal{G}_n$ denote the set of all undirected simple loopless graphs on the vertex set $V = \{v_1, v_2, \dots, v_n\}$. Let G and H be two graphs in $\mathcal{G}_n$. The union of G and H is the graph whose edge set is the union of the edge set of G and the edge set of H . That is, $G \cup H = (V, E(G) \cup E(H))$. The graph G contains the graph H if the edge set of H is a subset of the edge set of G and is denoted $H \subseteq G$ or $G \supseteq H$. If $G \supseteq H$ then we write $G \setminus H$ to denote the graph whose edge set consists of the edges of G that are not edges of H . Since the vertex set is fixed, we describe a graph G as larger (or smaller) than graph H when the edge set of G contains more (or fewer) elements than the edge set of H .

Let $G \in \mathcal{G}_n$. Let $f(G)$ be a two element labeling of the vertices of G .

In 2022, at a virtual presentation of the 53rd Southeastern International Conference on Combinatorics, Graph Theory & Computing, Brian Freyberg defined a red-blue coloring:

Definition 1.6. [7] Let G be a simple graph with each vertex colored red or blue. If every vertex has an equal number of red and blue neighbors, then the coloring is a red-blue coloring and G is a red-blue graph.

So, a graph has a red-blue labeling if each the neighborhood of each vertex is strongly (red,blue)-friendly.

We now define some graph labelings based on this definition.

Definition 1.7. Let $G = (V, E)$ be an undirected simple (loopless) graph with vertex set V and edge set E . The open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$. The closed neighborhood of v is $\overline{N(v)} = N(v) \cup \{v\}$.

Let $f : V \rightarrow \mathcal{X}$ be a labeling of the vertices of G . For $\xi \in \mathcal{X}$ and U a subset of the vertices of G , let $\xi(U)$ denote the vertices of U that are labeled ξ . So that, $\xi(V) = f^{-1}(\xi)$.

Definition 1.8. Let $f : V \rightarrow \mathcal{X}$ be labeling of $G = (V, E) \in \mathcal{G}_n$.

- The labeling f is a open Freyberg-Marr-labeling, an FM_O -labeling, if for each $v \in V$, for any $\xi, \chi \in \mathcal{X}$, $|\xi(N(v))| = |\chi(N(v))|$. That is, the open neighborhood of each vertex is strongly $\mathcal{X}$ -friendly.
- The labeling f is a closed Freyberg-Marr-labeling, an FM_C -labeling, if for each $v \in V$, for any $\xi, \chi \in \mathcal{X}$, $|\xi(\overline{N(v)})| = |\chi(\overline{N(v)})|$. That is, the closed neighborhood of each vertex is strongly $\mathcal{X}$ -friendly.

Note that if $\mathcal{X} = \{0, 1\}$, an FM_O -labeling is the red-blue labeling introduced by Brian Freyberg [7].

We now define another similar labeling that arises by using open or closed neighborhoods depending on the parity of the vertex degree:

Definition 1.9. A labeling of G , $f : V \rightarrow \mathcal{X}$ is a mixed Freyberg-Marr-labeling, an FM_M -labeling, if for each $v \in V$, for any $\xi, \chi \in \mathcal{X}$, $|\xi(N(v))| = |\chi(N(v))|$ if $\delta(v)$ is even, and $|\xi(\overline{N(v)})| = |\chi(\overline{N(v)})|$ if $\delta(v)$ is odd.

Henceforth, we shall assume that the vertex labeling set is $\{0, 1\}$.

Note that any graph with a vertex of odd degree is not FM_O -labelable and any graph with a vertex of even degree is not FM_C -labelable. Further every FM_O -labelable graph or FM_C -labelable graph is an FM_M -labelable graph. No graph except the trivial one ($V = \emptyset$) is both FM_O -labelable and FM_C -labelable, and a graph that is neither FM_O -labelable nor FM_C -labelable may or may not be FM_M -labelable.

Example 1.10.

1. Every cycle on an even number (≥ 4) of vertices is FM_O -labelable.
2. The cube (3-dimensional) is FM_C -labelable.
3. The fan graph on five vertices is neither FM_O -labelable graph nor FM_C -labelable but is FM_M -labelable.
4. The Petersen graph (as well as the fan graph on 6 vertices) is neither FM_O -labelable graph nor FM_C -labelable and is not FM_M -labelable.

The tesseract (the four dimensional cube) is FM_O -labelable. Probably every n dimensional cube is either FM_O - or FM_C -labelable. Fan graphs (k -fan for k vertices total) are neither FM_O - nor FM_C -labelable, they have vertices of both even and odd degree. but some are FM_M -labelable, the 5-fan and 8-fan in particular.

The only FM_M -labeling of a graph that is also a cordial labeling is an FM_O -labeling, since the number of edges whose incident vertices are labeled the same and the number of edges whose incident vertices are labeled differently differ precisely by the number of odd vertices, which in any graph is always an even number. Thus, an FM_C -labeling is never a cordial labeling.

A graph that we shall use below is Harary's random graph, a four cycle with a secant, see Figures 1 and 2.

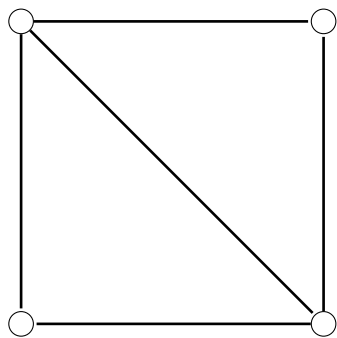


Figure 1. Harary's Random Graph

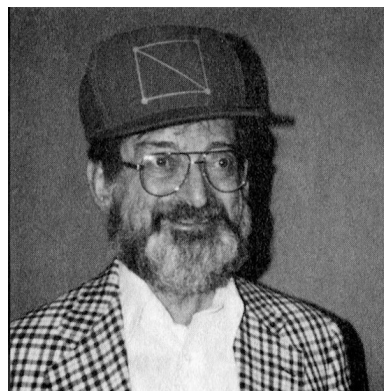


Figure 2. Frank Harary with Random Hat

1.1. Small and large FM labelable graphs.

Here we identify some large and small graphs that are labelable by one of the Freyberg-Marr labelings.

1.1.1. FM_O -labelable graphs

These graphs must have all vertices of even degree, so, K_n is not FM_O -labelable if n is even since all vertices have odd degree.

If n is odd, say there are more vertices of K_n labeled 0 than 1. Then each vertex labeled 0 of K_n has $\frac{n-1}{2}$ neighbors labeled 0 and $\frac{n-1}{2}$ neighbors labeled 1. Thus there are $\frac{n+1}{2}$ vertices labeled 0 and $\frac{n-1}{2}$ vertices labeled 1. Thus, each vertex labeled 1 has $\frac{n-1}{2} - 1$ neighbors labeled 1 and $\frac{n-1}{2} + 1$ neighbors labeled 0, so that K_n is not FM_O -labelable.

If n is even then $K_n \setminus \frac{n}{2} K_2$ is FM_O -labelable. If n is odd, $K_{n-1} \setminus \frac{n-1}{2} K_2$ is FM_O -labelable. However, when n is odd, there may be FM_O -labelable graphs with more edges.

The smallest number of edges in an FM_O -labelable graph is C_4 , the 4-cycle and it is the only FM_O -labelable graph on at most 4 edges.

Note that no star graph is FM_O -labelable.

1.1.2. FM_C -labelable graphs

These graphs must have all vertices of odd degree, so

K_n is not FM_C -labelable if n is odd. If n is even, K_n is FM_C -labelable, however given any edge graph E , $K_n \setminus E$ is not FM_C -labelable.

All edge graphs are FM_C -labelable. The only graph with two edges that is FM_C -labelable is $2K_2$, the only graph with three edges that is FM_C -labelable is $3K_2$, and the only graph with four edges that is FM_C -labelable is $4K_2$. There are only two graphs on five edges that are FM_C -labelable: $5K_2$ and the tree with degree sequence $(3, 3, 1, 1, 1, 1)$.

Note that no star graph is FM_C -labelable.

1.1.3. FM_M -labelable graphs

Since any FM_O -labelable or FM_C -labelable graph is FM_M -labelable, we have that for n even, K_n is FM_M -labelable. If n is odd then K_{n-1} is FM_M -labelable, but it is unknown at this time if there are larger FM_M -labelable graphs when n is odd.

For small graphs, no star graphs are FM_M -labelable. For graphs of at most four edges, only graphs of parallel edges, the 3-path, and C_4 are FM_M -labelable. There are six graphs with five edges that are FM_M -labelable, P_5 , the 4pan (a 4-cycle with one attached edge), the Harary random graph, the 4-cycle with a non-incident edge, the three path, P_3 , with two disjoint parallel edges and 5 parallel edges.

2. Linear preservers

A mapping $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ is said to be a linear operator if $T(G \cup H) = T(G) \cup T(H)$ where G and H are graphs in $\mathcal{G}_n$. Due to the fact that $\mathcal{G}_n$ is finite, we have the obvious Proposition:

Proposition 2.1. *Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator. then the following are equivalent:*

- T is injective.
- T is surjective.
- T is bijective,

Recall that a map on $\mathcal{G}_n$ is nonsingular means only that the only graph mapped to the edgeless graph is the edgeless graph.

Henceforth we let O denote the edgeless graph. Some authors prefer $\overline{K_n}$.

Definition 2.2. *Let $\mathcal{X}$ be a subset of $\mathcal{G}_n$. Then T is said to preserve the set $\mathcal{X}$ if whenever $U \in \mathcal{X}$, $T(U) \in \mathcal{X}$. We say T strongly preserves $\mathcal{X}$ if T preserves $\mathcal{X}$ and T preserves $\mathcal{G}_n \setminus \mathcal{X}$.*

Due to the nature of the algebraic structure of $\mathcal{G}_n$, addition of graphs being the union of the edge sets, any map that maps all of $\mathcal{G}_n$ to a single graph preserves every subset of $\mathcal{G}_n$ that contains that graph. Thus additional conditions on the mapping is required for any reasonable investigation, usually bijectivity or that the mapping be a linear operator that strongly preserves the set.

In [3] it was shown that $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ is a linear operator that strongly preserves

- the set of $(\mathbb{Z}_2, +)$ -cordial graphs for $n \geq 4$,
- the set of $(\mathbb{Z}_2, \times)$ -cordial graphs for $n \geq 5$, or
- the set of $(2, 3)$ -orientable graphs for $n \geq 6$

if and only if T is a vertex permutation.

The next sections use the following lemma from [1]:

Lemma 2.3. [1, Lemma 2.2] *If $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ is bijective, preserves $|E(G)|$, and maps 2-stars to 2-stars then T is a vertex permutation.*

2.1. Preservers of FM_O -labelable graphs

Lemma 2.4. *Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator. If T strongly preserves the set of FM_O -labelable graphs, then T is nonsingular.*

Proof. Suppose that T is singular. Then there is some edge graph, E , such that $T(E) = O$. Let C_4 be a 4-cycle such that E is not one of the edges of C_4 and connects two of the vertices of C_4 , so $C_4 \cup E$ is Harary's random graph H_4 . Now H_4 is not FM_O -labelable, while C_4 is, but here $T(C_4) = T(H_4)$, a contradiction. Thus T is nonsingular. $\square$

Lemma 2.5. *Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator. If T strongly preserves the set of FM_O -labelable graphs, then T maps edge graphs to edge graphs.*

Proof. Suppose that E is an edge graph, so that $|E| = 1$ and $|T(E)| > 1$. Let Q be a graph with maximal number of edges that is FM_O -labelable. Then, Q has only vertices of even degree. Suppose Q dominates E . Let $q = |Q|$. Then there are edge graphs $E_1 = E, E_2, E_3, \dots, E_k$ such that $T(E_1 \cup E_2 \cup \dots \cup E_k)$ is dominated by $T(Q)$ and such that $|T(E_1 \cup E_2 \cup \dots \cup E_k)| > k$. That is, there is some edge graph F dominated by Q such that $T(Q) = T(Q \setminus F)$. But Q is FM_O while $Q \setminus F$ is not since $Q \setminus F$ has two vertices of odd degree. Thus we have a contradiction. That is T maps edge graphs to edge graphs. $\square$

Lemma 2.6. *Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator. If T strongly preserves the set of FM_O -labelable graphs, then T is bijective.*

Proof. By Lemma 2.5 and Proposition 2.1, we only need show that the image of two edge graphs cannot be equal. Suppose that E and F are edge graphs and G is an FM_O -labelable graph that dominates $E \cup F$. Let $H = G \setminus (E \cup F)$. So $H \cup E$ is not FM_O -labelable and $H \cup E \cup F$ is FM_O -labelable. But $T(H \cup E \cup F) = T(H) \cup T(E) \cup T(F) = T(H) \cup T(E) = T(H \cup E)$. a contradiction. That is T is bijective. $\square$

A k star is a tree with k edges and diameter 2. It has one vertex of degree k and all other vertices are either isolated or degree 1.

Lemma 2.7. *Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator. If T strongly preserves the set of FM_O -labelable graphs, then T maps $(n-1)$ -stars to $(n-1)$ -stars.*

Proof. By Lemma 2.6, T is bijective. Suppose S is an $(n-1)$ -star and $T(S)$ is not. If $T(S)$ dominates a 3-path, then there is an edge graph E such that $T(S \cup E)$ dominates a 4-cycle. But then there is a 3-star S_3 such that $T(S_3 \cup E)$ is a 4-cycle, which is FM_O -labelable, but $S_3 \cup E$ is not FM_O -labelable, a contradiction. Thus, $T(S)$ does not dominate a 3-path. Thus $T(S)$ is a direct sum of C_3 's and stars. But for $n-1$ edges to be a direct sum of C_3 's and stars there can be only one block that is a star.

Let F be an edge graph whose image shares a vertex with a C_3 . Let H be an edge graph such that E, F and two edges of $S, U \cup V$, form a 4-cycle. Then $T(U \cup V \cup E \cup F)$ must be a 4-cycle (the only FM_O -labelable graph with 4 edges). But that is impossible, Adding two edges to a direct sum of C_3 's and stars can only dominate a C_4 if both edges are added to a star.

Thus T maps $(n-1)$ -stars to $(n-1)$ -stars. $\square$

Theorem 2.8. *Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator. If T strongly preserves the set of FM_O -labelable graphs, then T is a vertex permutation.*

Proof. By Lemma 2.6, T is bijective and by Lemma 2.7, T preserves $(n-1)$ -stars. Clearly, a bijective map preserves the number of edges in a graph and if it preserves $(n-1)$ -stars, it preserves 2-stars. Thus by Lemma 2.3, T is a vertex permutation. $\square$

2.2. Preservers of FM_C -labelable graphs

Lemma 2.9. *Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator. If T strongly preserves the set of FM_C -labelable graphs, then T is nonsingular.*

Proof. Suppose that T is singular, then there is some edge graph, E , such that $T(E) = O$. Let K_4 be the complete graph on four vertices with $(n - 4)$ isolated vertices such that E is one of the edges of K_4 . Now, K_4 is FM_C -labelable, while $K_4 \setminus E$ is not (it has two vertices of even degree). Further, $T(K_4) = T(K_4 \setminus E)$ since $T(E) = O$, a contradiction. Thus T is nonsingular. $\square$

Lemma 2.10. *Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator. If T strongly preserves the set of FM_C -labelable graphs, then T maps edge graphs to edge graphs.*

Proof. Suppose that E is a edge graph, so that $|E| = 1$, and suppose $|T(E)| > 1$. Let Q be a graph with maximal number of edges that is FM_C -labelable and suppose Q dominates E . Let $q = |Q|$. Then there are edge graphs $E_1 = E, E_2, E_3, \dots, E_k$ such that $T(E_1 \cup E_2 \cup \dots \cup E_k)$ is dominated by $T(Q)$ and such that $|T(E_1 \cup E_2 \cup \dots \cup E_k)| > k$. That is, there is some edge graph F dominated by Q such that $T(Q) = T(Q \setminus F)$. But Q is FM_O while $Q \setminus F$ is not, $Q \setminus F$ has two vertices of odd degree. Thus we have a contradiction. That is T maps edge graphs to edge graphs. $\square$

Lemma 2.11. *Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator. If T strongly preserves the set of FM_C -labelable graphs, then T is bijective.*

Proof. By Lemma 2.10 and Proposition 2.1, we only need show that the image of two edge graphs cannot be equal. Suppose that E and F are edge graphs such that $T(E) = T(F)$. Let G be an FM_C -labelable graph with minimum vertex degree at least 3 that dominates $E \cup F$. Then let $H = G \setminus (E \cup F)$. So $H \cup F$ is not FM_C -labelable (it has a vertex of even degree) and $H \cup E \cup F$ is FM_O -labelable. But $T(H \cup E \cup F) = T(H) \cup T(E) \cup T(F) = T(H) \cup T(E) = T(H \cup E)$. a contradiction. That is T is bijective. $\square$

Lemma 2.12. *Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator. If T strongly preserves the set of FM_C -labelable graphs, then T maps 2-stars to 2-stars.*

Proof. By Lemma 2.11, T is bijective. Suppose S is a 2-star. If $T(S)$ is not a 2-star, it is a pair of parallel edges. That is the image of a 2-star is either a 2-star or is FM_C -labelable. Since T strongly preserves FM_C -labelable graphs, $T(S)$ is a 2-star. $\square$

Theorem 2.13. *Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator. If T strongly preserves the set of FM_C -labelable graphs, then T is a vertex permutation.*

Proof. By Lemma 2.11, T is bijective and by Lemma 2.12 T preserves 2-stars. Clearly, a bijective map preserves the number of edges in a graph. Thus by Lemma 2.3, T is a vertex permutation. $\square$

2.3. Preservers of FM_M -labelable graphs

Lemma 2.14. *Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator. If T strongly preserves the set of FM_M -labelable graphs, then T is nonsingular.*

Proof. Let $C_3 \circ E$ be the graph with 4 edges, three of them forming a C_3 and the fourth edge incident to a vertex of the C_3 , sometimes called a 3-pan. Then $C_3 \circ E$ is FM_M -labelable, while C_3 is not. But if $T(E) = O$, $T(C_3 \circ E) = T(C_3)$, a contradiction, thus T is nonsingular. $\square$

A lemma we will use follows. It first appeared in [5, Lemma 2.5], but we include the proof here for completeness.

Lemma 2.15. *Let $\mathcal{K}$ be a finite set and $\phi : \mathcal{K} \rightarrow \mathcal{K}$ be any mapping. Then there is some power of ϕ that is idempotent.*

Proof. Since $\mathcal{K}$ is finite, and ϕ can be considered a subset of $\mathcal{K} \times \mathcal{K}$, the set $\{\phi, \phi^2, \phi^3, \dots, \phi^\ell, \dots\}$ must be finite. That is there are $a, b \in \mathbb{Z}_+$, the set of nonnegative integers, with $1 \leq a < b$ such that $\phi^a = \phi^b$. Let $d = b - a$. Since $\phi^a = \phi^b = \phi^{(a+d)}$, if $k \in \mathbb{Z}_+$ then $\phi^{(a+kd)} = \phi^{(a+d)+(k-1)d} = \phi^{(a+(k-1)d)}$, etc. So that for any $k \in \mathbb{Z}_+$, $\phi^{(a+kd)} = \phi^a$.

Suppose that $c \in \mathbb{Z}_+$ with $c \geq a$. Then $c = a + z$ for some $z \in \mathbb{Z}_+$. Thus, $\phi^{c+kd} = \phi^{(a+z+kd)} = \phi^{(a+kd)+z} = \phi^{(a+z)} = \phi^c$. That is, for any $c, k \in \mathbb{Z}_+$ with $c \geq a$, $\phi^{c+kd} = \phi^c$.

Now, $b > a$ and $d = b - a$ so $d \geq 1$. Thus, $ad \geq a$. It now follows that $\phi^{(ad+kd)} = \phi^{ad}$. For $k = a$ we have $(\phi^{ad})^2 = \phi^{2ad} = \phi^{(ad+ad)} = \phi^{ad}$. That is ϕ^{ad} is idempotent. $\square$

Lemma 2.16. Let $L : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be an idempotent linear operator. If L strongly preserves the set of FM_M -labelable graphs, then the image of an edge graph is an edge graph.

Proof. Suppose that L is idempotent and let E be an edge graph. From Lemma 2.14, $|L(E)| \geq 1$. Suppose $|L(E)| > 1$, say $L(E) \supseteq F \cup G$ where F and G are edge graphs and $F \neq E$.

Case 1. K_n is not FM_M -labelable. Let Q be a largest graph that is FM_M -labelable, so that if X is an edge graph not in Q , $Q \cup X$ is not FM_M -labelable. Further, by vertex permutation, assume that $E \in E(Q)$ and $F \notin E(Q)$. Then, $L(Q) = L(Q \cup E) = L(Q) \cup L(E) \supseteq L(Q) \cup F \cup G$. Hence $L(Q) = L^2(Q) \supseteq L^2(Q) \cup L(F) = L(Q) \cup L(F) = L^2(Q \cup F) = L(Q \cup F)$, a contradiction since Q is FM_M -labelable and any graph containing $Q \cup F$ is not FM_M -labelable.

Case 2. K_n is FM_M -labelable. Let Q be a largest graph that is not FM_M -labelable, so that if X is an edge graph not in Q , $Q \cup X$ is FM_M -labelable. Further, by vertex permutation, assume that $E \in E(Q)$ and $F \notin E(Q)$. Then, $L(Q) = L(Q \cup E) = L(Q) \cup L(E) \supseteq L(Q) \cup F \cup G$. Hence $L(Q) = L^2(Q) = L^2(Q) \cup L^2(E) \supseteq L(Q) \cup L(F) = L(Q \cup F)$, a contradiction since Q is not FM_M -labelable and any graph containing $Q \cup F$ is FM_M -labelable.

Having arrived at a contradiction in both cases, we have that $L(E)$ is an edge graph. $\square$

Lemma 2.17. Let $L : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be an idempotent linear operator. If L strongly preserves the set of FM_M -labelable graphs, then L is bijective.

Proof. By Lemma 2.1 we only need show L is injective and by Lemma 2.16 we only need show that the image of two edge graphs cannot be equal. So, suppose $L(E) = L(F)$ for edge graphs E and F . Let H be an edge graph which is not adjacent to F and such that $E \cup H$ is a 2-path. Then $E \cup H$ is not FM_M -labelable, while $F \cup G$ is FM_M -labelable. But now, $L(E \cup H) = L(E) \cup L(H) = L(F) \cup L(H) = L(F \cup H)$, a contradiction. $\square$

Lemma 2.18. Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator. If T strongly preserves the set of FM_M -labelable graphs, then T is bijective.

Proof. If T is not bijective, then T is not injective so that there are graphs X and Y , $X \neq Y$, such that $T(X) = T(Y)$. But then, any power of T has that property, so that $T^d(X) = T^d(Y)$ for d such that T^d is idempotent. But then, T^d is an idempotent linear operator that strongly preserves the set of graphs that are FM_M -labelable. Thus, T^d is bijective, a contradiction. $\square$

Lemma 2.19. Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator. If T strongly preserves the set of FM_M -labelable graphs, then T maps 2-stars to 2-stars.

Proof. By Lemma 2.18, T is bijective and hence T maps edge graphs to edge graphs. Suppose S is a 2-star. If $T(S)$ is not a 2-star, it must be a pair of parallel edges. That is the image of a 2-star is either a 2-star or is FM_M -labelable. If T maps a 2-star to a pair of parallel edges, then T maps a pair of parallel edges to a two star (since $\mathcal{G}_n$ is finite and T is bijective). Now a parallel pair of edges is FM_M -labelable, while a 2-star is not, thus the image of a parallel pair of edges must be a pair of parallel edges. Since T preserves FM_M -labelable graphs, $T(S)$ is a 2-star. $\square$

Theorem 2.20. *Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator. If T strongly preserves the set of FM_M -labelable graphs, then T is a vertex permutation.*

Proof. By Lemma 2.18, T is bijective and by Lemma 2.19 T preserves 2-stars. Clearly, a bijective map preserves the number of edges in a graph. Thus by Lemma 2.3, T is a vertex permutation. $\square$

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