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Adjacency spectrum and Wiener index of essential ideal graph of a finite commutative ring

Research Article

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Abstract: Let R be a commutative ring with unity. The essential ideal graph $\mathcal{E}_R$ of R, is a graph with a vertex set consisting of all nonzero proper ideals of R and two vertices I and K are adjacent if and only if $I + K$ is an essential ideal. In this paper, we study the adjacency spectrum of the essential ideal graph of the finite commutative ring $\mathbb{Z}_n$, for $n = \{p^m, p^{m_1}q^{m_2}\}$, where p, q are distinct primes, and $m, m_1, m_2 \in \mathbb{N}$. We show that 0 is an eigenvalue of the adjacency matrix of $\mathcal{E}_{\mathbb{Z}_n}$ if and only if either $n = p^2$ or n is not a product of distinct primes. We also determine all the eigenvalues of the adjacency matrix of $\mathcal{E}_{\mathbb{Z}_n}$ whenever n is a product of three or four distinct primes. Moreover, we calculate the topological indices, namely the Wiener index and hyper-Wiener index of the essential ideal graph of $\mathbb{Z}_n$ for different forms of n.

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Keywords: Essential ideal, Essential ideal graph, Eigenvalue, Wiener index, Hyper-Wiener index

1. Introduction

In recent decades, researchers have been exploring algebraic structures using graph theory properties. The notion of a graph connected to the zero divisors of a commutative ring was put forward by Beck [6] in 1998. However, the current definition and term for the *zero-divisor graph* were initially presented by Anderson and Livingston [3] in 1999. Following this, various studies were pursued on graphs defined on commutative rings by taking the ideals as vertices. Another graph, namely the *comaximal ideal graph* was introduced in [20] as a graph with vertices as the proper ideals of the ring R, and a pair of vertices I and K are adjacent if and only if $I + K = R$. Interested readers may refer to the papers [1, 7, 17] for

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more details. In 2018, J. Amjadi [2] introduced the *essential ideal graph* of a commutative ring, which is denoted as $\mathcal{E}_R$. The graph $\mathcal{E}_R$ has all nonzero proper ideals of R as the vertex set, and any two vertices I and K are adjacent if and only if $I+K$ is an essential ideal. It is worth mentioning that a proper ideal I of a ring R is said to be an *essential ideal* if it has a nonzero intersection with every other non-zero ideal of R.

In mathematical chemistry, molecular descriptors like the topological indices play a vital role. A topological index is an invariant of molecular graphs that can be used to study the properties of their constituent molecules. Among this, the *Wiener index* introduced by H. Wiener [19] is a well-known molecular descriptor, which, in particular, is used for the preliminary testing of drug molecules. A generalization of the Wiener index known as the *hyper-Wiener index* was introduced by M. Randic [15], and is widely used in biochemistry. Determining various topological indices of graphs associated with different algebraic structures has been an interesting area of research in the past few years. To get a better understanding of this, refer [5, 16]. Being motivated by the previous works, in this paper, we determine the Wiener and the hyper-Wiener index of $\mathcal{E}_{\mathbb{Z}_n}$, where $\mathbb{Z}_n$ is the ring of integers modulo n.

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. The *degree* of a vertex v, denoted by $deg(v)$, is defined as the number of edges that occur in v. In G, a vertex v is said to be universal, if it is adjacent to all other vertices. The *complete graph* K_n, is a graph in which any two vertices are adjacent. A graph G is a $k-$ *partite graph* if $V(G)$ can be partitioned into k subsets $V_1, V_2, \cdots, V_k$ (named partite sets) such that the vertices u and v form an edge in G if they belong to different partite sets. If, in addition, there exists an edge between every two vertices belonging to different partite sets, then graph G can be classified as *complete k-partite graph*. The graph denoted as $K_{m,n}$ represents a complete bipartite graph consisting of two sets with sizes m and n respectively. The *induced subgraph*, $G[S]$, is formed by taking the subset S of vertices from G, along with all the edges that connect vertices solely within S. The *complement* of a graph G is denoted by $\overline{G}$. A set of vertices in a graph G is *independent* if any two vertices in the set are nonadjacent. The *join* of two graphs, G_1 and G_2, represented as $G_1 \vee G_2$, is formed by adding edges between any two vertices v_1 and v_2, where $v_1 \in G_1$ and $v_2 \in G_2$. The *adjacency matrix* $A(G)$ of a graph G of order n is the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{i,j} = 1$, if v_i is adjacent to v_j in G and $a_{ij} = 0$ otherwise. The *eigenvalues* of a graph G are defined to be the eigenvalues of its adjacency matrix. The collection of all eigenvalues of G is called the *spectrum(adjacency spectrum)* of G. The *energy* of a graph G, denoted by $\mathbb{E}(G)$, is defined as the sum of the absolute values of the eigenvalues of $A(G)$. That is, $\mathbb{E}(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of G. A graph with energy greater than a complete graph is called *hyperenergetic*. Otherwise, it is called *non-hyperenergetic*. To delve into more definitions and results in ring, graph, and spectral graph theories, one can refer [4, 8, 11, 12, 18, 22].

The paper is organized as follows. In Section 2, we state the results that are needed for the subsequent sections. In Section 3, we find the eigenvalues of $\mathcal{E}_{\mathbb{Z}_n}$ for $n = p^m, m > 1$, and $n = p^{m_1} q^{m_2}$, where p and q are distinct primes with $p < q$ and m_1, m_2 are positive integers. We also prove that for the essential ideal graph of $\mathbb{Z}_n$, 0 is not an eigenvalue if and only if either $n = p^m$, $m > 2$ or n is a product of distinct primes. In Section 4, we calculate the Wiener index and the hyper-Wiener index of the essential ideal graph of $\mathbb{Z}_n$ for different values of n.

2. Preliminaries

The results shown below are beneficial for the parts that follow.

Lemma 2.1. *[21] Let M, N, P, Q be matrices and let Q be non-singular. Let $S = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$, then $\det S = \det Q \times \det(M - NQ^{-1}P)$. Here, $(M - NQ^{-1}P)$ is known as the* ***Schur complement*** *of M in S.*

Lemma 2.2. *[21] Let $S \in M_n(\mathbb{F})$ be a partitioned 2×2 block matrix $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$, where each matrix $S_{ii} \in M_{n_i}(\mathbb{F})$, $i = 1, 2$ and $n = n_1 + n_2$. If S_{11}, S_{22} and both Schur complements $S_{11} - S_{12}S_{22}^{-1}S_{21}$ and $S_{22} - S_{21}S_{11}^{-1}S_{12}$ are all invertible, then*

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}^{-1} = \begin{pmatrix} (S_{11} - S_{12}S_{22}^{-1}S_{21})^{-1} & -S_{11}^{-1}S_{12}(S_{22} - S_{21}S_{11}^{-1}S_{12})^{-1} \\ -(S_{22} - S_{21}S_{11}^{-1}S_{12})^{-1}S_{21}S_{11}^{-1} & (S_{22} - S_{21}S_{11}^{-1}S_{12})^{-1} \end{pmatrix}.$$

Proposition 2.3. *[14] Let $C_{(a,b,n)} = \begin{pmatrix} a & b & \cdots & \cdots & b \\ b & a & b & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{pmatrix}$ be a circulant matrix of order $n \times n$ with entries $a, b \in \mathbb{R}$. Then its determinant, denoted by δ, is given by $\delta = (a + (n-1)b)(a-b)^{n-1}$.*

Proposition 2.4. *[14] If the circulant matrix $C_{(a,b,n)}$ is nonsingular, then its inverse is given by $C_{(a,b,n)}^{-1} = \frac{1}{\delta}\begin{pmatrix} \delta_{n-1} & \Delta_{n-1} & \cdots & \Delta_{n-1} \\ \Delta_{n-1} & \delta_{n-1} & \cdots & \Delta_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \cdots & \cdots & \delta_{n-1} \end{pmatrix}$, where $\delta_{n-1} = (a + (n-2)b)(a-b)^{n-2}$ and $\Delta_{n-1} = -b(a-b)^{n-2}$.*

Theorem 2.5. *[10] If G is a regular graph of degree r with n vertices, then the characteristic polynomial of $\overline{G}$ is $P_{\overline{G}}(\lambda) = (-1)^n \frac{\lambda - n + r + 1}{\lambda + r + 1} P_G(-\lambda - 1)$.*

Theorem 2.6. *[10] Let G_1 and G_2 be two graphs of order n_1 and n_2 respectively. Then the characteristic polynomial of the join of G_1 and G_2 is given by*

$$\begin{aligned} P_{G_1 \vee G_2}(\lambda) = (-1)^{n_2} P_{G_1}(\lambda) P_{\overline{G_2}}(-\lambda - 1) + \\ (-1)^{n_1} P_{G_2}(\lambda) P_{\overline{G_1}}(-\lambda - 1) - (-1)^{n_1 + n_2} P_{\overline{G_1}}(-\lambda - 1) P_{\overline{G_2}}(-\lambda - 1). \end{aligned}$$

Observation 2.7. *[2] Let R be a commutative ring with nonzero unity. Then every proper essential ideal of R is a universal vertex in $\mathcal{E}_R$.*

For any composite integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $(k, \ \alpha_i) \in \mathbb{N}$, $(k, \alpha_1) \neq (1, 1)$, p_i's are distinct primes $(1 \leq i \leq k)$.

Theorem 2.8. *[13] For the essential ideal graph $\mathcal{E}_{\mathbb{Z}_n}$, $\mathcal{E}_{\mathbb{Z}_n} \cong H \vee K_m$, where H is a k-partite graph and K_m is a complete graph of order $m = \prod_{i=1}^{k} \alpha_i - 1$.*

3. Adjacency spectrum of essential ideal graph of $\mathbb{Z}_n$

In this section, we study the adjacency spectrum of the essential ideal graph of $\mathbb{Z}_n$. We obtain the spectrum for $n = p^{m_1}$ and $n = p^{m_1}q^{m_2}$, where p and q are distinct primes with $p < q$, and m_1, m_2 are positive integers. We also determine the adjacency spectrum of $\mathcal{E}_{\mathbb{Z}_n}$, when n is a product of three distinct primes and a product of four distinct primes. Throughout the section, by spectrum of $\mathcal{E}(\mathbb{Z}_n)$, we shall mean the adjacency spectrum of $\mathcal{E}(\mathbb{Z}_n)$.

We first prove the following result which provides a necessary and sufficient condition for an ideal to be an essential ideal of $\mathbb{Z}_n$.

Theorem 3.1. *Let $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ where $p_1 < p_2 < \cdots < p_k$ are distinct primes, and m_i is a non-negative integer for $1 \leq i \leq k$. Any nonzero ideal $I = \langle p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \rangle$ of $\mathbb{Z}_n$ is essential if and only if $r_i \neq m_i$ for any i.*

Proof. Assume that $I = \langle p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \rangle$ be a nonzero essential ideal of $\mathbb{Z}_n$. We need to prove that $r_i \neq m_i$ for any i, $1 \leq i \leq k$. Suppose that $r_i = m_i$ for some i; say 1. Then for the ideal $N = \langle p_2^{m_2} p_3^{m_3} \cdots p_k^{m_k} \rangle$, $I \cap N = \langle 0 \rangle$, contradicting the fact that I is essential.

Conversely, let $r_i \neq m_i$ for any i. That is, $I = \langle p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \rangle$, $0 \leq r_i \leq m_i - 1$ for $1 \leq i \leq k$ be a nonzero ideal. We need to prove that I is essential. If not, there is a nonzero ideal $L \neq I$ such that $I \cap L = \langle 0 \rangle$. But all the ideals of $\mathbb{Z}_n$ other than I will be in any one of the following sets.

$$A_{p_1^{m_1}} = \{\langle p_1^{m_1} p_2^{r_2} \cdots p_k^{r_k} \rangle; 0 \leq r_i \leq m_i \quad \text{for } 2 \leq i \leq k\}$$
$$A_{p_2^{m_2}} = \{\langle p_1^{r_1} p_2^{m_2} \cdots p_k^{r_k} \rangle; 0 \leq r_1 \leq m_1 - 1 \text{ and } 0 \leq r_i \leq m_i \quad \text{for } \; 3 \leq i \leq k\}.$$

In general, $A_{p_j^{m_j}} = \{\langle p_1^{r_1} p_2^{r_2} \cdots p_{j-1}^{r_{j-1}} p_j^{m_j} p_{j+1}^{r_{j+1}} \cdots p_k^{r_k} \rangle; 0 \leq r_i \leq m_i - 1, \;\text{ for }\; 1 \leq i \leq j-1 \;\text{ and } 0 \leq r_i \leq m_i \;\text{ for }\; j+1 \leq i \leq k\}; 1 \leq j \leq k$. Thus, L must be in any of the sets $A_{p_1^{m_1}}, A_{p_2^{m_2}}, \cdots, A_{p_k^{m_k}}$ so that its intersection with I is nonzero. This contradiction proves the result. □

Proposition 3.2. *Let $n = p^m$, $m > 2$ be a positive integer and p be any prime. Then the spectrum of the essential ideal graph $\mathbb{Z}_n$ is $\begin{pmatrix} m-2 & -1 \\ 1 & m-2 \end{pmatrix}$.*

Proof. If $n = p^m$, then all the nonzero proper ideals of $\mathbb{Z}_n$ are essential by Theorem 3.1, and hence $\mathcal{E}_{\mathbb{Z}_n}$is a complete graph. □

Corollary 3.3. *Let $n = p^m$, $m > 2$ be a positive integer and p be any prime. Then the energy of the essential ideal graph of $\mathbb{Z}_n$ is $2m - 4$.*

Theorem 3.4. *Let $n = p^{m_1} q^{m_2}$, where p and q are distinct primes with $p < q$ and m_1, m_2 are positive integers. Then the characteristic polynomial of $\mathcal{E}_{\mathbb{Z}_n}$ is given by $P_{\mathcal{E}_{\mathbb{Z}_n}}(\lambda) = \lambda^{m_1+m_2-2}(\lambda+1)^{m_1 m_2 - 2} P(\lambda)$, where $P(\lambda) = \lambda^3 + (2 - m_1 m_2)\lambda^2 + [(1 - m_1 m_2)(m_1 + m_2) - m_1 m_2]\lambda - {m_1}^2 {m_2}^2$.*

Proof. We can partition the vertex of $\mathcal{E}_{\mathbb{Z}_n}$ as follows:

$X = \{\langle p^r q^s \rangle : 0 \leq r \leq m_1 - 1,\; 0 \leq s \leq m_2 - 1 \text{ and } (r,s) \neq (0,0)\}$
$V_1 = \{\langle p^{m_1} q^s \rangle : 0 \leq s \leq m_2 - 1\}$ and
$V_2 = \{\langle p^r q^{m_2} \rangle : 0 \leq r \leq m_1 - 1\}$ so that $V(\mathcal{E}_{\mathbb{Z}_n}) \simeq X \cup V_1 \cup V_2$.

By Theorem 3.1 and Observation 2.7, $\mathcal{E}_{\mathbb{Z}_n}[X] \simeq K_{m_1 m_2 - 1}$. And, since V_1 and V_2 consist of independent vertices, $\mathcal{E}_{\mathbb{Z}_n}[V_1, V_2] \simeq K_{m_2, m_1}$ Thus, by Theorem 2.8, $\mathcal{E}_{\mathbb{Z}_n} \simeq K_{m_1 m_2 - 1} \vee K_{m_2, m_1}$.

To find the characteristic polynomial of $\mathcal{E}_{\mathbb{Z}_n}$, we take $G_1 = K_{m_1 m_2 - 1}$ and $G_2 = K_{m_2, m_1}$. Then using Theorems 2.5 and 2.6, we have,

$$\begin{aligned} P_{\mathcal{E}_{\mathbb{Z}_n}}(\lambda) &= (-1)^{m_1+m_2} P_{G_1}(\lambda) P_{\overline{G_2}}(-\lambda-1) + (-1)^{m_1 m_2 - 1} P_{G_2}(\lambda) P_{\overline{G_1}}(-\lambda-1) \\ &\quad - (-1)^{m_1 m_2 - 1 + m_1 + m_2} P_{\overline{G_1}}(-\lambda-1) P_{\overline{G_2}}(-\lambda-1). \end{aligned}$$

Here, $\overline{G_1}$ is the empty graph consisting of $m_1 m_2 - 1$ vertices and $\overline{G_2} = K_{m_2} \cup K_{m_1}$. Hence,

$$P_{\overline{G_1}}(-\lambda-1) = (-1)^{m_1 m_2 - 1}(\lambda+1)^{m_1 m_2 - 1},$$
$$P_{\overline{G_2}}(-\lambda-1) = (-1)^{m_1+m_2-2}(\lambda+m_1)(\lambda+m_2)\lambda^{m_1+m_2-2}.$$

Thus we have,

$$\begin{aligned}
P_{\mathcal{E}_{\mathbb{Z}_n}}(\lambda) &= (-1)^{m_1+m_2}(\lambda - m_1m_2+2)(\lambda+1)^{m_1m_2-2}(-1)^{m_1+m_2-2}(\lambda+m_1)\\
&\times(\lambda+m_2)\lambda^{m_1+m_2-2}+(-1)^{m_1m_2-1}\lambda^{m_1+m_2-2}(\lambda^2-m_1m_2)(-1)^{m_1m_2-1}\\
&\times(\lambda+1)^{m_1m_2-1}-(-1)^{m_1m_2-1+m_1+m_2}(-1)^{m_1m_2-1}(\lambda+1)^{m_1m_2-1}\\
&\times(-1)^{m_1+m_2-2}(\lambda+m_1)(\lambda+m_2)\lambda^{m_1+m_2-2}\\
&=\lambda^{m_1+m_2-2}(\lambda+m_1)(\lambda+m_2)(\lambda+1)^{m_1m_2-2}(\lambda-m_1m_2+2)+\lambda^{m_1+m_2-2}\\
&\times(\lambda^2-m_1m_2)(\lambda+1)^{m_1m_2-1}-\lambda^{m_1+m_2-2}(\lambda+1)^{m_1m_2-1}(\lambda+m_1)(\lambda+m_2).
\end{aligned}$$

On simplifying, we obtain

$$P_{\mathcal{E}_{\mathbb{Z}_n}}(\lambda)=\lambda^{m_1+m_2-2}(\lambda+1)^{m_1m_2-2}P(\lambda),$$

where

$$P(\lambda)=\lambda^3+(2-m_1m_2)\lambda^2+[(1-m_1m_2)(m_1+m_2)-m_1m_2]\lambda-{m_1}^2{m_2}^2.$$

□

Corollary 3.5. *Let $n=p^mq^m$,where p and q are distinct primes and $m>1$. Then the spectrum of $\mathcal{E}(\mathbb{Z}_n)$ is*

$$\begin{pmatrix}\frac{k+\sqrt{k^2+4m^3}}{2} & 0 & -1 & \frac{k-\sqrt{k^2+4m^3}}{2} & -m\\ 1 & 2m-2 & m^2-2 & 1 & 1\end{pmatrix}, \text{ where } k=(m^2+m-2).$$

Example 3.6. *Let $n=36=2^23^2$. The vertex set of $\mathcal{E}_{\mathbb{Z}_{36}}$ (see Fig.1) is $V=\{\langle 2\rangle,\langle 3\rangle,\langle 4\rangle,\langle 6\rangle,\langle 9\rangle,\langle 12\rangle,\langle 18\rangle\}$. It can be partitioned as $V=X\cup X_1\cup X_2$, where*

$$X=\{\langle 2\rangle,\langle 3\rangle,\langle 6\rangle\},V_1=\{\langle 4\rangle,\langle 12\rangle)\},V_2=\{\langle 9\rangle,\langle 18\rangle\}.$$

Since X contains all the proper essential ideals of $\mathbb{Z}_{36}$, the subgraph $\mathcal{E}_{\mathbb{Z}_{36}}[X]$ is K_3. Now, $\mathcal{E}_{\mathbb{Z}_{36}}[V_1,\ V_2]=K_{2,2}$ and hence $\mathcal{E}_{\mathbb{Z}_{36}}\simeq K_3\vee K_{2,2}$. Then the spectrum of $\mathcal{E}_{\mathbb{Z}_{36}}$ is

$$\begin{pmatrix}2+2\sqrt{3} & 0 & -1 & 2-2\sqrt{3} & -2\\ 1 & 2 & 2 & 1 & 1\end{pmatrix}$$

Corollary 3.7. *Let $n=p^mq^m$,where p and q are distinct primes and $m>1$. Then the energy of the essential ideal graph of $\mathbb{Z}_n$ is $k+\sqrt{k^2+4m^3}$, $k=(m^2+m-2)$.*

Lemma 3.8. *Let $n=p_1p_2\cdots p_k$, where $p_1,p_2,\cdots,p_k$ are distinct primes. Then any two vertices $\langle x\rangle$ and $\langle y\rangle$ of the essential ideal graph of $\mathbb{Z}_n$ are adjacent if and only if $gcd(x,y)=1$, provided x is the product of i distinct primes and y is the product of j distinct primes for $1\le i,j\le k-1$.*

Theorem 3.9. *Let $n=p_1p_2p_3$, p_i be a distinct prime for $1\le i\le 3$. Then the spectrum of $\mathcal{E}_{\mathbb{Z}_n}$ is*

$$\begin{pmatrix}1+\sqrt{2} & \frac{-1+\sqrt{5}}{2} & 1-\sqrt{2} & \frac{-1-\sqrt{5}}{2}\\ 1 & 2 & 1 & 2\end{pmatrix}.$$

Proof. The number of nonzero proper ideals of $\mathbb{Z}_n$ is $2^3-2=6$. Hence, the adjacency matrix is a 6×6 symmetric matrix. Then the corresponding six vertices of $\mathcal{E}_{\mathbb{Z}_n}$ can be partitioned as follows.

$V_1=\{\langle p_1\rangle,\langle p_2\rangle,\langle p_3\rangle\}$ and
$V_2=\{\langle p_2p_3\rangle,\langle p_1p_3\rangle,\langle p_1p_2\rangle\}$.

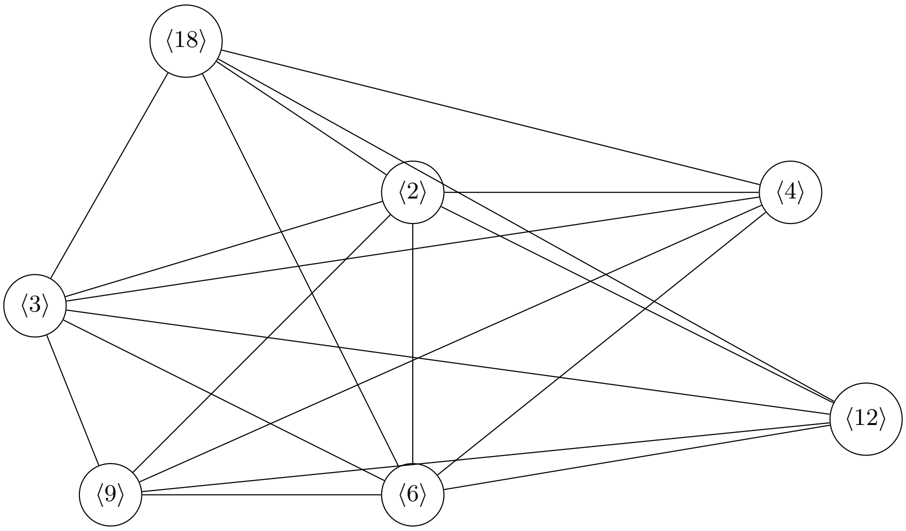

Figure 1. $\mathcal{E}_{\mathbb{Z}_{36}}$

By Lemma 3.8, we see that all vertices of V_1 are adjacent and they form the block $J - I$ of order 3, where J is a matrix having all entries 1, in the adjacency matrix of $\mathcal{E}_{\mathbb{Z}_n}$. Also, the vertices of V_2 are nonadjacent and each vertex is adjacent exactly to one of the vertices of V_1. Hence the vertices of V_1 and V_2 together form an identity block I of order 3 while the vertices of V_2 form a zero block of order 3. Then the adjacency matrix and characteristic polynomial of $\mathcal{E}_{\mathbb{Z}_n}$ are given by,

$$A = \begin{pmatrix} J_{3\times 3} - I_{3\times 3} & I_{3\times 3} \\ I_{3\times 3} & 0_{3\times 3} \end{pmatrix}$$

and

$$\det(A - \lambda I) = \det \begin{pmatrix} J - (\lambda+1)I & I \\ I & -\lambda I \end{pmatrix}$$

$$= \det(J - (\lambda+1)I) \times \det(-\lambda I - I(J - (\lambda+1)I)^{-1}I),$$

where $J - (\lambda+1)I$ is the circulant matrix $C_{(-\lambda,1,3)}$. By Propositions 2.3 and 2.4, its determinant and inverse are given by

$$\delta = (2-\lambda)(\lambda+1)^2$$

and

$$C^{-1}_{(-\lambda,1,3)} = \frac{1}{(2-\lambda)(\lambda+1)^2} \begin{pmatrix} (\lambda^2-1) & (\lambda+1) & (\lambda+1) \\ (\lambda+1) & (\lambda^2-1) & (\lambda+1) \\ (\lambda+1) & (\lambda+1) & (\lambda^2-1) \end{pmatrix}$$

$$= \frac{1}{(2-\lambda)(\lambda+1)} \begin{pmatrix} (\lambda-1) & 1 & 1 \\ 1 & (\lambda-1) & 1 \\ 1 & 1 & (\lambda-1) \end{pmatrix}.$$

Also,

$$-\lambda I - I(J-(\lambda+1)I)^{-1}I = \begin{pmatrix} -\lambda - \frac{(\lambda-1)}{(2-\lambda)(\lambda+1)} & \frac{-1}{(2-\lambda)(\lambda+1)} & \frac{-1}{(2-\lambda)(\lambda+1)} \\ \frac{-1}{(2-\lambda)(\lambda+1)} & -\lambda - \frac{(\lambda-1)}{(2-\lambda)(\lambda+1)} & \frac{-1}{(2-\lambda)(\lambda+1)} \\ \frac{-1}{(2-\lambda)(\lambda+1)} & \frac{-1}{(2-\lambda)(\lambda+1)} & -\lambda - \frac{(\lambda-1)}{(2-\lambda)(\lambda+1)} \end{pmatrix},$$

is the circulant matrix $C_{(-\lambda-\frac{(\lambda-1)}{(2-\lambda)(\lambda+1)},\frac{-1}{(2-\lambda)(\lambda+1)},3)}$.
Again by Proposition 2.3,

$$\det(-\lambda I-(J-(\lambda+1)I)^{-1})=\frac{(-\lambda(2-\lambda)-1)(-\lambda(\lambda+1)+1)^2}{(2-\lambda)(\lambda+1)^2},\text{ and hence}$$
$$\det(A-\lambda I)=\frac{(2-\lambda)(\lambda+1)^2(\lambda^2-2\lambda-1)(\lambda^2+\lambda-1)^2}{(2-\lambda)(\lambda+1)^2} = (\lambda^2-2\lambda-1)(\lambda^2+\lambda-1)^2.$$

By solving the two quadratic polynomials, we obtain the required spectrum. □

Corollary 3.10. *Let $n=p_1p_2p_3$, p_i be a distinct prime for $1\le i\le 3$. Then*

1. *The energy of the essential ideal graph of $\mathbb{Z}_n$ is $2(\sqrt{2}+\sqrt{5})$.*
2. *The graph $\mathcal{E}_{\mathbb{Z}_n}$ is non-hyperenergetic.*

Theorem 3.11. *Let $n=p_1p_2p_3p_4$, p_i be a distinct prime for $1\le i\le 4$. Then the spectrum of the essential ideal graph of $\mathbb{Z}_n$ is*

$$\begin{pmatrix} \frac{5+\sqrt{21}}{2} & 1 & \frac{5-\sqrt{21}}{2} & \frac{-3+\sqrt{5}}{2} & -1 & \frac{-3-\sqrt{5}}{2} \\ 1 & 5 & 1 & 3 & 1 & 3 \end{pmatrix}.$$

Proof. To find the adjacency matrix, we first partition the vertex set of $\mathcal{E}_{\mathbb{Z}_n}$ as follows.
$V_1=\{\langle p_1\rangle,\langle p_2\rangle,\langle p_3\rangle,\langle p_4\rangle\}$
$V_2=\{\langle p_1p_2\rangle,\langle p_1p_3\rangle,\langle p_1p_4\rangle,\langle p_2p_3\rangle,\langle p_2p_4\rangle,\langle p_3p_4\rangle\}$
$V_3=\{\langle p_1p_2p_3\rangle,\langle p_1p_2p_4\rangle,(p_1p_3p_4\rangle,\langle p_2p_3p_4\rangle\}$.
Then by Lemma 3.8, we observe that all vertices of V_1 form the block matrix $J-I$ of order 4 and each vertex of V_2 is adjacent exactly to one of the vertices of V_2, which will form the block matrix $C=\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\1&0&0&0&0&0\end{pmatrix}$. Also, each vertex of V_1 is adjacent exactly to three vertices of V_2 and one of the vertices of V_3, forming the blocks $B=\begin{pmatrix}0&0&0&1&1&1\\0&1&1&0&0&1\\1&0&1&0&1&0\\1&1&0&1&0&0\end{pmatrix}$ of order 4×6 and I, identity matrix of order 4, respectively. Finally, vertices of V_2 together with vertices of V_3 form zero block of order 6×4 and vertices of V_3 constitute zero blocks of order 4 to the adjacency matrix of $\mathcal{E}_{\mathbb{Z}_n}$. Hence the adjacency matrix is

$$A=\begin{pmatrix}(J-I)_{4\times 4} & B_{4\times 6} & I_{4\times 4}\\ B^T_{6\times 4} & C_{6\times 6} & 0_{6\times 4}\\ I_{4\times 4} & 0_{4\times 6} & 0_{4\times 4}\end{pmatrix}.$$

Then

$$A-\lambda I=\left(\begin{array}{cc|c} J-(\lambda+1)I & B & I\\ B^T & C-\lambda I & 0\\ \hline I & 0 & -\lambda I\end{array}\right)=\begin{pmatrix}M & N\\ P & Q\end{pmatrix}.$$

By Lemma 2.1, if $\lambda \neq 0$,

$$\det(A - \lambda I) = \det Q \times \det(M - NQ^{-1}P).$$

Now,

$$\det(M - NQ^{-1}P) = \det \begin{pmatrix} (J - (\lambda+1)I + \frac{1}{\lambda} I)_{4\times 4} & B_{4\times 6} \\ B^T & D - \lambda I_{6\times 6} \end{pmatrix} \quad (1)$$

where $D - \lambda I_{6\times 6} = \begin{pmatrix} -\lambda I_{3\times 3} & E_{3\times 3} \\ E_{3\times 3} & -\lambda I_{3\times 3} \end{pmatrix}$, $E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Applying Lemmas 2.1, 2.2 and Proposition 2.3, we have

$$\det(D - \lambda I) = \det \begin{pmatrix} -\lambda I_{3\times 3} & E_{3\times 3} \\ E_{3\times 3} & -\lambda I_{3\times 3} \end{pmatrix} = (\lambda^2 - 1)^3$$

and for $\lambda \neq \pm 1$,

$$(D - \lambda I)^{-1} = \begin{pmatrix} \frac{\lambda}{1-\lambda^2} I & \frac{1}{1-\lambda^2} E \\ \frac{1}{1-\lambda^2} E & \frac{\lambda}{1-\lambda^2} I \end{pmatrix}$$

$$\text{and } B(D - \lambda I)^{-1}B^T = \begin{pmatrix} \frac{3\lambda}{1-\lambda^2} & \frac{2+\lambda}{1-\lambda^2} & \frac{2+\lambda}{1-\lambda^2} & \frac{2+\lambda}{1-\lambda^2} \\ \frac{2+\lambda}{1-\lambda^2} & \frac{3\lambda}{1-\lambda^2} & \frac{2+\lambda}{1-\lambda^2} & \frac{2+\lambda}{1-\lambda^2} \\ \frac{2+\lambda}{1-\lambda^2} & \frac{2+\lambda}{1-\lambda^2} & \frac{3\lambda}{1-\lambda^2} & \frac{2+\lambda}{1-\lambda^2} \\ \frac{2+\lambda}{1-\lambda^2} & \frac{2+\lambda}{1-\lambda^2} & \frac{2+\lambda}{1-\lambda^2} & \frac{3\lambda}{1-\lambda^2} \end{pmatrix} = C_{(\frac{3\lambda}{1-\lambda^2}, \frac{2+\lambda}{1-\lambda^2}, 4)}.$$

And, Equation (1) is,

$$\begin{aligned} \det(M - NQ^{-1}P) &= (\lambda^2 - 1)^3 \times \det C_{(\frac{\lambda^4 - 5\lambda^2 + 1}{\lambda(1-\lambda^2)}, \frac{-\lambda^2 - \lambda - 1}{1-\lambda^2}, 4)} \\ &= (\lambda^2 - 1)^3 \left(\frac{\lambda^4 - 3\lambda^3 - 8\lambda^2 - 3\lambda + 1}{\lambda(1-\lambda^2)} \right) \left(\frac{\lambda^4 + \lambda^3 - 4\lambda^2 + \lambda + 1}{\lambda(1-\lambda^2)} \right)^3. \end{aligned}$$

Hence,

$$\begin{aligned} \det(A - \lambda I) &= \lambda^4 \times (\lambda^2 - 1)^3 \left(\frac{\lambda^4 - 3\lambda^3 - 8\lambda^2 - 3\lambda + 1}{\lambda(1-\lambda^2)} \right) \left(\frac{\lambda^4 + \lambda^3 - 4\lambda^2 + \lambda + 1}{\lambda(1-\lambda^2)} \right)^3 \\ &= (\lambda^8 - 9\lambda^7 + 26\lambda^6 - 29\lambda^5 + 29\lambda^3 - 26\lambda^2 + 9\lambda - 1)(\lambda^2 + 3\lambda + 1)^3. \end{aligned}$$

□

Corollary 3.12. *Let $n = p_1p_2p_3p_4$, p_i be a distinct prime for $1 \leq i \leq 4$. Then*

1. *The energy of the essential ideal graph of $\mathbb{Z}_n$ is 20.*
2. *The graph $\mathcal{E}_{\mathbb{Z}_n}$ is non-hyperenergetic.*

Theorem 3.13. *Let $n = p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$, where $p_1, p_2, \cdots, p_k$ are distinct primes and m_i is a non negative integer for $1 \leq i \leq k$. Then 0 is not an eigenvalue of $\mathcal{E}_{\mathbb{Z}_n}$ if and only if either $k = 1$ and $m_1 > 2$ or $m_i = 1$ for $1 \leq i \leq k$.*

Proof. Assume that 0 is not an eigenvalue of the adjacency matrix of $\mathcal{E}_{\mathbb{Z}_n}$. If $n = p^m$, $m > 2$, then we are done. Suppose that $n \neq p^m$, $m > 2$. Then we need to prove that $m_i = 1$ for all $i = 1, 2, \cdots, k$. If possible, suppose $m_i > 1$ for atleast one i, say m_1. Without loss of generality, we assume that $n = p_1^{m_1} p_2 \cdots p_k$. Then by Theorem 3.1, the set of all essential ideals of $\mathbb{Z}_n$ is given by,
$X = \{\langle p_1\rangle, \langle p_1^2\rangle, \cdots, \langle p_1^{m_1-1}\rangle\}$. By Observation 2.7, these are the universal vertices of the essential ideal graph of $\mathbb{Z}_n$. Now consider the vertices $I = \langle p_1^{m_1-1} p_2 \cdots p_k\rangle$ and $L = \langle p_2 p_3 \cdots p_k\rangle$. In $\mathcal{E}_{\mathbb{Z}_n}$, I and L are nonadjacent and are adjacent to any other vertex K if and only if their sums $I + K$ and $L + K$ is an element of the set X. By elementary number theory, K can be either an element of the set X or the ideal $\langle p_1^{m_1}\rangle$. In other words, the adjacency and non-adjacency of the two vertices I and L are the same. Then the rows and columns corresponding to the vertices I and L in the adjacency matrix are the same. Hence the matrix is singular and zero is an eigenvalue.

Conversely, by Proposition 3.2, the result is obvious when $n = p^m$, $m > 2$. Now, let $n = p_1 p_2 p_3 \cdots p_{k-1} p_k$. We shall index the rows and columns of the adjacency matrix of $\mathcal{E}_{\mathbb{Z}_n}$ in the following way:

Let us consider the set $S = \{p_1, p_2, p_3, \ldots, p_{k-1}, p_k\}$. Clearly, S has k elements. We first list the vertices of the form $\langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle, \ldots, \langle p_{k-1}\rangle, \langle p_k\rangle$. That is, we choose one element at a time from S. Next, we shall consider the vertices of the form $\langle p_1p_2\rangle, \langle p_1p_3\rangle, \cdots, \langle p_1p_k\rangle, \cdots, \langle p_{k-1}p_k\rangle$. That is, we shall choose two elements at a time from S. Clearly, we shall have $\binom{k}{2}$ such vertices. This process continues until we have exhausted all the vertices of $\mathcal{E}_{\mathbb{Z}_n}$. Thus, in the end, we shall choose vertices of the form $\langle p_2p_3 \cdots p_{k-1}p_k\rangle$, $\langle p_1p_3 \cdots p_{k-1}p_k\rangle, \cdots, \langle p_1p_2 \cdots p_{k-1}\rangle$. That is, we choose $k-1$ elements at a time from S making a total of $\binom{k}{k-1}$ such vertices. Using the above indexing and Lemma 3.8, the adjacency matrix of $\mathcal{E}_{\mathbb{Z}_n}$ will be of the following form:

$$\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & I_{k\times k} \\
\cdots & \cdots & \cdots & \cdots & \cdots & I_{\binom{k}{2}\times\binom{k}{2}} & 0 \\
\cdots & \cdots & \cdots & \cdots & I_{\binom{k}{3}\times\binom{k}{3}} & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & 0 & \vdots & \vdots & \vdots \\
\cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & I_{\binom{k}{k-2}\times\binom{k}{k-2}} & \vdots & \vdots & \vdots & \vdots & \vdots \\
I_{\binom{k}{k-1}\times\binom{k}{k-1}} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{2}$$

Note that the matrix in (2) is non-singular. Therefore, 0 is not an eigenvalue of the adjacency matrix of $\mathcal{E}(\mathbb{Z}_n)$. This proves the result. □

4. The Wiener and hyper-Wiener index of the essential ideal graph of $\mathbb{Z}_n$

In this section, we compute the Wiener index and the hyper-Wiener index of $\mathcal{E}(\mathbb{Z}_n)$ for various n.

Definition 4.1. *The **Wiener index** of a graph G is the sum of all distances between any pair of vertices of G. That is,*

$$W(G) = \sum_{u,v\in V(G)} d(u,v) = \frac{1}{2}\sum_{u\in V(G)} d_G(u),$$

where $d_G(u)$ is the sum of distances between u and all other vertices of $V(G)$.

Definition 4.2. *The* ***hyper-Wiener index*** *of a graph G is defined as*

$$WW(G) = \frac{1}{2}W(G) + \frac{1}{2}\sum_{u,v \in V(G)} d^2(u,v).$$

Proposition 4.3. *Let $n = p^m$, $m > 1$ is a positive integer. Then*

$$W(\mathcal{E}_{\mathbb{Z}_n}) = WW(\mathcal{E}_{\mathbb{Z}_n}) = \binom{m-1}{2}.$$

Proof. By Lemma 3.1 the essential ideal graph $\mathcal{E}_{\mathbb{Z}_n}$ is complete if $n = p^m$ and hence

$$W(\mathcal{E}_{\mathbb{Z}_n}) = \frac{1}{2}\sum_{\langle x\rangle \in V(\mathcal{E}_{\mathbb{Z}_n})} d_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle),$$

where

$$d_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle) = \sum_{\langle x\rangle \in V(\mathcal{E}_{\mathbb{Z}_n})\ \langle y\rangle \neq \langle x\rangle} d(\langle x\rangle, \langle y\rangle) = m - 2.$$

Also,

$$WW(\mathcal{E}_{\mathbb{Z}_n}) = \frac{1}{2}\binom{m-1}{2} + \frac{1}{4}\sum_{\langle x\rangle \in V(\mathcal{E}_{\mathbb{Z}_n})} d^2_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle),$$

where $d^2_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle)$ is the sum of squares of distances between $\langle x\rangle$ and all other vertices of $\mathcal{E}_{\mathbb{Z}_n}$. Hence, $WW(\mathcal{E}_{\mathbb{Z}_n}) = \frac{1}{2}\binom{m-1}{2} + \frac{1}{4}(m-1)(m-2) = \binom{m-1}{2}$. □

Theorem 4.4. *Let $n = p^{m_1}q^{m_2}$, where $p < q$ are distinct primes and m_1, m_2 are positive integers. Then the Wiener index of the essential ideal graph of $\mathbb{Z}_n$ is*

$$W(\mathcal{E}_{\mathbb{Z}_n}) = \frac{1}{2}[m_1m_2(m_1m_2 - 1) + (m_1 + m_2)(2m_1m_2 - 4) + 2(1 + m_1^2 + m_2^2)].$$

Proof. First we partition the vertex set of $\mathcal{E}_{\mathbb{Z}_n}$ as follows :

$X = \{\langle p^r q^s\rangle : 0 \leq r \leq m_1 - 1,\ 0 \leq s \leq m_2 - 1 \text{ and } (r,s) \neq (0,0)\}$
$V_1 = \{\langle p^{m_1} q^s\rangle : 0 \leq s \leq m_2 - 1\}$ and
$V_2 = \{\langle p^r q^{m_2}\rangle : 0 \leq r \leq m_1 - 1\}$.

Then X is the set of all essential ideals of $\mathbb{Z}_n$ and induces a complete subgraph $K_{m_1m_2-1}$. The vertices of V_1 and V_2 are independent and induce a complete bipartite graph K_{m_2,m_1}. Thus $\mathcal{E}_{\mathbb{Z}_n} \simeq K_{m_1m_2-1} \vee K_{m_2,m_1}$. For every $\langle x\rangle \in X$, the sum of the distances to any vertex $\langle y\rangle \in V(\mathcal{E}_{\mathbb{Z}_n})$ can be obtained as $\sum_{\langle y\rangle \in X, \langle y\rangle \neq \langle x\rangle} d(\langle x\rangle,\langle y\rangle) + \sum_{\langle y\rangle \in V_1} d(\langle x\rangle,\langle y\rangle) + \sum_{\langle y\rangle \in V_2} d(\langle x\rangle,\langle y\rangle)$
$= \sum_{\langle y\rangle \in X, y\neq x} 1 + \sum_{\langle y\rangle \in V_1} 1 + \sum_{\langle y\rangle \in V_2} 1$
$= m_1m_2 + m_1 + m_2 - 2$.

For every $\langle x\rangle \in V_1$, the sum of the distances to any vertex $\langle y\rangle \in V(\mathcal{E}_{\mathbb{Z}_n})$ can be determined as $\sum_{\langle y\rangle \in X} d(\langle x\rangle,\langle y\rangle) + \sum_{\langle y\rangle \in V_1, \langle y\rangle \neq \langle x\rangle} d(\langle x\rangle,\langle y\rangle) + \sum_{\langle y\rangle \in V_2} d(\langle x\rangle,\langle y\rangle)$
$= \sum_{\langle y\rangle \in X} 1 + \sum_{\langle y\rangle \in V_1, y\neq x} 2 + \sum_{\langle y\rangle \in V_2} 1$
$= m_1m_2 + m_1 + 2m_2 - 3$.

Similarly, for every $\langle x\rangle \in V_2$, the sum of the distances to any vertex $\langle y\rangle \in V(\mathcal{E}_{\mathbb{Z}_n})$ is $m_1m_2 + 2m_1 + m_2 - 3$. Hence the Wiener index of the graph $\mathcal{E}_{\mathbb{Z}_n}$ is given by $W(\mathcal{E}_{\mathbb{Z}_n}) = \frac{1}{2}[\sum_{\langle x\rangle \in X}(m_1m_2 + m_1 + m_2 - 2) + \sum_{\langle x\rangle \in V_1}(m_1m_2 + m_1 + 2m_2 - 3)$
$+ \sum_{\langle x\rangle \in V_2}(m_1m_2 + 2m_1 + m_2 - 3)]$
$= \frac{1}{2}[(m_1m_2 - 1)((m_1m_2 + m_1 + m_2 - 2)) + m_2(m_1m_2 + m_1 + 2m_2 - 3)$
$+ m_1(m_1m_2 + 2m_1 + m_2 - 3)]$
$= \frac{1}{2}[m_1m_2(m_1m_2 - 1) + (m_1 + m_2)(2m_1m_2 - 4) + 2(1 + m_1^2 + m_2^2)]$. □

Corollary 4.5. *Let $n = p^m q^m$, where p and q are distinct primes and $m > 1$. Then the Wiener index of the essential ideal graph of $\mathbb{Z}_n$ is*

$$W(\mathcal{E}_{\mathbb{Z}_n}) = \frac{m^4 + 4m^3 + 3m^2 - 8m + 2}{2}.$$

Theorem 4.6. *Let $n = p^{m_1} q^{m_2}$, where $p < q$ are distinct primes and $m_1,\ m_2$ are positive integers. Then the hyper-Wiener index of the essential ideal graph of $\mathbb{Z}_n$ is*

$$WW(\mathcal{E}_{\mathbb{Z}_n}) = \frac{1}{2}[m_1m_2(m_1m_2 - 1) + (m_1 + m_2)(2m_1m_2 - 5) + 3(m_1^2 + m_2^2) + 2].$$

Proof. By definition,

$$WW(\mathcal{E}_{\mathbb{Z}_n}) = \frac{1}{2}W(\mathcal{E}_{\mathbb{Z}_n}) + \frac{1}{4}\sum_{\langle x\rangle \in V(\mathcal{E}_{\mathbb{Z}_n})} d^2_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle) \tag{3}$$

where

$$d^2_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle) = \sum_{\langle y\rangle \in V(\mathcal{E}_{\mathbb{Z}_n})} d^2(\langle x\rangle, \langle y\rangle)$$

That is, the sum of squares of distances between the vertex $\langle x\rangle$ and all other vertices of $\mathcal{E}_{\mathbb{Z}_n}$. Now, take the same partition of $V(\mathcal{E}_{\mathbb{Z}_n})$ into $X \cup V_1 \cup V_2$ described in the proof of Theorem 4.4.
Case 1: $\langle x\rangle \in X$
Since $d(\langle x\rangle, \langle y\rangle) = 1$ for any $\langle y\rangle \in X,\ V_1\ or\ V_2$, $d^2_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle) = m_1m_2 + m_1 + m_2 - 2$ Case 2: $\langle x\rangle \in V_1$
$d^2(\langle x\rangle, \langle y\rangle) = \begin{cases} 1, & \text{when } \langle y\rangle \in X \ or\ V_2 \\ 4, & \text{when } \langle y\rangle \in V_1\ and\ \langle y\rangle \neq \langle x\rangle \end{cases}$.
Thus $d^2_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle) = m_1m_2 + m_1 + 4m_2 - 5$.
Case 3: $\langle x\rangle \in V_2$
Here, $d^2(\langle x\rangle, \langle y\rangle) = \begin{cases} 1, & \text{when } \langle y\rangle \in X \ or\ V_1 \\ 4, & \text{when } \langle y\rangle \in V_2\ and\ \langle y\rangle \neq \langle x\rangle \end{cases}$.
Then, $d^2_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle) = m_1m_2 + m_2 + 4m_1 - 5$.
From Equation (3) and Theorem 4.4, we get the required result. □

Next, we calculate the Wiener and hyper-Wiener indices of $\mathcal{E}_{\mathbb{Z}_n}$ for $n = p_1p_2\cdots p_k$, where $p_1 < p_2 < \cdots < p_k$ are distinct primes, using the idea of equitable partition of vertices.

Definition 4.7. *[9] For a graph G, A partition of vertices $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ is said to be an equitable partition if each vertex in V_i has the same number of neighbors in V_j for any $i, j \in \{1, 2, \cdots, k\}$.*

For this, consider the set $S = \{p_1, p_2, p_3, \ldots, p_{k-1}, p_k\}$. Then going through the same process as in the proof of Theorem 3.13, we can exhaust all the vertices of $\mathcal{E}_{\mathbb{Z}_n}$. Also, we can see that the vertices of $\mathcal{E}_{\mathbb{Z}_n}$ can be partitioned into an equitable partition. That is,
$V_1 = \{\langle p_i\rangle : 1 \le i \le k\}$
$V_2 = \{\langle p_ip_j\rangle : 1 \le i \le k-1 and\ i+1 \le j \le k\}$
$V_3 = \{\langle p_ip_jp_l\rangle : 1 \le i \le k-2,\ i+1 \le j \le k-1 and\ j+1 \le l \le k\}$
$\vdots$
$V_{(k-1)} = \{\langle p_1p_2p_3\cdots p_{k-1}\rangle, \langle p_1p_2p_3\cdots p_{k-2}p_k\rangle, \cdots, \langle p_2p_3\cdots p_{k-1}p_k\rangle\}$.
Clearly $|V_1| = \binom{k}{1},\ |V_2| = \binom{k}{2}, \cdots,$ and $|V_{(k-1)}| = \binom{k}{k-1}$.

By Lemma 3.8, we can see that any vertex in V_1 has $\binom{k-1}{1}$ neighbors in V_1, $\binom{k-1}{2}$ neighbors in $V_2, \cdots, \binom{k-1}{k-1}$ neighbors in V_{k-1}. In general, any vertex of the set V_t has $\binom{k-t}{1}$ neighbors in V_1, $\binom{k-t}{2}$ neighbors in $V_2, \cdots, \binom{k-t}{k-1}$ neighbors in V_{k-1} respectively. Hence this makes an equitable partition of $V(\mathcal{E}_{\mathbb{Z}_n})$ into sets $V_1,\ V_2,\ \cdots,\ V_{k-1}$.

Theorem 4.8. *Let $n = p_1p_2\cdots p_k$ where $p_1, p_2, \cdots, p_k$ are distinct primes. Then the Wiener index of the essential ideal graph $\mathcal{E}_{\mathbb{Z}_n}$ is*

$$W(\mathcal{E}_{\mathbb{Z}_n}) = \frac{1}{2}\sum_{t=1}^{k-1}\binom{k}{t}[2^{k+1} + 2^t - 2^{k-t} - 7].$$

Proof. By definition,

$$W(\mathcal{E}_{\mathbb{Z}_n}) = \frac{1}{2}\sum_{\langle x\rangle \in V(\mathcal{E}_{\mathbb{Z}_n})} d_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle),$$

where $d_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle)$ is the sum of distances between the vertex $\langle x\rangle$ and all other vertices of $\mathcal{E}_{\mathbb{Z}_n}$.

Let $\langle x\rangle \in V_t$, for $1 \le t \le k-1$. Then,

$$d_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle) = \sum_{\langle y\rangle \in V_1} d(\langle x\rangle, \langle y\rangle) + \sum_{\langle y\rangle \in V_2} d(\langle x\rangle, \langle y\rangle) + \cdots + \sum_{\langle y\rangle \in V_t,\ \langle y\rangle \neq \langle x\rangle} d(\langle x\rangle, \langle y\rangle)$$
$$\sum_{\langle y\rangle \in V_{t+1}} d(\langle x\rangle, \langle y\rangle) + \cdots + \sum_{\langle y\rangle \in V_{k-t}} d(\langle x\rangle, \langle y\rangle) + \sum_{\langle y\rangle \in V_{k-t+1}} d(\langle x\rangle, \langle y\rangle) +$$
$$\cdots + \sum_{\langle y\rangle \in V_{k-1}} d(\langle x\rangle, \langle y\rangle) \tag{4}$$

By Lemma 3.8, if $\langle y\rangle \in V_s$ for $1 \le s \le k-t$ then,
$d(\langle x\rangle, \langle y\rangle) = \begin{cases} 1 & if gcd(x,y) = 1 \\ 2 & if gcd(x,y) \neq 1 \end{cases}$ and if $\langle y\rangle \in V_{k-t+s}$ for $1 \le s \le t-1$ then,

$$d(\langle x\rangle, \langle y\rangle) = \begin{cases} 3 & if gcd(x,y) = product\ of\ s\ distinct\ primes \\ 2 & otherwise \end{cases}.$$

Then, for $1 \le s \le k-t$; $s \neq t$,

$$\sum_{\langle y\rangle \in V_s} d(\langle x\rangle, \langle y\rangle) = 2\binom{k}{s} - \binom{k-t}{s}$$

and for $1 \le s \le t-1$,

$$\sum_{\langle y\rangle \in V_{k-t+s}} d(\langle x\rangle, \langle y\rangle) = 2\binom{k}{k-t+s} + \binom{t}{s}.$$

Hence by Equation (4),

$$d_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle) = \sum_{s=1 s\neq t}^{k-t} [2\binom{k}{s} - \binom{k-t}{s}] + [2\binom{k}{t} - \binom{k-t}{t} - 2] + \sum_{s=1}^{t-1}[2\binom{k}{k-t+s} + \binom{t}{s}]$$

$$= 2\sum_{s=1}^{k-1}\binom{k}{s} - \sum_{s=1}^{k-t}\binom{k-t}{s} + \sum_{s=1}^{t-1}\binom{t}{s} - 2 = 2^{k+1} + 2^t - 2^{k-t} - 7.$$

Hence,

$$W(\mathcal{E}_{\mathbb{Z}_n}) = \frac{1}{2}[\sum_{\langle x\rangle \in V_1} d_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle) + \sum_{\langle x\rangle \in V_2} d_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle) + \cdots + \sum_{\langle x\rangle \in V_{k-1}} d_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle)]$$

$$= \frac{1}{2}\sum_{t=1}^{k-1}\binom{k}{t}[2^{k+1} + 2^t - 2^{k-t} - 7].$$

□

Theorem 4.9. *Let $n = p_1 p_2 \cdots p_k$ where $p_1 < p_2 < \cdots < p_k$ are distinct primes. Then the hyper-Wiener index of the essential ideal graph $\mathcal{E}_{\mathbb{Z}_n}$ is*

$$WW(\mathcal{E}_{\mathbb{Z}_n}) = \frac{1}{2}\sum_{t=1}^{k-1}\binom{k}{t}[3\times 2^k - 2\times 2^{k-t} + 3\times 2^t - 13].$$

Proof. By definition,

$$WW(\mathcal{E}_{\mathbb{Z}_n}) = \frac{1}{2}W(\mathcal{E}_{\mathbb{Z}_n}) + \frac{1}{4}\sum_{\langle x\rangle\in V(\mathcal{E}_{\mathbb{Z}_n})} d^2_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle) \tag{5}$$

where $d^2_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle)$ is the sum of squares distances between $\langle x\rangle$ and all other vertices of $\mathcal{E}_{\mathbb{Z}_n}$.
Let $\langle x\rangle$ be a vertex of V_t, for $1 \le t \le k-1$. For $\langle y\rangle \in V_s$, $1 \le s \le k-t$,

$$d^2(\langle x\rangle,\langle y\rangle) = \begin{cases} 1, & \textit{if } gcd(x,y) = 1 \\ 4, & \textit{if } gcd(x,y) \neq 1 \end{cases}$$

and for $\langle y\rangle \in V_{k-t+s}$, $1 \le s \le t-1$,

$$d^2(\langle x\rangle,\langle y\rangle) = \begin{cases} 9 & if\ gcd(x,y) = \textit{product of } s \textit{ distinct primes} \\ 4 & \textit{otherwise} \end{cases}.$$

Then,

$$\begin{aligned} d^2_{\mathcal{E}_{\mathbb{Z}_n}}(\langle x\rangle) = \sum_{\langle y\rangle\in V_1} d^2(\langle x\rangle,\langle y\rangle) + \sum_{\langle y\rangle\in V_2} d^2((x),(y)) + \cdots + \sum_{\langle y\rangle\in V_t,\ (y)\neq(x)} d^2(\langle x\rangle,\langle y\rangle) + \\ \cdots + \sum_{\langle y\rangle\in V_{k-t}} d^2(\langle x\rangle,\langle y\rangle) + \cdots + \sum_{\langle y\rangle\in V_{k-1}} d^2(\langle x\rangle,\langle y\rangle) \end{aligned} \tag{6}$$

For $1 \le s \le k-t$; $s \neq t$,

$$\sum_{\langle y\rangle\in V_s} d^2(\langle x\rangle,\langle y\rangle) = 4\binom{k}{s} - 3\binom{k-t}{s}$$

and for $1 \le s \le t-1$,

$$\sum_{\langle y\rangle\in V_{k-t+s}} d^2(\langle x\rangle,\langle y\rangle) = 4\binom{k}{k-t+s} + 5\binom{t}{s}.$$

Finally, for $\langle y\rangle \in V_t$,

$$\sum_{\langle y\rangle\in V_t} d^2(\langle x\rangle,\langle y\rangle) = 4\binom{k}{t} - 3\binom{k-t}{t} - 4.$$

Hence by Equations (5) and (6) and Theorem 4.8, we get the required result. □

5. Conclusion

In this paper, the adjacency spectrum of the essential ideal graph of finite commutative ring $\mathbb{Z}_n$, for $n = \{p^m, p^{m_1}q^{m_2}\}$, where p, q are distinct primes, and $m, m_1, m_2 \in \mathbb{N}$ is determined. All the eigenvalues of $\mathcal{E}_{\mathbb{Z}_n}$ whenever n is a product of three or four distinct primes are computed. Further, we have established a characterization of the ring $\mathbb{Z}_n$ for which 0 is an eigenvalue of $\mathcal{E}_{\mathbb{Z}_n}$. In addition, the topological indices, namely the Wiener index and hyper-Wiener index of the essential ideal graph of $\mathbb{Z}_n$ for different forms of n are calculated.

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