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Positive harmonic functions on biregular trees

Research Article

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Abstract: We show that if f is a positive harmonic function on a biregular tree which has maximal growth along an infinite path in the tree, then every harmonic function g on the tree with $0 \leq g \leq f$ is a multiple of f, thus generalizing a result of Cartier about regular trees.

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1. Introduction

A tree is homogeneous (or regular) if all its vertices have the same degree; it is *biregular* if any vertices x and y whose distance is even have the same degree, which we will assume greater than two. Regular and biregular trees are infinite.

A complex valued function f defined on the vertices of a graph is *harmonic at a vertex* x if its value at x is the arithmetical mean of its values at the neighbours of x; the function is *harmonic on the graph* if it is harmonic at every vertex of the graph. The study of harmonic functions on graphs is connected to such diverse domains as probability [7], potential theory [1] or harmonic analysis.

In particular, since the seminal work of Cartier [4] the properties of harmonic functions on regular trees have been thoroughly investigated (see for example [5], [1] and the references therein). Although biregular trees are quite straightforward generalizations of regular trees, they have not attracted a similar interest.

In a preceding paper, we have shown that if h is a positive harmonic function on a biregular tree $\mathbb{T}$ of degrees $q + 1$ and $r + 1$, and x, y are adjacent vertices with x of degree $q + 1$, then

$$\frac{q+1}{q(r+1)}\, h(x) \leq h(y) \leq \frac{r(q+1)}{(r+1)}\, h(x). \tag{1}$$

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[6, Proposition 3.3, p. 76]. Moreover if $\chi = (\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$ is an infinite path in $\mathbb{T}$ (using the terminology of [3]), there exists one and only one positive harmonic function f on $\mathbb{T}$ with $f(x_0) = 1$ which has maximal growth on χ [6, Conclusion 3.5, p. 78]. Here we will show that such a function is also maximal in the following sense:

Theorem 1.1. *If g is a harmonic function on $\mathbb{T}$ with $0 \leq g \leq f$, then it is a multiple of f.*

Remark 1.2. *This was proved by Cartier for regular trees [4, Corollary 2.6 p. 236]. But contrary to Cartier, here we shall use only elementary tools.*

2. Positive harmonic functions

Proposition 2.1. *Given an infinite path $\chi = (\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$ in $\mathbb{T}$ with x_0 of degree $q+1$, there exists one and only one positive harmonic function f on $\mathbb{T}$ with $f(x_0) = 1$ such that f has maximal growth along χ. On χ, f is given by*

$$f(x_{2k}) = (qr)^k \quad \textit{and} \quad f(x_{2k+1}) = (qr)^k \, \frac{r(q+1)}{r+1}.$$

and on a vertex y not in χ it is defined as follows: let x be the vertex in χ closest to y, and n the distance between x and y; then

$$f(y) = f(x) \cdot (qr)^{-k} \quad \textit{if } n = 2k,$$

$$f(y) = f(x) \cdot (qr)^{-k} \, \frac{q+1}{q(r+1)} \quad \textit{if } n = 2k+1.$$

Proof. See [6, Conclusion 3.5, p. 78]. □

Let $\chi = (\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$ be an infinite path in $\mathbb{T}$ with x_0 of degree $q+1$ and f the function defined as in Proposition 2.1.

Corollary 2.2. *If $\zeta = (\dots, z_{-2}, z_{-1}, z_0, z_1, z_2, \dots)$ is an infinite path in $\mathbb{T}$ such that $f(z_0) = 1$ and $\lim_{j \to +\infty} f(z_j) = +\infty$, then there exists $m \in \mathbb{Z}$ with $z_j = x_j$ for all $j \geq m$.*

Corollary 2.3. *If $\zeta = (\dots, z_{-2}, z_{-1}, z_0, z_1, z_2, \dots)$ is an infinite path in $\mathbb{T}$ such that there exists $m \in \mathbb{Z}$ with $z_j = x_j$ for all $j \geq m$, then $f(z_j) = f(x_j)$ for all $j \in \mathbb{Z}$.*

It follows also from the definition of f that it has maximal growth between any two adjacent vertices x and y: if $\deg x = q+1$ and $\deg y = r+1$, then

$$f(y) = \frac{q+1}{q(r+1)} \, f(x) \quad \text{or} \quad f(y) = \frac{r(q+1)}{(r+1)} \, f(x);$$

and if $\deg x = r+1$ and $\deg y = q+1$, then

$$f(y) = \frac{r+1}{r(q+1)} \, f(x) \quad \text{or} \quad f(y) = \frac{q(r+1)}{(q+1)} \, f(x).$$

Conversely, if a positive harmonic function g on $\mathbb{T}$ has maximal growth between any two adjacent vertices, then there exists an infinite path in $\mathbb{T}$ along which g has maximal growth. Indeed, a given vertex z_0 in $\mathbb{T}$ has at least a neighbour z_{-1} with $g(z_{-1}) < g(z_0)$ and a neighbour z_1 with $g(z_1) > g(z_0)$, by the harmonicity of g; similarly, there exist at least a neighbour z_{-2} of z_{-1} with $g(z_{-2}) < g(z_{-1})$ and a neighbour z_2 of z_1 with $g(z_2) > g(z_1)$; in this way we can construct step by step the needed infinite path $(\dots, z_{-2}, z_{-1}, z_0, z_1, z_2, \dots)$.

We are now ready to prove Theorem 1.1. So, let us take an harmonic function g on $\mathbb{T}$ such that $0 \leq g \leq f$.

If $g = 0$ or $g = f$, g is a mutiple of f. If $g(z) = 0$ on a vertex z of $\mathbb{T}$, then g is necessarily null on the whole tree, by the minimum principle, similar to the one on $\mathbb{R}^n$ [2, p. 71]. The function $f - g$ is positive harmonic on $\mathbb{T}$; if $(f - g)(z) = 0$ on a vertex z of $\mathbb{T}$, then $f - g$ is necessarily null on the whole tree, by the same minimum principle. Hence we can assume that $0 < g < f$ on all $\mathbb{T}$.

Firstly we suppose that g has maximal growth between any two adjacent vertices in $\mathbb{T}$. From the discussion held above we deduce that there exists an infinite path $\zeta = (\dots, z_{-2}, z_{-1}, z_0, z_1, z_2, \dots)$ on which it has maximal growth. Then $\lim_{j\to+\infty} g(z_j) = +\infty$ and, since $g < f$, $\lim_{j\to+\infty} f(z_j) = +\infty$. From Corollary 2.3 follows that there exist $m, n \in \mathbb{Z}$ with $z_j = x_{m+j}$ for all $j \geq n$. And then g is a multiple of f.

Next, we suppose that g has not everywhere maximal growth; this means that there exist two adjacent vertices x and y satisfying one of the followings

i) $\deg x = q + 1$, $\deg y = r + 1$ and

$$\frac{q+1}{q(r+1)}\, g(x) < g(y) \quad \text{or} \quad g(y) < \frac{r(q+1)}{(r+1)}\, g(x);$$

ii) $\deg x = r + 1$, $\deg y = q + 1$ and

$$\frac{r+1}{r(q+1)}\, g(x) < g(y) \quad \text{or} \quad g(y) < \frac{q(r+1)}{(q+1)}\, g(x).$$

It will suffice to study the case i).

Using Corollary 2.2, we may change the path χ without changing the function f so that $x = x_{2k}$ for some $k \in \mathbb{Z}$ (and then $f(x) = f(x_{2k}) = (qr)^k$). We put

$$\beta := g(x_{2k}) = g(x).$$

We suppose now that

$$g(x_{2k+1}) = \frac{r(q+1)}{r+1}\, g(x_{2k}) = \frac{r(q+1)}{r+1}\, \beta.$$

Let $x_{2k-1}, y_1, \dots, y_{q-1}$ be the other neighbours of x_{2k}. If g takes the same value γ on the vertices $x_{2k-1}, y_1, \dots, y_{q-1}$, then the harmonicity of g in x_{2k}:

$$\frac{1}{q+1}\left[g(x_{2k+1}) + g(x_{2k-1}) + g(y_1) + \dots + g(y_{q-1})\right] = g(x_{2k})$$

can be written

$$\frac{1}{q+1}\left[\frac{r(q+1)}{r+1}\, \beta + q\gamma\right] = \beta$$

or

$$\frac{r}{r+1}\, \beta + \frac{q}{q+1}\, \gamma = \beta;$$

hence

$$\frac{q}{q+1}\, \gamma = \frac{r+1-r}{r+1}\, \beta$$

and finally

$$\gamma = \frac{q+1}{q(r+1)}\, \beta,$$

from which we deduce that g does take this value γ on all vertices $x_{2k-1}, y_1, \ldots, y_{q-1}$, because in the contrary it would take at least once a value inferior to γ, contradicting (1). But then a vertex y as in i) above does not exist. We conclude that

$$g(x_{2k+1}) < \frac{r(q+1)}{r+1}\, g(x_{2k}) = \frac{r(q+1)}{r+1}\, \beta.$$

So there exists $0 < \alpha < 1$ with

$$g(x_{2k+1}) = \frac{r(q+1)}{r+1}\, \alpha\beta;$$

and then we may find an integer $\ell \in \mathbb{N}$ such that

$$\frac{qr}{(qr)^\ell - 1}\left[\frac{(qr)^k}{\beta} - 1\right] < 1 - \alpha, \tag{2}$$

since $\beta = g(x_{2k}) < f(x_{2k}) = (qr)^k$.

Let T be the subtree of $\mathbb{T}$ formed by all the paths $\zeta = (z_0, z_1, z_2, \ldots)$ with $z_0 = x_{2k}$ and $z_1 \neq x_{2k+1}$; in particular $(x_{2k}, x_{2k-1}, x_{2k-2}, \ldots)$ is a path in T. We write $S(x_{2k}, j)$ the sphere in $\mathbb{T}$ of centre x_{2k} and radius j, that is the set of vertices in $\mathbb{T}$ whose distance to x_{2k} is j. An easy recurrence shows that

$$|S(x_{2k}, j) \cap T| = q^{\lfloor (j+1)/2 \rfloor} \cdot r^{\lfloor j/2 \rfloor}$$

for all $j \in \mathbb{N}$ (where $\lfloor r \rfloor = \max\{m \in \mathbb{Z} : m \leq r\}$ if $r \in \mathbb{R}$). In particular

$$|S(x_{2k}, 2\ell) \cap T| = (qr)^\ell.$$

An automorphism of $\mathbb{T}$ which fixes x_{2k} and all vertices out of T sends any $S(x_{2k}, j) \cap T$ to itself. Hence we can identify the group of permutations of $\{1, 2, \ldots, (qr)^\ell\}$ to a subgroup $\mathcal{S}$ of automorphisms of $\mathbb{T}$ which fix x_{2k} and all vertices out of T and permute the vertices of $S(x_{2k}, 2\ell) \cap T$.

Given an automorphism σ of $\mathbb{T}$ and a function ϕ defined on $\mathbb{T}$ we put

$$\sigma(\phi)(y) := \phi(\sigma(y))$$

for all $y \in \mathbb{T}$. This defines a function $\sigma(\phi)$ on $\mathbb{T}$ which is positive if ϕ is positive and harmonic if ϕ is harmonic. Also, if $\phi < \phi'$ then $\sigma(\phi) < \sigma(\phi')$.

We choose now

$$h := \frac{1}{|\mathcal{S}|} \sum_{\sigma \in \mathcal{S}} \sigma(g);$$

it is positive harmonic on $\mathbb{T}$. Since $g < f$, $\sigma(g) < \sigma(f)$ for all $\sigma \in \mathcal{S}$ and then

$$h < \frac{1}{|\mathcal{S}|} \sum_{\sigma \in \mathcal{S}} \sigma(f) = f$$

by the definition of f. Moreover, if y, y' are two vertices in T whose distances to x_{2k} are equal and not more than 2ℓ, then $h(y) = h(y')$. Finally, because any $\sigma \in \mathcal{S}$ fixes x_{2k} and x_{2k+1},

$$h(x_{2k+1}) = g(x_{2k+1}) = \frac{r(q+1)}{r+1}\, \alpha\beta$$

and

$$h(x_{2k}) = g(x_{2k}) = \beta.$$

We then put, for all $j \in \mathbb{N}$ with $0 \le j \le 2\ell$,

$$\beta_j := h(x_{2k-j});$$

in fact $\beta_j = h(y)$ for any $y \in S(x_{2k}, j) \cap T$.

We will now prove that if n is odd

$$\beta_n = \frac{(q+1)\beta}{q^{(n+1)/2}\, r^{(n-1)/2}(r+1)} \left[\sum_{j=0}^{n} q^{\lfloor j/2 \rfloor} r^{\lfloor (j+1)/2 \rfloor} - \alpha \sum_{j=1}^{n} q^{\lfloor j/2 \rfloor} r^{\lfloor (j+1)/2 \rfloor} \right],$$

and if n is even

$$\beta_n = \frac{\beta}{q^{n/2}\, r^{n/2}} \left[\sum_{j=0}^{n} q^{\lfloor j/2 \rfloor} r^{\lfloor (j+1)/2 \rfloor} - \alpha \sum_{j=1}^{n} q^{\lfloor j/2 \rfloor} r^{\lfloor (j+1)/2 \rfloor} \right],$$

by strong induction on $0 \le n \le 2\ell$. The case $n = 0$ is immediate:

$$\beta_0 = \frac{\beta}{q^0\, r^0} \left[q^{\lfloor 0 \rfloor} r^{\lfloor 1/2 \rfloor} \right] = \beta.$$

The value β_1 is obtained using the harmonicity of h at x_{2k}: if $x_{2k+1}, x_{2k-1}, y_1, \ldots, y_{q-1}$ are the neighbours of x_{2k},

$$\frac{1}{q+1}[h(x_{2k+1}) + h(x_{2k-1}) + h(y_1) + \ldots h(y_{q-1})] = h(x_{2k})$$

can be written

$$\frac{1}{q+1}\left[\frac{r(q+1)}{r+1}\, \alpha\beta + q\beta_1 \right] = \beta$$

or

$$\frac{r}{r+1}\, \alpha\beta + \frac{q}{q+1}\beta_1 = \beta;$$

hence

$$\frac{q}{q+1}\beta_1 = \frac{(r+1)\beta - \alpha r \beta}{r+1}$$

and finally

$$\beta_1 = \frac{(q+1)\beta}{q(r+1)}\, [r + 1 - \alpha r],$$

which establishes the case $n = 1$. Then we suppose $n \ge 2$ and the assertion true for $0, \ldots, n-1$. Firstly the case n even: the value β_n is obtained by using the harmonicity of h in $x_{2k-(n-1)} = x_{2k-n+1}$, which is of degree $r + 1$: if $x_{2k-n+2}, x_{2k-n}, y_1, \ldots, y_{r-1}$ are the neighbours of x_{2k-n+1},

$$\frac{1}{r+1}[h(x_{2k-n+2}) + h(x_{2k-n}) + h(y_1) + \ldots h(y_{r-1})] = h(x_{2k-n+1})$$

can be written

$$\frac{1}{r+1}\, [\beta_{n-2} + r\beta_n] = \beta_{n-1};$$

hence

$$r\beta_n = (r+1)\beta_{n-1} - \beta_{n-2}$$

that is, by the induction hypothesis and since n is even,

$$\begin{aligned}
r\beta_n &= \frac{(q+1)\beta}{q^{n/2}\,r^{(n-2)/2}}\left[\sum_{j=0}^{n-1} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor} - \alpha\sum_{j=1}^{n-1} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor}\right] \\
&\quad - \frac{\beta}{q^{(n-2)/2}\,r^{(n-2)/2}}\left[\sum_{j=0}^{n-2} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor} - \alpha\sum_{j=1}^{n-2} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor}\right] \\
&= \frac{(q+1)\beta}{q^{n/2}\,r^{(n-2)/2}}\left[\sum_{j=0}^{n-1} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor} - \alpha\sum_{j=1}^{n-1} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor}\right] \\
&\quad - \frac{q\beta}{q^{n/2}\,r^{(n-2)/2}}\left[\sum_{j=0}^{n-2} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor} - \alpha\sum_{j=1}^{n-2} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor}\right] \\
&= \frac{\beta}{q^{n/2}\,r^{(n-2)/2}}\left[\sum_{j=0}^{n-1} q^{\lfloor j/2\rfloor+1} r^{\lfloor (j+1)/2\rfloor} - \alpha\sum_{j=1}^{n-1} q^{\lfloor j/2\rfloor+1} r^{\lfloor (j+1)/2\rfloor}\right. \\
&\quad\quad \left. + \sum_{j=0}^{n-1} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor} - \alpha\sum_{j=1}^{n-1} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor}\right] \\
&\quad - \frac{\beta}{q^{n/2}\,r^{(n-2)/2}}\left[\sum_{j=0}^{n-2} q^{\lfloor j/2\rfloor+1} r^{\lfloor (j+1)/2\rfloor} - \alpha\sum_{j=1}^{n-2} q^{\lfloor j/2\rfloor+1} r^{\lfloor (j+1)/2\rfloor}\right] \\
&= \frac{\beta}{q^{n/2}\,r^{(n-2)/2}}\left[q^{\lfloor (n-1)/2\rfloor+1} r^{\lfloor n/2\rfloor} - \alpha q^{\lfloor (n-1)/2\rfloor+1} r^{\lfloor n/2\rfloor}\right. \\
&\quad\quad \left. + \sum_{j=0}^{n-1} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor} - \alpha\sum_{j=1}^{n-1} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor}\right] \\
&= \frac{\beta}{q^{n/2}\,r^{(n-2)/2}}\left[\sum_{j=0}^{n} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor} - \alpha\sum_{j=1}^{n} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor}\right]
\end{aligned}$$

using the fact that $\lfloor (n-1)/2\rfloor + 1 = (n-2)/2+1 = n/2 = \lfloor n/2\rfloor$ and $\lfloor n/2\rfloor = n/2 = \lfloor (n+1)/2\rfloor$. Hence

$$\beta_n = \frac{\beta}{q^{n/2}\,r^{n/2}}\left[\sum_{j=0}^{n} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor} - \alpha\sum_{j=1}^{n} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor}\right]$$

and the case n even is established. The case n odd can be handled in a similar way.

In particular, the case $n = 2\ell$ is now established:

$$\beta_{2\ell} = \frac{\beta}{q^{\ell}\,r^{\ell}}\left[\sum_{j=0}^{2\ell} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor} - \alpha\sum_{j=1}^{2\ell} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor}\right].$$

But

$$\begin{aligned}\sum_{j=1}^{2\ell} q^{\lfloor j/2\rfloor} r^{\lfloor (j+1)/2\rfloor} &= r + qr + qr^2 + q^2r^2 + \cdots + q^{\ell-1}r^{\ell} + q^{\ell}r^{\ell} \\ &= (r+qr)\sum_{i=0}^{\ell-1}(qr)^i \\ &= (r+qr)\frac{(qr)^{\ell}-1}{qr-1}.\end{aligned}$$

Therefore

$$\beta_{2\ell} = \frac{\beta}{(qr)^{\ell}}\left[(r+qr)\frac{(qr)^{\ell}-1}{qr-1} + 1 - \alpha(r+qr)\frac{(qr)^{\ell}-1}{qr-1}\right].$$

From $h < f$ follows $h(x_{2k-2\ell}) < f(x_{2k-2\ell})$, that is $\beta_{2\ell} < (qr)^{k-l}$ or

$$\frac{\beta}{(qr)^{\ell}}\left[(r+qr)\frac{(qr)^{\ell}-1}{qr-1} + 1 - \alpha(r+qr)\frac{(qr)^{\ell}-1}{qr-1}\right] < \frac{(qr)^k}{(qr)^{\ell}}.$$

Hence

$$\frac{\beta}{qr-1}\left[(r+qr)((qr)^{\ell}-1) + qr - 1 - \alpha(r+qr)((qr)^{\ell}-1)\right] < (qr)^k.$$

Then

$$(r+qr)((qr)^{\ell}-1) + qr - 1 - \alpha(r+qr)((qr)^{\ell}-1) < \frac{(qr-1)(qr)^k}{\beta}$$

and

$$(r+qr)((qr)^{\ell}-1) + qr - 1 - \frac{(qr-1)(qr)^k}{\beta} < \alpha(r+qr)((qr)^{\ell}-1),$$

from which we deduce

$$1 + \frac{(qr-1) - (qr-1)(qr)^k/\beta}{(r+qr)((qr)^{\ell}-1)} < \alpha$$

and further

$$\begin{aligned}1-\alpha &< (qr-1)\frac{(qr)^k/\beta - 1}{(r+qr)((qr)^{\ell}-1)} \\ &< \frac{qr}{(r+qr)((qr)^{\ell}-1)}\left[\frac{(qr)^k}{\beta} - 1\right] \\ &< \frac{qr}{(qr)^{\ell}-1}\left[\frac{(qr)^k}{\beta} - 1\right],\end{aligned}$$

in contradiction to our choice (2) of ℓ. We conclude that g must have maximal growth everywhere; but then we have already shown that it is a multiple of f: the theorem is proved.

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