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On energies of graphs with given independence number and families of hyperenergetic graphs

Research Article

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Abstract: Let G be a simple graph of order n and $\mathscr{L}(G) \equiv \mathscr{L}^1(G)$ its line graph. Then, the iterated line graph of G is defined recursively as $\mathscr{L}^2(G) \equiv \mathscr{L}(\mathscr{L}(G)), \mathscr{L}^3(G) \equiv \mathscr{L}(\mathscr{L}^2(G)), \ldots, \mathscr{L}^k(G) \equiv \mathscr{L}\left(\mathscr{L}^{k-1}(G)\right)$. The energy $\mathcal{E}(G)$ is the sum of absolute values of the eigenvalues of G. In this paper, it is derived a sharp upper bound for the energy of the line graph of a connected graph G of order n and independence number not less than α where $1 \leq \alpha \leq n-2$. This bound is attained, if and only if, G is isomorphic to the complete split graphs $SK_{n,\alpha}$. It is also determined a lower bound for the energy of the line graph of a graph G of order n and independence number α. For $1 \leq \alpha \leq n-1$ and $\mathcal{H} = \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor + 1} \bigcup \left(\alpha + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor - n\right) K_{\lfloor \frac{n}{\alpha} \rfloor}$, the equality holds, if and only if $G \cong \mathcal{H}$. As a consequence, families of hyperenergetic graphs are determined. Also, a lower bound for the energy of the iterated line of a graph G of order n and independence number α is given and, for $1 \leq \alpha \leq n-1$, the equality holds, if and only if, $G \cong \alpha K_{\lfloor \frac{n}{\alpha} \rfloor}$. Additionally, an upper bound for the incidence energy of connected graphs G of order n and independence number not less than α is presented. Moreover, an upper bound on the Laplacian energy-like of the complement $\overline{G}$ of G is presented. For $1 \leq \alpha \leq n-1$, the bound is attained, if and only if, $G \cong \mathcal{H}$. Finally, a Nordhaus-Gaddum type relation is given.

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1. Introduction

In this section, some notation and framework are introduced. Let G be an undirected simple graph with vertex set $V(G)$ of cardinality n, where each $v_i, i \in \{1, \ldots, n\}$ denotes a vertex of G and the edge set is $E = E(G)$, of cardinality m, where each $e_j, j \in \{1, \ldots, m\}$ represents an edge of G. Sometimes, to simplify the notation we represent a vertex v_i just by its index i and then an edge is just represented by ij. In this case we say that i is adjacent to j, or that i and j are neighbors. The degree of a vertex v and its set of neighbors are denoted by d_v and $N_G(v)$, respectively. The graph G is *bipartite* if its vertex set can be splitted into two disjoint sets V_1 and V_2, such that every edge connects a vertex in V_1 to one vertex in V_2, and there are no edges between vertices in the disjoint sets, that is $V_1 \cap V_2 = \emptyset$. A bipartite graph (V_1, V_2, E) is said to be *complete*, usually denoted by K_{pq}, if $|V_1| = p$, $|V_2| = q$ and $ij \in E$ for all $i \in V_1$ and $j \in V_2$. Let $A(G)$ and $D(G)$ be the *adjacency matrix* and the *diagonal matrix of vertex degrees* of G, respectively. The *complement* of a graph G is represented by $\overline{G}$. The *Laplacian matrix* of G is the matrix $L(G) = D(G) - A(G)$ and the *signless Laplacian matrix* is $Q(G) = A(G) + D(G)$. The eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are called the *eigenvalues, Laplacian eigenvalues* and *signless Laplacian eigenvalues* of G, respectively. In this work, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the eigenvalues of $A(G)$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and $q_1 \geq q_2 \geq \cdots \geq q_n$ are the eigenvalues of $L(G)$ and $Q(G)$, respectively. The *line graph*, $\mathscr{L}(G)$, is the graph whose vertex set is in one-to-one correspondence with the edge set of G where two vertices are adjacent, if and only if, the corresponding edges in G have a common vertex. The energy of the line graph of a graph G and its relations with other concepts of graph energies were earlier studied for instance in [17, 27]. The *join* of two vertex disjoint graphs G_1 and G_2 is the graph obtained from the disjoint union $G_1 \cup G_2$, by adding new edges from each vertex in G_1 to every vertex in G_2. It is usually denoted by $G_1 \vee G_2$.

This graph operation can be generalized in the following way: Let H be a graph of order k and $V(H) = \{1, 2, \ldots, k\}$. Let $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ be a set of pairwise vertex disjoint graphs. Here, each vertex $j \in V(H)$ is assigned to the graph $G_j \in \mathcal{F}$. Let G be the graph obtained from the graphs $G_1, G_2, \ldots, G_k$, and the edges connecting each vertex of G_i with all the vertices of G_j, for all edge $ij \in E(H)$. That is, G is the graph with vertex set

$$V(G) = \bigcup_{i=1}^{k} V(G_i)$$

and edge set

$$E(G) = \left(\bigcup_{i=1}^{k} E(G_i)\right) \bigcup \left(\bigcup_{ij \in E(H)} \{uv : u \in V(G_i), v \in V(G_j)\}\right).$$

This graph is called the $H-join$ (or *generalized composition*) of the graphs $G_1, G_2, \ldots, G_k$, [9, 10, 47], and it is denoted by

$$G = \bigvee_{H} \{G_j : 1 \leq j \leq k\}.$$

As usual, K_n, P_n, C_n and S_n denote the *complete graph, path, cycle* and *star* on n vertices, respectively. A cocktail party graph is a graph consisting of two rows of paired vertices in which all vertices but the paired ones are connected with an edge. An *independent set* is a set of vertices in a graph where no two of which are adjacent. The *independence number* of a graph G, denoted by $\alpha(G)$, or just α, if there is no ambiguity, is the number of vertices of a largest independent set in G. The *vertex connectivity*, denoted by $\kappa(G)$, is the minimum number of vertices whose deletion disconnects G. The minimum number of edges that disconnects G is called *edge connectivity*. Moreover, a *complete split graph* $SK_{n,\alpha}$ is a graph that can be partitioned into an independent set on α vertices and a clique on $n-\alpha$ vertices such that every vertex in the independent set is adjacent to every vertex in the clique. Clearly, $SK_{n,\alpha} \cong \bigvee_{P_2} \{\overline{K_\alpha}, K_{n-\alpha}\}$. For other undefined notations and terminology from graph theory, the readers are referred to [4].

The *incidence matrix* of G is the $n \times m$ matrix $\mathbb{I}(G)$ whose (i,j)-entry is 1 if v_i is incident to e_j and 0 otherwise. It is known, [11], that

$$\mathbb{I}(G)\mathbb{I}(G)^T = D(G) + A(G) = Q(G). \tag{1}$$

The definition of *energy of a graph*, $\mathcal{E}(G)$, was introduced by I. Gutman in 1978, [23], as the sum of the absolute values of its eigenvalues,

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Although in mathematical literature, this quantity was formally put forward in 1978, its chemical roots go back to the 1940s and it is used to approximate the total π-electron energy of a molecule [24, 38]. Note that the concept of graph energy is defined for all graphs, and many mathematicians studied it without being restricted by any chemistry-caused constrains; see for instance the recent papers [18, 22, 29, 32, 33, 40, 48, 49] and the references cited therein. Besides applications in Chemistry, this concept can also be applied in medical sciences, see for instance [31] where the authors use it in glaucoma detection with machine learning techniques. V. Nikiforov [42] proposed an extension of the graph energy concept: for a $p \times q$ matrix M with real entries and singular values $s_1, s_2, \ldots, s_p$, the energy of M is defined as

$$\mathcal{E}(M) = \sum_{i=1}^{p} s_i.$$

Note that if M is real and symmetric then the singular values are the absolute values of its eigenvalues. The *Laplacian energy* of a graph G with n vertices and m edges is defined in [28] as follows

$$LE(G) = \sum_{j=1}^{n} \left| \mu_j - \frac{2m}{n} \right|.$$

The Laplacian energy is nowadays widely study; see for instance [15, 16, 28, 44] and all the references cited therein. Similarly, the *signless Laplacian energy* of a graph G with n vertices and m edges is defined in [27] by

$$LE^+(G) = \sum_{j=1}^{n} \left| q_j - \frac{2m}{n} \right|.$$

Notice that the definition of energy of a matrix given in [42] is in perfectly harmony with the ordinary graph energy as it is easily seen that $\mathcal{E}(G) = \mathcal{E}(A(G))$, $LE(G) = \mathcal{E}\left(L(G) - \frac{2m}{n}I\right)$, $LE^+(G) = \mathcal{E}\left(Q(G) - \frac{2m}{n}I\right)$. Details on the properties of the two last previous graph energies can be found for instance in [1, 12, 15, 28, 52]. Therefore, attending to previous observations, the concept of graph energy was extended to non symmetric and even to non square matrices and gave rise to new concepts related with different energies like skew energy, incidence energy, see e.g. [2, 25, 26].

The energy of the line graph of a graph G and its relations with the other graph energies were earlier studied for instance in [17, 27]. Some recent work that relates the signless Laplacian energy of a graph and the energy of its line graph as function of some invariant parameters (like the first Zagreb index, the clique number, n, m) is done in [21]. Moreover, in [37], a sharp upper bound for the energy of the line graph of a graph G having vertex connectivity less than or equal to a positive number was obtained. In addition, upper bounds on the energy in terms of the edge connectivity, the inertia and the matching number of G were presented and a new family of *hyperenergetic* graphs (graphs for which its energy is greater than the energy of the complete graph) was given.

In [25, 34], the authors introduce the notion of *incidence energy* $IE(G)$ of G as the sum of the singular values of the incidence matrix $\mathbb{I}(G)$. From these facts and (1), it follows that

$$IE(G) = \sum_{i=1}^{n} \sqrt{q_i}.$$

Analogously, the *Laplacian-energy like* $LEL(G)$ of G is defined as

$$LEL(G) = \sum_{i=1}^{n} \sqrt{\mu_i}.$$

This energy was presented in [39], and some properties can be seen there.

In [19], a relation between the energy and the energy of its line graph is presented and some bounds on the energy of the line graph were obtained. Moreover, results related to *equienergetic* (non-isomorphic graphs that have the same energy) and hyperenergetic graphs were also given.

Let now $\mathcal{F}_n$ be the family of simple undirected connected graphs on n vertices and $k \in \mathbb{N}$. Consider the set $\mathcal{V}_n^k = \{G \in \mathcal{F}_n : \kappa(G) \leq k\}$. If $G \in \mathcal{V}_n^k$ then the following upper bounds for $IE(G)$ and $LEL(G)$ can be seen in [45] and [53], respectively:

$$\begin{aligned} LEL(G) &\leq k\sqrt{n} + \sqrt{k} + (n-k-2)\sqrt{n-1}, \\ IE(G) &\leq k\sqrt{n-2} + (n-k-2)\sqrt{n-3} + \sqrt{n-2+\frac{k}{2}+\frac{1}{2}\sqrt{(2n-k)^2+16(k-n+1)}} \\ &+ \sqrt{n-2+\frac{k}{2}-\frac{1}{2}\sqrt{(2n-k)^2+16(k-n+1)}}. \end{aligned}$$

The equality holds for both bounds, if and only if, $G \cong \bigvee_{P_3}\{K_1, K_k, K_{n-k-1}\}$. Throughout this paper, $Sp(M)$ is the spectrum associated to a square matrix M. When M is the adjacency matrix of a graph the used notation is $Sp(G)$. This paper is organized in the following way. In Section 1, besides the main concepts and notation used throughout the paper we present some recent work that motivated the authors. In Section 2, the spectrum of the line graph of the complete split graph $SK_{n,\alpha}$ is presented and its negative eigenvalues and respective multiplicities are discussed.

In Section 3, when G is connected, an upper bound for $\mathcal{E}(\mathscr{L}(G))$ in terms of n and the independence number α is given and it is shown that the equality is attained in the complete split graphs $G \cong SK_{n,\alpha}$. Additionally, we present conditions for which $\mathscr{L}(K_{n,\alpha})$ is hyperenergetic reforcing the idea from Walikar et al, [50], that K_n its not the only graph that has maximum energy, a conjecture firstly posed in [23]. Moreover, we derive a lower bound for the energy of the line graph of graphs of order n and independence number α. It is shown that the bound is attained, if and only if, $G \cong \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor + 1} \bigcup \left(\alpha + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor - n\right) K_{\lfloor \frac{n}{\alpha} \rfloor}$. In Section 4, a lower bound for the energy of the iterated line graph of graphs of order n and independence number α is given and, it is shown that the equality holds, if and only if, $G \cong \alpha K_{\lfloor \frac{n}{\alpha} \rfloor}$. In Section 5, an upper bound for the incidence energy of connected graphs G of order n and independence number not less than α is given. In Section 6, an upper bound for the Laplacian energy-like of the complement of a graph G of order n and independence number α is presented. Again, it is shown that the bound is attained, if and only if, $G \cong \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor + 1} \bigcup \left(\alpha + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor - n\right) K_{\lfloor \frac{n}{\alpha} \rfloor}$. Finally, a Nordhaus-Gaddum type relation is given.

2. Spectrum of the line graph $\mathscr{L}(SK_{n,\alpha})$

In this section, we present the signless Laplacian spectrum of the complete split graph $SK_{n,\alpha}$ viewed as a join of two graphs and then, as a consequence, the spectrum of $\mathscr{L}(SK_{n,\alpha})$ is presented. The multiplicities of its eigenvalues are also presented.

Recall the definition of H-join and consider H a graph of order k. Let $\{G_1, \ldots, G_k\}$ be a family of graphs where each G_j is of order n_j, for $j \in \{1, \ldots, n\}$. We label its vertices with the labels $1, 2, \ldots, n-1, n$, starting with the vertices of G_1, then with the vertices of G_2 and so on and finally ending with the vertices of G_k. The next theorem describes the spectrum of the signless Laplacian matrix of the H-join of graphs when $\{G_1, \ldots, G_k\}$ is a family of regular graphs. In [9, Theorem 5], the spectrum of the adjacency matrix of the $H-join$ of regular graphs is obtained. The version of this result for the signless Laplacian matrix is given below and its proof is similar. Recently, in [46] the spectrum of the adjacency matrix of the H- join of a family of arbitrary graphs was studied.

Theorem 2.1. *[37] If $G \cong \bigvee_H \{G_j : 1 \leq j \leq k\}$ where G_j is a r_j-regular graph of order n_j, for all $j = 1, 2, \ldots, k$, then*

$$Sp(Q(G)) = \bigcup_{G_j \neq K_1} \{s_j + \lambda : \lambda \in Sp(Q(G_j)) - \{2r_j\}\} \bigcup Sp(Q_k(G))$$

where $Q_k(G)$ is a matrix of order $k \times k$ given by

$$Q_k(G) = \begin{bmatrix} s_1 + 2r_1 & \delta_{12}\sqrt{n_1 n_k} & \cdots & \delta_{1k}\sqrt{n_1 n_k} \\ \delta_{12}\sqrt{n_1 n_2} & s_2 + 2r_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \delta_{(k-1)k}\sqrt{n_{k-1} n_k} \\ \delta_{1k}\sqrt{n_1 n_2} & \cdots & \delta_{(k-1)k}\sqrt{n_{k-1} n_k} & s_k + 2r_k \end{bmatrix}$$

where $\delta_{ij} = 1$ if $ij \in E(H)$ and 0 otherwise, and

$$s_i = \sum_{ji \in V(H)} n_j$$

for $i = 1, 2, \ldots, k$.

In the next result the signless Laplacian eigenvalues of the graph $SK_{n,\alpha}$ are presented in an explicit way. We must refer that these eigenvalues were also presented in [30]. However, we believe that the list of eigenvalues is the one presented below as, from computational results, the list presented in [30] does not work, for instance for the graph $SK_{7,3}$ its signless Laplacian eigenvalues are $\frac{13+\sqrt{73}}{2} \approx 10.772$, $\frac{13-\sqrt{73}}{2} \approx 2.228$, 5 with multiplicity 3 and 4 with multiplicity 2. The signless Laplacian eigenvalues of the graph $SK_{n,\alpha}$ can also be seen in the reference [14, Theorem 3.2].

However, here we regard $SK_{n,\alpha} \cong \overline{K_\alpha} \vee K_{n-\alpha} \cong \bigvee_{P_2}\{\overline{K}_\alpha, K_{n-\alpha}\}$ and then from Theorem 2.1 the signless Laplacian eigenvalues of the graphs $SK_{n,\alpha}$ are obtained in the following lemma:

Lemma 2.2. *The signless Laplacian eigenvalues of $SK_{n,\alpha}$ where $2 \leq \alpha \leq n-2$ are given by*

$$\begin{cases} 3n/2 - \alpha - 1 + \sqrt{(n-2)^2 + 4\alpha(n-\alpha)}/2, & \\ 3n/2 - \alpha - 1 - \sqrt{(n-2)^2 + 4\alpha(n-\alpha)}/2, & \\ n-2 & n-\alpha-1 \text{ times}, \\ n-\alpha & \alpha-1 \text{ times}. \end{cases}$$

Lemma 2.3. *[5, 27] Let G be a graph of order n with $m \geq 1$ edges. Let q_i be the $i-th$ largest signless Laplacian eigenvalue of G and $\lambda_i(\mathscr{L}(G))$ the $i-th$ largest eigenvalue of the line graph $\mathscr{L}(G)$. Then,*

$$q_i = \lambda_i(\mathscr{L}(G)) + 2,$$

for $i = 1, 2, \ldots, k$, where $k = min\{n, m\}$. In addition, if $m > n$, then $\lambda_i(\mathscr{L}(G)) = -2$ for $i \geq n+1$ and if $n > m$, then $q_i = 0$ for $i \geq m+1$.

From Lemma 2.2 and Lemma 2.3, the following result is an immediate consequence.

Corollary 2.4. *The eigenvalues of* $\mathscr{L}(SK_{n,\alpha})$ *where* $2 \leq \alpha \leq n-2$ *are given by*

$$\begin{cases} 3n/2 - \alpha - 3 + \sqrt{(n-2)^2 + 4\alpha(n-\alpha)}/2, & \\ 3n/2 - \alpha - 3 - \sqrt{(n-2)^2 + 4\alpha(n-\alpha)}/2, & \\ n-4 & n-\alpha-1 \text{ times}, \\ n-\alpha-2 & \alpha - 1 \text{ times}, \\ -2 & m-n \text{ times}. \end{cases}$$

Theorem 2.5. *The negative eigenvalues of the graph* $\mathscr{L}(SK_{n,\alpha})$ *where* $2 \leq \alpha \leq n-2$ *are* -2 *as eigenvalue with multiplicity* $m-n$*, and additionally, when* $n < \alpha + \sqrt{\alpha} + 2$*, the eigenvalue* $3n/2 - \alpha - 3 - \sqrt{(n-2)^2 + 4\alpha(n-\alpha)}/2$ *is also negative and a simple characteristic root.*

Proof. Since $2 \leq \alpha \leq n-2$, we get

$$3n/2 - \alpha - 3 + \sqrt{(n-2)^2 + 4\alpha(n-\alpha)}/2 \geq 0,$$

$$n-4 \geq 0 \text{ and } n-\alpha-2 \geq 0.$$

Suppose $n - \alpha - 2 \geq \sqrt{\alpha}$ then $\alpha^2 + 3\alpha - 2n\alpha + n^2 + 4 - 4n \geq 0$, which is equivalent to:

$$(3n - 2\alpha - 6)^2 \geq (n-2)^2 + 4\alpha(n-\alpha) \geq 0.$$

Therefore, $3n/2 - \alpha - 3 \geq \sqrt{(n-2)^2 + 4\alpha(n-\alpha)}/2$, and in consequence, using the previous inequality it is easy to conclude that:

$$3n/2 - \alpha - 3 - \sqrt{(n-2)^2 + 4\alpha(n-\alpha)}/2 \geq 0.$$

Now, suppose $n < \alpha + \sqrt{\alpha} + 2$, using the previous argument, we get

$$3n/2 - \alpha - 3 - \sqrt{(n-2)^2 + 4\alpha(n-\alpha)}/2 < 0.$$

Thus, Theorem 2.5 follows from Corollary 2.4. □

3. Bounds on the energy of line graphs and a family of hyperenergetic graphs

In this section, bounds for the energy of the line graph of a graph are presented. Some of the bounds are determined for connected graphs with independence number not less than a positive number α. The equality cases are discussed. Additionally, we present conditions for which $\mathscr{L}(K_{n,\alpha})$ is hyperenergetic reforcing the idea from Walikar et al, [50], that K_n its not the only graph that has maximum energy, a conjecture firstly posed in [23].

The following result reveals that the energy of the line graph of a connected graph strictly increases when it is added an edge to the original graph.

Lemma 3.1. *[37] Let* G *be a connected graph on* n *vertices non-isomorphic to the complete graph* K_n*. Then,*

$$\mathcal{E}(\mathscr{L}(G)) < \mathcal{E}(\mathscr{L}(G+e)).$$

The next results are immediate consequences of previous lemma. However, the next corollary can also be seen in [27] and it is clear that the equality holds, if and only if, $H \cong G$.

Corollary 3.2. *[27] If H is a subgraph of a graph G, then*

$$\mathcal{E}(\mathscr{L}(H)) \leq \mathcal{E}(\mathscr{L}(G)).$$

Recalling that the signless Laplacian eigenvalues of the complete graph K_n are $2n-2$ and $n-2$ with multiplicity $n-1$, attending to Lemma 2.3 and Corollary 3.1 we can write:

Corollary 3.3. *Let G be a connected graph of order $n \geq 4$. Then,*

$$\mathcal{E}(\mathscr{L}(G)) \leq 2n^2 - 6n. \tag{2}$$

Equality holds in (2), if and only if, $G \cong K_n$.

Lemma 3.4. *The number of edges of the graph $SK_{n,\alpha}$ is*

$$\frac{(n-\alpha)(n+\alpha-1)}{2}.$$

Proof. The number of edges of the graph K_n is $\dfrac{n(n-1)}{2}$. Then, the number of edges of the graph $SK_{n,\alpha}$ is given by $m = \dfrac{(n-\alpha)(n-\alpha-1)}{2} + \alpha(n-\alpha) = \dfrac{(n-\alpha)(n+\alpha-1)}{2}$.

□

We remark that recently in [8, Lemma 2.7] the energy of line graph of the graph $K_r \vee \overline{K_{n-r}}$ for $2 \leq r \leq \frac{n-1}{2}$ was studied. In this work, under some simpler conditions, the energy of the line graph of the same graph viewed as a P_2- join of the family of graphs $\{\overline{K}_\alpha, K_{n-\alpha}\}$ is also presented in the proof of the next theorem. These conditions allowed us to determine a family of hyperenergetic graphs in Theorem 3.9. Note that, if $\alpha = n - r$ and $2 \leq r \leq \frac{n-1}{2}$ then $\dfrac{n+1}{2} \leq \alpha \leq n-2$ which implies that the energy determined for $\mathscr{L}(SK_{n,\alpha})$ at the proof of Theorem 3.5 generalizes the energy determined in [8, Lemma 2.7], as the condition used is $1 \leq \alpha \leq n-2$.

Theorem 3.5. *Let G be a connected graph of order $n \geq 4$ and independence number not less than α. Let $1 \leq \alpha \leq n-2$. Then,*

$$\mathcal{E}(\mathscr{L}(G)) \leq \begin{cases} 2(n^2 - 3n - \alpha^2 + \alpha) & if \;\; n \geq \alpha + \sqrt{\alpha} + 2 \\ 2n^2 - 9n + 6 - 2\alpha^2 + 4\alpha + \sqrt{(n-2)^2 + 4\alpha(n-\alpha)} & if \;\; n < \alpha + \sqrt{\alpha} + 2. \end{cases} \tag{3}$$

Equality holds in (3), if and only if, $G \cong SK_{n,\alpha}$.

Proof. Let G be a connected graph of order $n \geq 4$ and independence number not less than a positive integer α. Firstly consider $\alpha = 1$. Then $G \cong K_n$. Let $n \geq \alpha + \sqrt{\alpha} + 2$. We want to prove that $\mathcal{E}(\mathscr{L}(G)) \leq 2n^2 - 6n$. Since $K_{n,1} \cong K_n$, by Corollary 3.3, the result holds. Consider now $2 \leq \alpha \leq n-2$ and let G be a graph with $\alpha(G) = \alpha_1$ such that $\mathscr{L}(G)$ has largest energy among all connected graphs $\mathscr{L}(H)$, where H has order n and independence number $\alpha_1 \geq \alpha$. Let S be an independent set of G with cardinality $\alpha(G)$. Suppose that $G \ncong SK_{n,\alpha(G)}$, then there are two vertices non-adjacent $u, v \in V(G)$. We can assume two situations, $u \in S$ and $v \in V(G) - S$ or $u, v \in V(G) - S$. In both cases, a graph $G_1 \cong G + e$ where e is an edge connecting the vertices u, v can be constructed. By Lemma 3.1, $\mathcal{E}(\mathscr{L}(G)) < \mathcal{E}(\mathscr{L}(G_1))$, which is a contradiction to the maximality of G. Then, $G \cong SK_{n,\alpha(G)}$.

From Corollary 2.4, the following is obtained:

$$\begin{aligned}\mathcal{E}\big(\mathscr{L}\big(SK_{n,\alpha(G)}\big)\big) &= (\alpha(G)-1)|n-\alpha(G)-2| + (n-\alpha(G)-1)|n-4| + (m-n)|-2| \\ &+ \left|3n/2-\alpha(G)-3+\sqrt{(n-2)^2+4\alpha(G)(n-\alpha(G))}/2\right| \\ &+ \left|3n/2-\alpha(G)-3-\sqrt{(n-2)^2+4\alpha(G)(n-\alpha(G))}/2\right|.\end{aligned}$$

Suppose $n \geq \alpha + \sqrt{\alpha} + 2$. From Theorem 2.5 and Lemma 3.4, we get:

$$\begin{aligned}\mathcal{E}\big(\mathscr{L}\big(SK_{n,\alpha(G)}\big)\big) &= (\alpha(G)-1)(n-\alpha(G)-2) + (n-\alpha(G)-1)(n-4) \\ &+ (n-\alpha(G))(n+\alpha(G)-1) - 2n \\ &+ 3n/2-\alpha(G)-3+\sqrt{(n-2)^2+4\alpha(G)(n-\alpha(G))}/2 \\ &+ 3n/2-\alpha(G)-3-\sqrt{(n-2)^2+4\alpha(G)(n-\alpha(G))}/2.\end{aligned}$$

Then,

$$\mathcal{E}\big(\mathscr{L}\big(SK_{n,\alpha(G)}\big)\big) = 2(n^2-3n-\alpha^2(G)+\alpha(G)). \tag{4}$$

Define the function $f(x) = -x^2 + x + n^2 - 3n$ where $x \geq \alpha$. Clearly, f is strictly decreasing for $x \geq 1$. Consequently, $\mathcal{E}(\mathscr{L}(G)) \leq \mathcal{E}\big(\mathscr{L}\big(SK_{n,\alpha}\big)\big)$, for all connected graphs G of order n and independence number not less than α. Suppose that $n < \alpha + \sqrt{\alpha} + 2$. By Theorem 2.5, the eigenvalue $3n/2 - \alpha(G) - 3 - \sqrt{(n-2)^2 + 4\alpha(G)(n-\alpha(G))}/2$ is negative and then

$$\mathcal{E}\big(\mathscr{L}\big(SK_{n,\alpha(G)}\big)\big) = 2n^2 - 9n + 6 - 2\alpha(G)^2 + 4\alpha(G) + \sqrt{(n-2)^2 + 4\alpha(G)(n-\alpha(G))}. \tag{5}$$

Consider the functions $g(x) = -2x^2 + 4x + 2n^2 - 9n + 6$ and $h(x) = \sqrt{(n-2)^2 + 4x(n-x)}$ where $x \geq \alpha$. Clearly, g is strictly decreasing for $x \geq 1$ and h is strictly decreasing for $x \geq \dfrac{n}{2}$. The conditions $n < \alpha + \sqrt{\alpha} + 2$ and $2 \leq \alpha \leq n-2$ imply that if $n \geq 6$ then $\alpha \geq 3$.

Thus, $\sqrt{\alpha} \leq \alpha - 1$, which implies that $\alpha \geq \dfrac{n}{2}$, i.e., $g+h$ is strictly decreasing for $x \geq \alpha$. Taking into account the hypothesis $2 \leq \alpha \leq n-2$, it remains to consider the cases:

(a) $n = 4$ and $\alpha = 2$;

(b) $n = 5$ and $\alpha = 3$;

(c) $n = 5$ and $\alpha = 2$.

In cases (a) and (b) the condition $\alpha \geq \dfrac{n}{2}$ is verified. Thereby, $g+h$ is strictly decreasing for $x \geq \alpha$. If condition (c) is verified, that is, $n = 5$ and $\alpha = 2$, it is easy to see that among all integers the values for x for which the function $h(x) = \sqrt{9 + 4x(5-x)}$ attains its maximum value are $x = 2 = \alpha$ and $x = 3 = \alpha + 1$. Hence, the function $g+h$ attains it maximum value in $x = 2 = \alpha$. Consequently, $\mathcal{E}(\mathscr{L}(G)) \leq \mathcal{E}\big(\mathscr{L}\big(SK_{n,\alpha}\big)\big)$, for all connected graphs G of order n and independence number not less than α. The equality in (3) holds, if and only if, $G \cong SK_{n,\alpha}$.

The proof is complete. □ □

It remains to study the case $\alpha = n-1$. Recalling that $S_n \cong SK_{n,\alpha}$ with $\alpha = n-1$, and its signless Laplacian eigenvalues are n, 1 with multiplicity $n-2$ and 0, attending to Lemma 2.3, the next remark gives, for $n \geq 4$, an exact value for $\mathcal{E}\left(\mathscr{L}(S_n)\right)$.

Remark 3.6. *For $n \geq 4$, $\mathcal{E}(\mathscr{L}(S_n)) = 2(n-2)$.*

The next definition can be seen in [38].

Definition 3.7. *[38] A graph G on n vertices is said to be hyperenergetic graph if*

$$\mathcal{E}(G) \geq \mathcal{E}(K_n) = 2(n-1). \tag{6}$$

It was conjectured in [23] that all graphs have energy at most $2(n-1)$. Nevertheless, this was disproved by Walikar et al in [50] and the following result was presented:

Theorem 3.8. *[50] For $n \geq 5$, the line graph of K_n is hyperenergetic. For $n \geq 4$, the line graph of the complete bipartite graph $K_{n,n}$ is hyperenergetic. For $n \geq 6$, the line graph of the cocktail party graph with n vertices is hyperenergetic.*

As a consequence of the previous theorem we can state the following:

Theorem 3.9. *The line graph of the graph $SK_{n,\alpha}$ is hyperenergetic,*

(a) *if $\alpha \geq 2$ and $n \geq \alpha + \sqrt{\alpha} + 2$,*

(b) *if $2 \leq \alpha \leq n-3$ and $n < \alpha + \sqrt{\alpha} + 2$,*

(c) *if $\alpha = n-1$ and $n \geq 4$,*

(d) *if $\alpha = 1$ and $n \geq 5$.*

Proof. (a) Suppose $\alpha \geq 2$ and $n \geq \alpha + \sqrt{\alpha} + 2$. Then, $n \geq \alpha + 4$. Taking $\alpha(G) = \alpha$ in (4), we have $\mathcal{E}\big(\mathscr{L}\big(SK_{n,\alpha}\big)\big) = 2(n^2 - 3n - \alpha^2 + \alpha)$. By Lemma 3.4, the number of edges of $SK_{n,\alpha}$, is the number of vertices of $\mathscr{L}(SK_{n,\alpha})$, that is,

$$\frac{(n-\alpha)(n+\alpha-1)}{2}.$$

Imposing the condition (6), the following inequality is obtained

$$2(n^2 - 3n - \alpha^2 + \alpha) \geq 2\left(\frac{(n-\alpha)(n+\alpha-1)}{2} - 1\right).$$

From the previous inequality, we get $n^2 - 5n \geq \alpha^2 - \alpha - 2$, which is equivalent to:

$$n(n-5) \geq (\alpha+1)(\alpha-2).$$

Since $n \geq \alpha + 4$,
the previous inequality is always true and therefore the graph $\mathscr{L}(SK_{n,\alpha})$ is hyperenergetic.

(b) Consider now $n < \alpha + \sqrt{\alpha} + 2$. Suppose $3 \leq \alpha \leq n-3$ and take $\alpha(G) = \alpha$ in the expression in (5), we get $\mathcal{E}(\mathscr{L}(SK_{n,\alpha})) = 2n^2 - 9n + 6 - 2\alpha^2 + 4\alpha + \sqrt{(n-2)^2 + 4\alpha(n-\alpha)}$. Imposing the condition (6), the following inequality is obtained

$$2n^2 - 9n + 6 - 2\alpha^2 + 4\alpha + \sqrt{(n-2)^2 + 4\alpha(n-\alpha)} \geq 2\left(\frac{(n-\alpha)(n+\alpha-1)}{2} - 1\right). \tag{7}$$

Since the inequality

$$2n^2 - 8n - 2\alpha^2 + 4\alpha + 4 \geq 2\left(\frac{(n-\alpha)(n+\alpha-1)}{2} - 1\right),$$

it is equivalent to

$$(n-6)(n-1) \geq \alpha(\alpha-3),$$

the graph $\mathscr{L}\big(SK_{n,\alpha}\big)$ is hyperenergetic.
Suppose $\alpha = 2$ then $n = 5$. Hence, the inequality (7) is true. Thereby, the graph $\mathscr{L}\big(SK_{n,\alpha}\big)$ is hyperenergetic.

(c) For $\alpha = n-1$, follows from Remark 3.6 as $\mathcal{E}(\mathscr{L}(SK_{n,n-1})) = \mathcal{E}(\mathscr{L}(S_n)) = 2(n-2)$, and that $\mathscr{L}(S_n)$ has $n-1$ vertices.

(d) For $\alpha = 1$, we get $SK_{n,\alpha} \cong K_n$. Since $n \geq 5$, by Theorem 3.8 the proof is complete.

□

To illustrate the previous theorem, in next example we present two graphs that are hyperenergetic as they verify the conditions (a) and (b) of Theorem 3.9.

Example 3.10. *At Fig.1 the graphs $SK_{7,3}$ and $SK_{5,2}$ are depicted. The condition* (a) of Theorem 3.9 is fulfilled in the graph $SK_{7,3}$ and the condition (b) of Theorem 3.9 is fulfilled in the graph $SK_{5,2}$. In fact, in $SK_{7,3}$ we have $\alpha \geq 2$ and $n = 7 \geq 5+\sqrt{3} = \alpha+\sqrt{\alpha}+2$ then $\mathcal{E}(\mathscr{L}(SK_{7,3})) = 2(n^2-3n-\alpha^2+\alpha) = 44$. Additionally, $2(m-1) = 34$. Thus, $\mathscr{L}(SK_{7,3})$ is a hyperenergetic graph. Considering the graph $SK_{5,2}$, the condition (b) is satisfied as $2 \leq \alpha \leq n-3$ and $n = 5 < 4+\sqrt{2} = \alpha+\sqrt{\alpha}+2$ then $\mathcal{E}(\mathscr{L}(SK_{n,\alpha})) = 2n^2-9n+6-2\alpha^2+4\alpha+\sqrt{(n-2)^2+4\alpha(n-\alpha)} \approx 16.744563$. Moreover, $2(m-1) = 16$. Therefore, $\mathscr{L}(SK_{5,2})$ is a hyperenergetic graph.

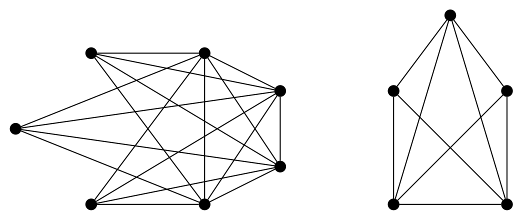


Figure 1. The complete split graphs $SK_{7,3}$ and $SK_{5,2}$.

The next remark and lemma will be an important tool to present a lower bound for the energy of the line graph of a graph of order n and independence number α.

Lemma 3.11. *[36] Let G be a graph of order n and independence number α where $1 \leq \alpha \leq n-1$. Then,*

$$\left(n-\alpha\left\lfloor\frac{n}{\alpha}\right\rfloor\right)K_{\lfloor\frac{n}{\alpha}\rfloor+1}\bigcup\left(\alpha+\alpha\left\lfloor\frac{n}{\alpha}\right\rfloor-n\right)K_{\lfloor\frac{n}{\alpha}\rfloor},$$

is a subgraph of G.

Theorem 3.12. *Let G be a graph of order n and independence number equals to a positive integer α. Let $1 \leq \alpha \leq n-1$. Then,*

$$\mathcal{E}(\mathscr{L}(G)) \geq \begin{cases} 0 & if \quad \left\lfloor\frac{n}{\alpha}\right\rfloor = 1 \\ 4(n-2\alpha) & if \quad \left\lfloor\frac{n}{\alpha}\right\rfloor = 2,3 \\ 2\left((2n-\alpha)\left\lfloor\frac{n}{\alpha}\right\rfloor - 2n - \alpha\left\lfloor\frac{n}{\alpha}\right\rfloor^2\right) & if \quad \left\lfloor\frac{n}{\alpha}\right\rfloor \geq 4. \end{cases} \tag{8}$$

The equality holds in (8), if and only if, $G \cong \left(n-\alpha\left\lfloor\frac{n}{\alpha}\right\rfloor\right)K_{\lfloor\frac{n}{\alpha}\rfloor+1}\bigcup\left(\alpha+\alpha\left\lfloor\frac{n}{\alpha}\right\rfloor-n\right)K_{\lfloor\frac{n}{\alpha}\rfloor}$.

Proof. By Lemma 3.1 and Lemma 3.11, we get

$$\mathcal{E}(\mathscr{L}(G)) \geq \left(n-\alpha\left\lfloor\frac{n}{\alpha}\right\rfloor\right)\mathcal{E}\left(\mathscr{L}\left(K_{\lfloor\frac{n}{\alpha}\rfloor+1}\right)\right)+\left(\alpha+\alpha\left\lfloor\frac{n}{\alpha}\right\rfloor-n\right)\mathcal{E}\left(\mathscr{L}\left(K_{\lfloor\frac{n}{\alpha}\rfloor}\right)\right),$$

for all graphs of order n and independence number equals to α, the equality holds, if and only if, $G \cong \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor + 1} \bigcup \left(\alpha + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor - n\right) K_{\lfloor \frac{n}{\alpha} \rfloor}$.

The proof is complete. □

The next lemma gives an expression for the number of edges of the graph $\mathcal{H}$, where

$$\mathcal{H} = \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor + 1} \bigcup \left(\alpha + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor - n\right) K_{\lfloor \frac{n}{\alpha} \rfloor}.$$

Lemma 3.13. *The number of edges of the graph* $\mathcal{H} = \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor + 1} \bigcup \left(\alpha + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor - n\right) K_{\lfloor \frac{n}{\alpha} \rfloor}$ *is*

$$m = \frac{\left\lfloor \frac{n}{\alpha} \right\rfloor \left(2n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor - \alpha\right)}{2}.$$

Proof. The graph $\mathcal{H}$ has $\left\lfloor \frac{n}{\alpha} \right\rfloor \left(\alpha + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor - n\right)$ vertices of degree $\left\lfloor \frac{n}{\alpha} \right\rfloor - 1$ and $\left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) \left(\left\lfloor \frac{n}{\alpha} \right\rfloor + 1\right)$ vertices of degree $\left\lfloor \frac{n}{\alpha} \right\rfloor$. Thus,

$$2m = \left\lfloor \frac{n}{\alpha} \right\rfloor \left(\alpha + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor - n\right) \left(\left\lfloor \frac{n}{\alpha} \right\rfloor - 1\right) + \left\lfloor \frac{n}{\alpha} \right\rfloor \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) \left(\left\lfloor \frac{n}{\alpha} \right\rfloor + 1\right).$$

Hence,

$$m = \frac{\left\lfloor \frac{n}{\alpha} \right\rfloor \left(2n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor - \alpha\right)}{2}.$$

□

Lemma 3.14. *[19] Let G be a graph of order n with m edges and p pendant vertices. If $m > 2n - p - 1$, then $\mathscr{L}(G)$ is hyperenergetic.*

Remark 3.15. *For $n \geq 5$, the number of edges of the complete graph K_n is not less than $2n$.*

The next result is an immediate consequence of Remark 3.15, Lemma 3.13 and Lemma 3.14.

Theorem 3.16. *Let $0 \leq \alpha \leq n - 1$. The graph* $\mathscr{L}\left(\left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor + 1} \bigcup \left(\alpha - n + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor}\right)$ *is hyperenergetic if the following conditions are verified*

1. $\left\lfloor \frac{n}{\alpha} \right\rfloor \geq 5$
2. $\left\lfloor \frac{n}{\alpha} \right\rfloor = 4$ *and* $\frac{2n+1}{10} \geq \alpha$
3. $n = 3$ *and* $\alpha = 1$
4. $n = 2$ *and* $\alpha = 1$.

Proof. Let $G \cong \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor + 1} \bigcup \left(\alpha - n + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor}$.

1. Denote by $m(G)$, the number of edges of the graph G. Let $\left\lfloor \frac{n}{\alpha} \right\rfloor \geq 5$. By Remark 3.15,

$$\begin{aligned} m(G) &= \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) m\left(K_{\lfloor \frac{n}{\alpha} \rfloor + 1}\right) + \left(\alpha - n + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) m\left(K_{\lfloor \frac{n}{\alpha} \rfloor}\right) \\ &\geq 2\left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right)\left(\left\lfloor \frac{n}{\alpha} \right\rfloor + 1\right) + 2\left(\alpha - n + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right)\left\lfloor \frac{n}{\alpha} \right\rfloor \\ &= 2n. \end{aligned}$$

By Lemma 3.14, the graph $\mathscr{L}(G)$ is hyperenergetic.

2. Let $\left\lfloor \frac{n}{\alpha} \right\rfloor = 4$. From Theorem 3.12 and imposing the condition (6), the following inequality is obtained

$$4(3n - 10\alpha) \geq 8n - 20\alpha - 2.$$

From the previous inequality is equivalent to

$$\frac{2n+1}{10} \geq \alpha.$$

Thereby, the graph $\mathscr{L}(G)$ is hyperenergetic.

3. Let $\left\lfloor \frac{n}{\alpha} \right\rfloor = 3$. Imposing the condition (6) and again from Theorem 3.12, the following inequality is obtained

$$4(n - 2\alpha) \geq 2(3n - 6\alpha - 1).$$

The previous inequality is equivalent to

$$\alpha \geq \frac{n-1}{2}.$$

Since $\left\lfloor \frac{n}{\alpha} \right\rfloor = 3$, the previous inequality is equivalent to have $n = 3$ and $\alpha = 1$.

4. Consider $\left\lfloor \frac{n}{\alpha} \right\rfloor = 2$, the proof is similar as in 3.

□

Example 3.17. *At Fig. 2 the graph it is depicted a graph $\mathcal{H}$ where the vertices in equal colors and sizes represent the independent sets. Therefore, there are 5 independent sets of order 3 and an independent set of order 1 in the conditions (a), (b) and (c) from Lemma 3.11. Then $n = 16$, $\alpha = 3$, and $\left\lfloor \frac{n}{\alpha} \right\rfloor = 5$. Theorem 3.16 asserts that $\mathscr{L}(\mathcal{H})$ is hyperenergetic.*

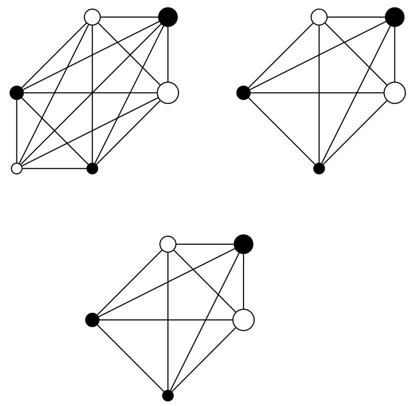
4. Bounds for the energy of iterated line graphs

The iterated line graph of a connected graph G is defined recursively as $\mathscr{L}^{k+1}(G) \equiv \mathscr{L}\left(\mathscr{L}^k(G)\right)$ for $k \geq 1$ (assuming that $\mathscr{L}^k(G)$ is nonempty). It is consistent to say that $\mathscr{L}(G) \equiv \mathscr{L}^1(G)$ and $G \equiv \mathscr{L}^0(G)$. The basic properties of iterated line graph sequences can be found in [6, 7]. In recent years the research of these type of graphs have been widely studied. The diameter and radius of the iterated line graphs were studied in [35] and in [41] the authors studied distance properties of these graphs. In [51], Xiong and Liu characterize the graphs for which $\mathscr{L}^i(G)$ is Hamiltonian with $i \geq 2$. In [43], spectra and energies of iterated line graphs of regular graphs were studied. In another areas we can see, for instance that the graphs $\mathscr{L}^1(G)$ and $\mathscr{L}^2(G)$ are very useful in Chemistry when discussion molecular conformation, [20]. In this section, a lower bound for the energy of the iterated line graph is presented and the equality case is discussed. The next corollary is a consequence of Corollary 3.2.

Corollary 4.1. *Let G be a graph not empty on n vertices. Then, for all $k \geq 1$,*

$$\mathcal{E}\left(\mathscr{L}^{k+1}(G - e)\right) < \mathcal{E}\left(\mathscr{L}^{k+1}(G)\right).$$

If the graph is regular with degree $r \geq 3$ then, for any $k \geq 1$, the exact value for the energy of the iterated line graphs was obtained in [43].



$\mathcal{H} \cong K_6 \cup 2K_5$

Figure 2. The line graph of the graph $\mathcal{H}$ depicted in the figure is an example of an hyperenergetic graph. In fact, $\mathcal{E}(\mathscr{L}(\mathcal{H})) = 76 \geq 68 = 2(m-1)$.

Lemma 4.2. *[43] Let G be a regular graph of order n and degree $r \geq 3$. Then, for any $k \geq 1$,*

$$\mathcal{E}\left(\mathscr{L}^{k+1}(G)\right) = 2n(r-2)\prod_{i=0}^{k-1}\left(2^i r - 2^{i+1} + 2\right).$$

Theorem 4.3. *Let G be a graph on n vertices and independence number α where $1 \leq \alpha \leq n-1$. Then, for any $k \geq 1$,*

$$\mathcal{E}\left(\mathscr{L}^{k+1}(G)\right) \geq 2n\left(\left\lfloor\frac{n}{\alpha}\right\rfloor - 3\right)\prod_{i=0}^{k-1}\left(2^i\left(\left\lfloor\frac{n}{\alpha}\right\rfloor - 3\right) + 2\right). \tag{9}$$

Equality holds in (9), if and only if, $G \cong \alpha K_{\lfloor \frac{n}{\alpha} \rfloor}$.

Proof. By Lemma 3.11, G has a regular subgraph H of degree $\left\lfloor\frac{n}{\alpha}\right\rfloor - 1$. By Lemma 4.2,

$$\mathcal{E}\left(\mathscr{L}^{k+1}(H)\right) = 2n\left(\left\lfloor\frac{n}{\alpha}\right\rfloor - 3\right)\prod_{i=0}^{k-1}\left(2^i\left(\left\lfloor\frac{n}{\alpha}\right\rfloor - 3\right) + 2\right).$$

By Corollary 3.2,

$$\mathcal{E}\left(\mathscr{L}^{k+1}(G)\right) \geq 2n\left(\left\lfloor\frac{n}{\alpha}\right\rfloor - 3\right)\prod_{i=0}^{k-1}\left(2^i\left(\left\lfloor\frac{n}{\alpha}\right\rfloor - 3\right) + 2\right),$$

with equality, if and only if, $H \cong G$. In addition, $H \cong G$, if and only if, $n = \alpha\left\lfloor\frac{n}{\alpha}\right\rfloor$, ($n$ is multiple of α), i.e. $G \cong \alpha K_{\lfloor \frac{n}{\alpha} \rfloor}$. □

Corollary 4.4. *Let G be a graph on n vertices and independence number α where $1 \leq \alpha \leq n-1$. Then,*

$$\mathcal{E}\left(\mathscr{L}^{2}(G)\right) \geq 2n\left(\left\lfloor\frac{n}{\alpha}\right\rfloor - 3\right)\left(\left\lfloor\frac{n}{\alpha}\right\rfloor - 1\right). \tag{10}$$

Equality holds in (10), if and only if, $G \cong \alpha K_{\lfloor \frac{n}{\alpha} \rfloor}$.

5. Upper bounds on the incidence energy of a graph

In [45], the authors presented an upper bound for the incidence energy of a graph G of order n having a vertex connectivity less than or equal to k, and proved that the bound is sharp when $G \cong K_k \vee (K_1 \cup K_{n-k-1})$. Inspired by their work, in this section we present an upper bound for the incidence energy of a connected graph with independence number not less than α. The equality case is also discussed. The next result presented in [45], shows in analogy to the energy, that for a connected graph G the incidence energy strictly increases when an edge is added.

Lemma 5.1. *[45] Let G be a connected graph of order n and non isomorphic to the complete graph. Then,*

$$IE(G) < IE(G+e).$$

Corollary 5.2. *Let G be a connected graph of order $n \geq 3$. Then,*

$$IE(G) \leq \sqrt{2n-2} + (n-1)\sqrt{n-2}. \tag{11}$$

Equality holds in (11), if and only if, $G \cong K_n$.

Theorem 5.3. *Let G be a connected graph of order n and independence number not less than α. Let*

$$\begin{aligned} \mathscr{C}(n,\alpha) &= (\alpha-1)\sqrt{n-\alpha} + (n-\alpha-1)\sqrt{n-2} \\ &+ \sqrt{3n/2-\alpha-1+\sqrt{(n-2)^2+4\alpha(n-\alpha)}/2} \\ &+ \sqrt{3n/2-\alpha-1-\sqrt{(n-2)^2+4\alpha(n-\alpha)}/2}. \end{aligned}$$

Then,

$$IE(G) \leq \mathscr{C}(n,\alpha). \tag{12}$$

Equality holds in (12), if and only if, $G \cong SK_{n,\alpha}$.

Proof. Let G be a connected graph of order n and independence number not less than α. We first consider $\alpha = 1$. Since $\mathscr{C}(n,1) = \sqrt{2n-2} + (n-1)\sqrt{n-2}$ and $K_n \cong SK_{n,1}$ by Corollary 5.2, the result is true for $\alpha = 1$. Now, let $\alpha = n-1$ then $G \cong S_n$. Since $\mathscr{C}(n,n-1) = n + \sqrt{n} - 2 = IE(S_n)$ and $S_n \cong SK_{n,n-1}$, the result is true for $\alpha = n-1$. Consider now $2 \leq \alpha \leq n-2$. Suppose that G has the largest incidence energy among all the connected graphs of order n and independence number $\alpha(G) \geq \alpha$. Let S be an independent set of G with cardinality $\alpha(G)$. Suppose that $G \ncong SK_{n,\alpha}$, then there are two vertices non-adjacent $u, v \in V(G)$. We can assume $u \in S$ and $u \in V(G) - S$ or $u, v \in V(G) - S$. In both cases a graph $G_1 \cong G + e$ where e is an edge connecting the vertices u, v can be constructed. By Lemma 5.1, $IE(G) < IE(G_1)$ which is a contradiction to the maximality of G. Then, $G \cong SK_{n,\alpha(G)}$. By Lemma 2.2, we obtain

$$IE\left(SK_{n,\alpha}\right) = \mathscr{C}(n,\alpha(G)).$$

Let us define the function

$$\begin{aligned} f(x) &= (x-1)\sqrt{n-x} + (n-x-1)\sqrt{n-2} \\ &+ \sqrt{3n/2-x-1+\sqrt{(n-2)^2+4x(n-x)}/2} \\ &+ \sqrt{3n/2-x-1-\sqrt{(n-2)^2+4x(n-x)}/2}, \end{aligned}$$

where $1 \leq x \leq n-1$. In this interval f is strictly decreasing. Consequently, $IE(G) \leq IE(SK_{n,\alpha})$, for all connected graphs of order n and independence number not less than α. Equality holds in (12), if and only if, $G \cong SK_{n,\alpha}$. The proof is complete. □

6. An upper bound on the Laplacian-energy like of a graph

In this section, we present an upper bound on the Laplacian energy like of the complement $\overline{G}$ of a graph G of order n and independence number α where $1 \leq \alpha \leq n-1$. Equality holds, if and only if, $G \cong \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor + 1} \bigcup \left(\alpha - n + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor}$. Moreover, a Nordhaus-Gaddum type relation, [3], is given.

We recall that if $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ are the Laplacian eigenvalues of G, then, $0 \leq n - \mu_1 \leq \cdots \leq n - \mu_{n-1}$ are the Laplacian eigenvalues of $\overline{G}$.

Additionally, the next result shows that the Laplacian energy like of a graph strictly increases when an edge is added.

Lemma 6.1. *[53]. Let G be a graph of order n non isomorphic to the complete graph. Then,*

$$LEL(G) < LEL(G + e).$$

The next result gives an upper bound for $LEL(G)$ in terms of n and the independence number α and can be seen in [13].

Theorem 6.2. *[13] Let G be a connected graph of order n and independence number α. Then,*

$$LEL(G) \leq (n - \alpha)\sqrt{n} + (\alpha - 1)\sqrt{n - \alpha}. \tag{13}$$

Equality holds in (13), if and only if, $G \cong SK_{n,\alpha}$.

Theorem 6.3. *Let G be a graph of order n and independence number α where $1 \leq \alpha \leq n-1$. Then,*

$$LEL(\overline{G}) \leq \left(\alpha - n + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) \left(\left\lfloor \frac{n}{\alpha} \right\rfloor - 1\right) \sqrt{n - \left\lfloor \frac{n}{\alpha} \right\rfloor} + \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) \left\lfloor \frac{n}{\alpha} \right\rfloor \sqrt{n - \left\lfloor \frac{n}{\alpha} \right\rfloor - 1}. \tag{14}$$

Equality holds in (14), if and only if, $G \cong \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor + 1} \bigcup \left(\alpha - n + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor}$.

Proof. By Lemma 6.1, Lemma 3.11, we can see that if G has the minimal Laplacian-energy like among all the graphs H on n vertices and independence number α, then

$$G \cong \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor + 1} \bigcup \left(\alpha - n + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor}.$$

It is clear that in this case, if

$$Sp\left(L\left(G\right)\right) = \left(\underbrace{0, \ldots, 0}_{\alpha}, \underbrace{\left\lfloor \frac{n}{\alpha} \right\rfloor, \ldots, \left\lfloor \frac{n}{\alpha} \right\rfloor}_{\left(\alpha - n + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right)\left(\left\lfloor \frac{n}{\alpha} \right\rfloor - 1\right)}, \underbrace{\left\lfloor \frac{n}{\alpha} \right\rfloor + 1, \ldots, \left\lfloor \frac{n}{\alpha} \right\rfloor + 1}_{\left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right)\left\lfloor \frac{n}{\alpha} \right\rfloor}\right),$$

then,

$$Sp\left(L\left(\overline{G}\right)\right) = \left(\underbrace{0, \ldots, 0}_{\alpha}, \underbrace{n - \left\lfloor \frac{n}{\alpha} \right\rfloor, \ldots, n - \left\lfloor \frac{n}{\alpha} \right\rfloor}_{\left(\alpha - n + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right)\left(\left\lfloor \frac{n}{\alpha} \right\rfloor - 1\right)}, \underbrace{n - \left\lfloor \frac{n}{\alpha} \right\rfloor - 1, \ldots, n - \left\lfloor \frac{n}{\alpha} \right\rfloor - 1}_{\left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right)\left\lfloor \frac{n}{\alpha} \right\rfloor}\right).$$

Taking into account that $\sqrt{x}$, is a real increasing function for $x \geq 0$, we can say that for a graph H, $LEL\left(\overline{H}\right)$ increases while the Laplacian eigenvalues of $\overline{H}$ increase. Moreover, if the Laplacian eigenvalues of H decrease then the Laplacian eigenvalues of $\overline{H}$ increase. We conclude that

$$LEL(\overline{G}) \leq \left(\alpha - n + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) \left(\left\lfloor \frac{n}{\alpha} \right\rfloor - 1\right) \sqrt{n - \left\lfloor \frac{n}{\alpha} \right\rfloor} + \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) \left\lfloor \frac{n}{\alpha} \right\rfloor \sqrt{n - \left\lfloor \frac{n}{\alpha} \right\rfloor - 1}.$$

for all graphs G of order n and independence number α. The equality in (14) holds, if and only if, $G \cong \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor + 1} \bigcup \left(\alpha - n + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right) K_{\lfloor \frac{n}{\alpha} \rfloor}$. The proof is complete. □

The next corollary gives a Nordhaus-Gaddum type relation, [3].

Corollary 6.4. *Let G be a graph of order n and independence number α where $1 \le \alpha \le n-1$. Then,*

$$\begin{aligned} LEL(G) + LEL\left(\overline{G}\right) &\le (n-\alpha)\sqrt{n} + (\alpha - 1)\sqrt{n-\alpha} \\ &+ \left(\alpha - n + \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right)\left(\left\lfloor \frac{n}{\alpha} \right\rfloor - 1\right)\sqrt{n - \left\lfloor \frac{n}{\alpha} \right\rfloor} \\ &+ \left(n - \alpha \left\lfloor \frac{n}{\alpha} \right\rfloor\right)\left\lfloor \frac{n}{\alpha} \right\rfloor \sqrt{n - \left\lfloor \frac{n}{\alpha} \right\rfloor - 1} \end{aligned}$$

The equality holds, if and only if, $G \cong K_n$.

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