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Binary vertex labelings of graphs and digraphs

Research Article

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Abstract: A $(0,1)$-labeling of a set is said to be *friendly* if the number of elements of the set labeled 0 and the number labeled 1 differ by at most 1. Let g be a labeling of the edge set of a graph that is induced by a labeling f of the vertex set. If both g and f are friendly then f is said to be a *cordial* labeling of the graph. This concept extended to directed graphs is called $(2,3)$-cordiality of digraphs. We investigate the labelings that are both cordial for a graph and $(2,3)$-cordial for an orientation of it. We also consider the same problem for other known binary graph labelings.

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Keywords: Friendly labeling, Cordial graph, $(2,3)$-cordial digraph, Freyberg-Marr labeling

1. Introduction

A $(0,1)$-labeling of the vertices of a graph is cordial if the zeros and ones are evenly distributed over the vertices and if the edges are also evenly labeled by zeros and ones, where that labeling is the absolute difference of the vertex labelings of the incident vertices. This concept was introduced by Cahit [2] in 1987. Since then there have been a couple of further binary vertex labelings of graphs defined, see [1, 5]. In this article we define some related labelings and investigate the relationships among them.

Let $\mathcal{G}_n$ denote the set of simple (loopless) undirected graphs on the vertex set $V = \{v_1, v_2, \dots v_n\}$, and let $\mathcal{D}_n$ denote the set of simple (loopless) directed graphs on the same vertex set V. In this article we need to specify the concept of a friendly labeling.

Definition 1.1. *Let $\mathcal{Z}$ be a set and label the set with entries from the set $\mathcal{A}$. Let f be the mapping $f : \mathcal{Z} \to \mathcal{A}$ that preforms this labeling. This labeling of a set is called* friendly *if $-1 \leq |f^{-1}(i)| - |f^{-1}(j)| \leq 1$ for any $i, j \in \mathcal{A}$. To specify that the indexing set is $\mathcal{A}$ we say the labeling f is $\mathcal{A}$-friendly. We say that f is k-*friendly *if it is $\mathcal{Z}_k$-friendly where $\mathcal{Z}_k$ is the set of integers modulo k, or if $\mathcal{A}$ is any set of k elements. A labeling is said to be* balanced *if $|f^{-1}(i)| = |f^{-1}(j)|$ for any $i, j \in \mathcal{A}$.*

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2. Cordial graphs and digraphs

Definition 2.1. *Let $\mathcal{A}$ be a set and G be an undirected graph with vertex set V and edge set E. Let $f : V \to \mathcal{A}$ be an $\mathcal{A}$-friendly labeling of the* non-isolated *vertices of V and let $g : E \to \mathcal{A}$ be an induced labeling of E, that is given an edge uv, $g(uv) = \hat{g}(f(u), f(v))$ where $\hat{g} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. Note that $\hat{g}$ must be symmetric to be well defined. The graph G with the $\mathcal{A}$-friendly labeling f of the non-isolated vertices of G and induced edge labeling g is said to be* $\mathcal{A}$-cordial *if g is also an $\mathcal{A}$-friendly mapping. Also, G is* k-cordial *if $\mathcal{A}$ is $\mathcal{Z}_k$, and we say G is* cordial *if $\mathcal{A}$ is $\mathcal{Z}_2$.*

If the induced mapping $g : E \to \mathcal{A}$ is defined to be $g(uv) = f(u) * f(v)$ where $*$ is a binary operation on $\mathcal{A}$ and G is $\mathcal{A}$-cordial with the friendly vertex label f, and induced labeling g, we say G is $(\mathcal{A}, *)$-cordial.

Remark 2.2. *It should be noted that $(\mathcal{Z}_2, +)$-cordial and $(\mathcal{Z}_2, -)$-cordial are the same and in fact is the usual definition of cordial where f is a $(0, 1)$ labeling and $g(u, v) = |f(u) - f(v)|$ using real arithmetic. Further, the restriction that f is a friendly labeling of the non-isolated vertices, not necessarily all vertices, is required because, for example, if all vertices were labeled, the graph $2K_2$ would be $(\mathcal{Z}_2, +)$-cordial as a graph on five or more vertices but not on four vertices.*

For digraphs we have a similar definition, however, as digraphs are not usually symmetric, we do not require the induced mapping g to be symmetric:

Definition 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be sets and let $G = (V, A) \in \mathcal{D}_n$ be an directed graph with vertex set V and arc set A. Let $f : V \to \mathcal{A}$ be an $\mathcal{A}$-friendly labeling of the non-isolated vertices of V and let $g : A \to \mathcal{B}$ be an induced labeling of A, that is, given an arc $\overrightarrow{uv}$, $g(\overrightarrow{uv}) = \hat{g}(f(u), f(v))$ where $\hat{g} : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$. The graph G with the $\mathcal{A}$-friendly labeling f of the non-isolated vertices and induced edge labeling g is said to be* $(\mathcal{A}, \mathcal{B})$-cordial *if g is a $\mathcal{B}$-friendly mapping. When $\mathcal{A} = \mathcal{Z}_k$ and $\mathcal{B} = \mathcal{Z}_\ell$ we say that G is (k, ℓ)-cordial. In particular we say that G is $(2, 3)$-cordial if $\mathcal{A} = \mathcal{Z}_2$, $\mathcal{B} = \mathcal{Z}_3$ and $g(\overrightarrow{uv}) = f(v) - f(u)$. Note that in this case g is anti-symmetric.*

Note that in the above definitions, if $\hat{g}$ is the binary mapping corresponding to one of the binary operations on $\mathcal{A}$ or $\mathcal{B}$ we indicate that by placing that operator in the notation. For example a digraph is $(\mathcal{A}, \mathcal{B}, -)$-cordial indicates that the induced labeling g is $g(\overrightarrow{uv}) = f(v) - f(u)$. So a digraph is $(2, 3, -)$-cordial means that the arc labelings are $f(v) - f(u)$ for the arc $\overrightarrow{uv}$. In this specific case we usually drop the minus sign and write $(2, 3)$-cordial. A graph with an $\mathcal{A}$-friendly vertex labeling is *product cordial* if it is $(\mathcal{A}, \times)$-cordial. Unless specified otherwise, product cordial graphs have the vertex set labeled with $\{0, 1\}$.

Definition 2.4. *A graph $G \in \mathcal{G}_n$ is said to be* $(2, 3)$-orientable *if some orientation of the edges of G yeilds a digraph that is $(2, 3)$-cordial.*

Some interest has lately been shown in the cordiality of the friendship graphs, introduced by Erdős, Rényi and Sós, [3], with a proof of the friendship theorem: *If any two distinct vertices of a simple graph have exactly one common neighbor then some vertex is adjacent to all other vertices.* Such graphs are known as friendship graphs, F_n on $2n + 1$ vertices. See Figure 1. It is known, [4], that not all friendship graphs are cordial and some are. Notably F_3, F_5 and F_8 are cordial, however, F_4 and F_6 are not cordial. We consider the $(2, 3)$-orientability in the following example.

Example 2.5. *Let F_n denote the friendship graph on $2n + 1$ vertices, that is F_n is the graph consisting of n triangles sharing a common vertex. See Figure 1. It is easily seen that by labeling the central vertex 0 and the other two vertices of each triangle with one vertex labeled 0 and the other vertex labeled 1, the labeling is friendly. Further, orienting each triangle as a directed 3-cycle yields one edge of each triangle labeled 0, one labeled 1, and one labeled -1. So that the oriented graph has n arcs labeled 0, n arcs labeled 1, and n arcs labeled -1. That is, F_n is $(2, 3)$-orientable,*

Theorem 2.6. *Let G be a connected graph in $\mathcal{G}_n$ with at least 6 edges. Let $f : V \to \{0, 1\}$ be a vertex labeling of G. Then, f is never both a cordial labeling of G and a $(2, 3)$-orientable labeling of G unless the number of edges in G is odd and at most 7.*

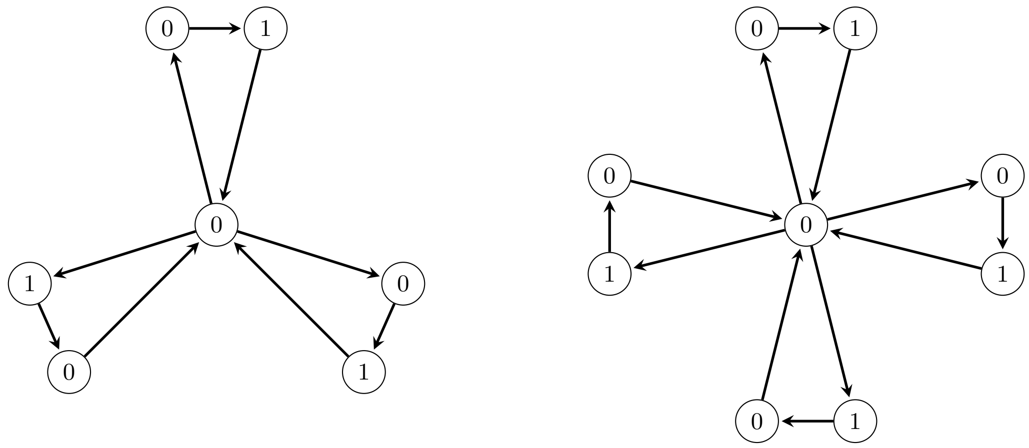


Figure 1. Oriented F_3 and F_4 which are $(2,3)$-cordial.

Proof. Suppose $G \in \mathcal{G}_n$ has at least six edges and is cordial with cordial vertex labeling $f : V \to \{0,1\}$.

Suppose the number of edges in G is even, $m = 2\ell$. Then, the number of edges whose incident vertices are labeled differently is ℓ, and the number of edges whose incident vertices are labeled the same is ℓ. If f is also a vertex labeling of G yielding a $(2,3)$-orientable graph, then the number of edges labeled differently must be approximately $\frac{2}{3}m$. Thus, we must have $\frac{2}{3}m - 1 \leq \ell$ or $\frac{4\ell}{3} \leq \ell + 1$. That is $\ell \leq 3$ or $m \leq 6$. Since $m \geq 6$, $m = 6$. We now have a graph that has six edges that has exactly 3 edges labeled the same. But a $(2,3)$-orientable graph with 6 edges must have exactly 2 edges whose incident vertices are labeled the same. Thus, m cannot be six and hence, m is not even.

Now suppose the number of edges in G, m, is odd, $m = 2\ell + 1$, and at least 9. Then, since f is a cordial labeling of the vertices of G we must have that the number of edges labeled the same is ℓ or $\ell + 1$. If f is also a vertex labeling of G yielding a $(2,3)$-orientable graph, then the number of edges labeled differently must be more than the number of edges whose incident vertices are labeled the same. Thus the number of edges labeled differently is $\ell + 1$ and the number of edges labeled the same is ℓ. Since approximately $\frac{1}{3}m$ of the edges in a $(2,3)$-oriented graph must be labeled the same, we must have $\frac{1}{3}m + 1 \geq \ell$ or $\frac{2\ell+1}{3} \geq \ell - 1$. That is $\ell \leq 4$ or $m \leq 9$. Thus $m = 9$.

Now, suppose $G \in \mathcal{G}_n$ is a graph with an odd number of edges and with a vertex labeling f that is a cordial labeling and induces a $(2,3)$-orientable graph. From above, the number of edges in G is nine. If G has nine edges, then four of them have incident vertices labeled the same and five differently. But this is impossible as G, having f being a labeling that yields a $(2,3)$-orientable graph with nine edges, must have six edges whose incident vertices are labeled differently. Thus, $m \leq 7$. □

3. Graphs which have a friendly vertex labeling that is both cordial and $(2,3)$-orientable

We only consider connected graphs. Also there are no graphs with more than 7 edges that have a friendly vertex labeling that is both cordial and $(2,3)$-orientable. We list all graphs with 7 or fewer edges that have a friendly vertex labeling that is both cordial and $(2,3)$-orientable.

3.1. Graphs with one edge

The only connected graph with one edge is K_2 and any friendly labeling yields a cordial graph and a $(2,3)$-orientable graph.

3.2. Graphs with two edges

The only connected graph with 2 edge is P_2, the path with two edges. Label the vertices 0,0,1 to get a labeling that yields a cordial graph and a $(2,3)$-orientable graph.

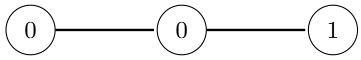


Figure 2. Graphs with two edges

3.3. Graphs with three edges

There are three connected graphs with three edges: K_3, P_3 and the 3-star. All three have a friendly vertex labeling that is both cordial and $(2,3)$-orientable:

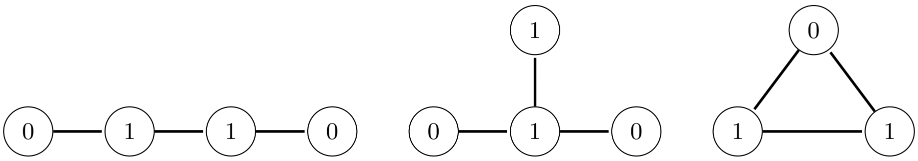
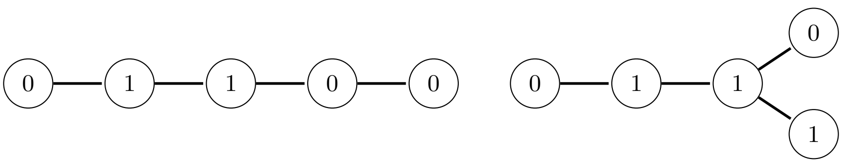


Figure 3. Graphs with three edges

3.4. Graphs with four edges

There are five connected graphs with four edges, all have a friendly vertex labeling that is both cordial and $(2,3)$-orientable.



3.5. Graphs with five edges

Here the number of vertices must be 4, 5, or 6.

If G has 4 vertices, only the random graph ($K_4 \setminus e$) where e is any edge) has 5 edges. It has a friendly vertex labeling that is both cordial and $(2,3)$-orientable:

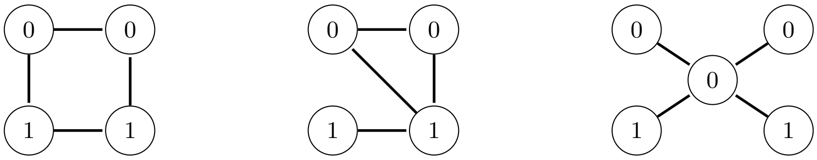
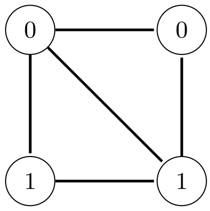
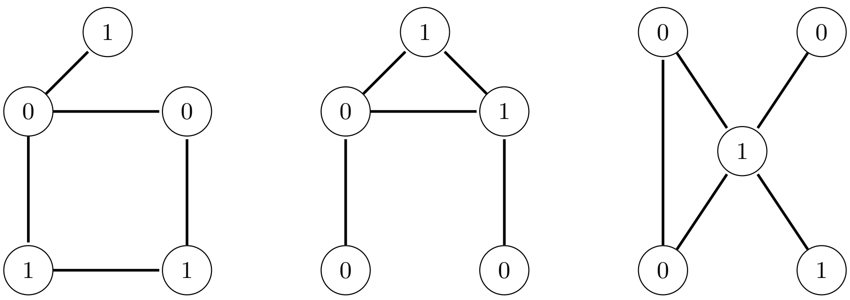


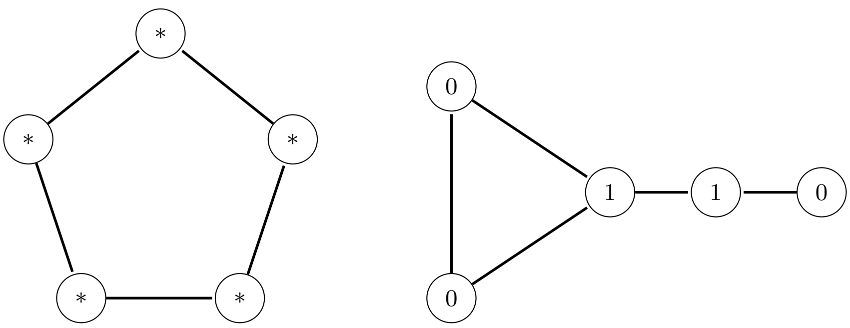
Figure 4. Graphs with four edges



If G has 5 vertices.

There are five graphs with 5 edges on five vertices. All except the 5-cycle have a friendly vertex labeling that is both cordial and $(2,3)$-orientable (The 5-cycle is not $(2,3)$-orientable with any friendly vertex labeling.):





If G has 6 vertices, the only graphs with 5 edges are trees, There are six of these, all have a friendly vertex labeling that is both cordial and $(2,3)$-orientable:

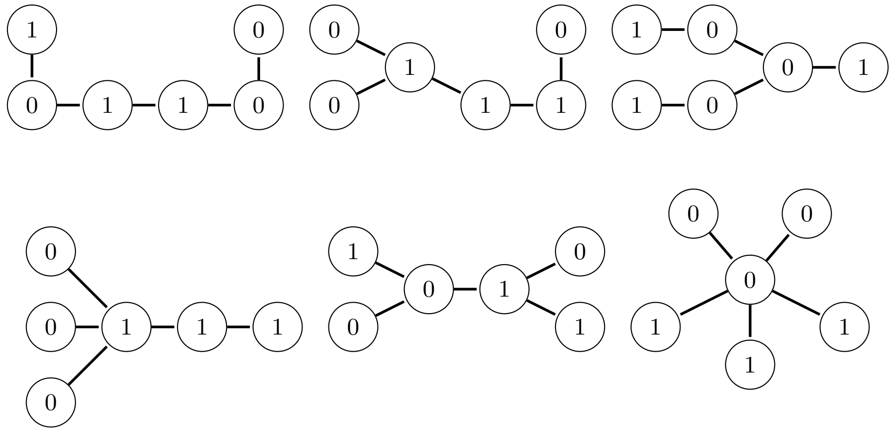


Figure 5. Graphs with five edges

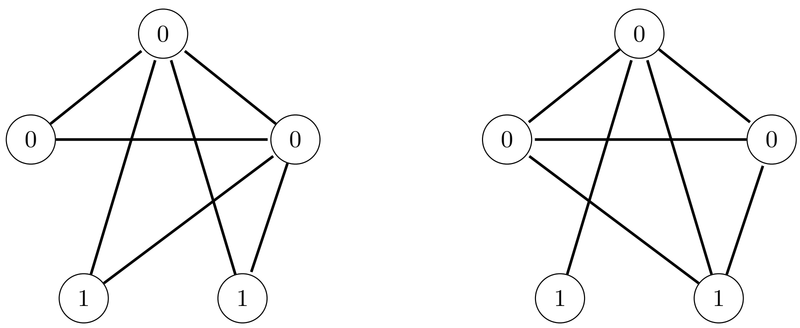
3.6. Graphs with six edges

There are no graphs with six edges that have a friendly vertex labeling that is both cordial and $(2, 3)$-orientable. A cordial labeling would have 3 edges labeled 0 and 3 not labeled 0, so the labeling cannot be $(2, 3)$-orientable.

3.7. Graphs with seven edges

There are 79 non isomorphic connected graphs with 7 edges. Since the graph has 7 edges and is connected there must be at least 5 vertices and at most 8.

There are four connected graphs with seven edges on 5 vertices, all have a friendly vertex labeling that is both cordial and $(2, 3)$-orientable.



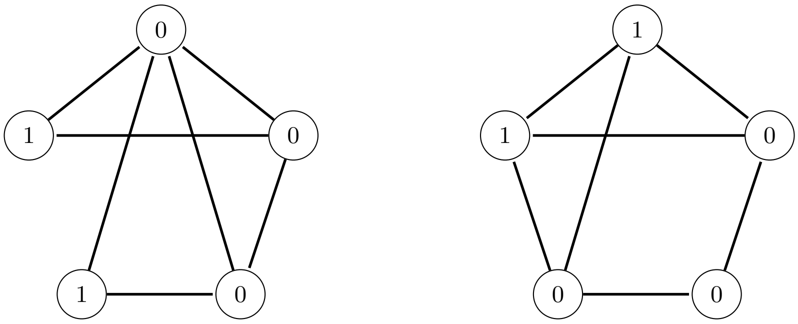
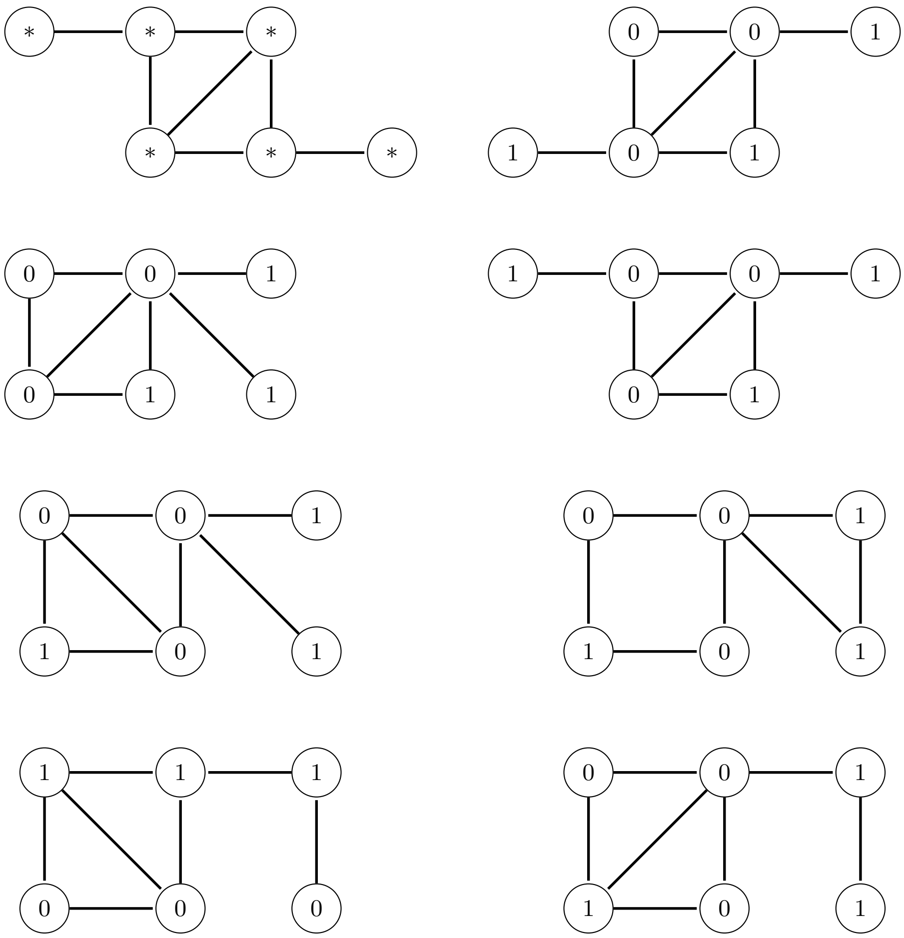


Figure 6. Graphs with seven edges

There are 19 non-isomorphic graphs with 7 edges on 6 vertices. All but one have a friendly vertex labeling that is both cordial and $(2,3)$-orientable.



There are 33 non-isomorphic graphs with 7 edges on 7 vertices. All have a friendly vertex labeling that is both cordial and $(2,3)$-orientable.

There are 23 non-isomorphic graphs with 7 edges on 8 vertices, all trees of course. All have a friendly vertex labeling that is both cordial and $(2,3)$-orientable.

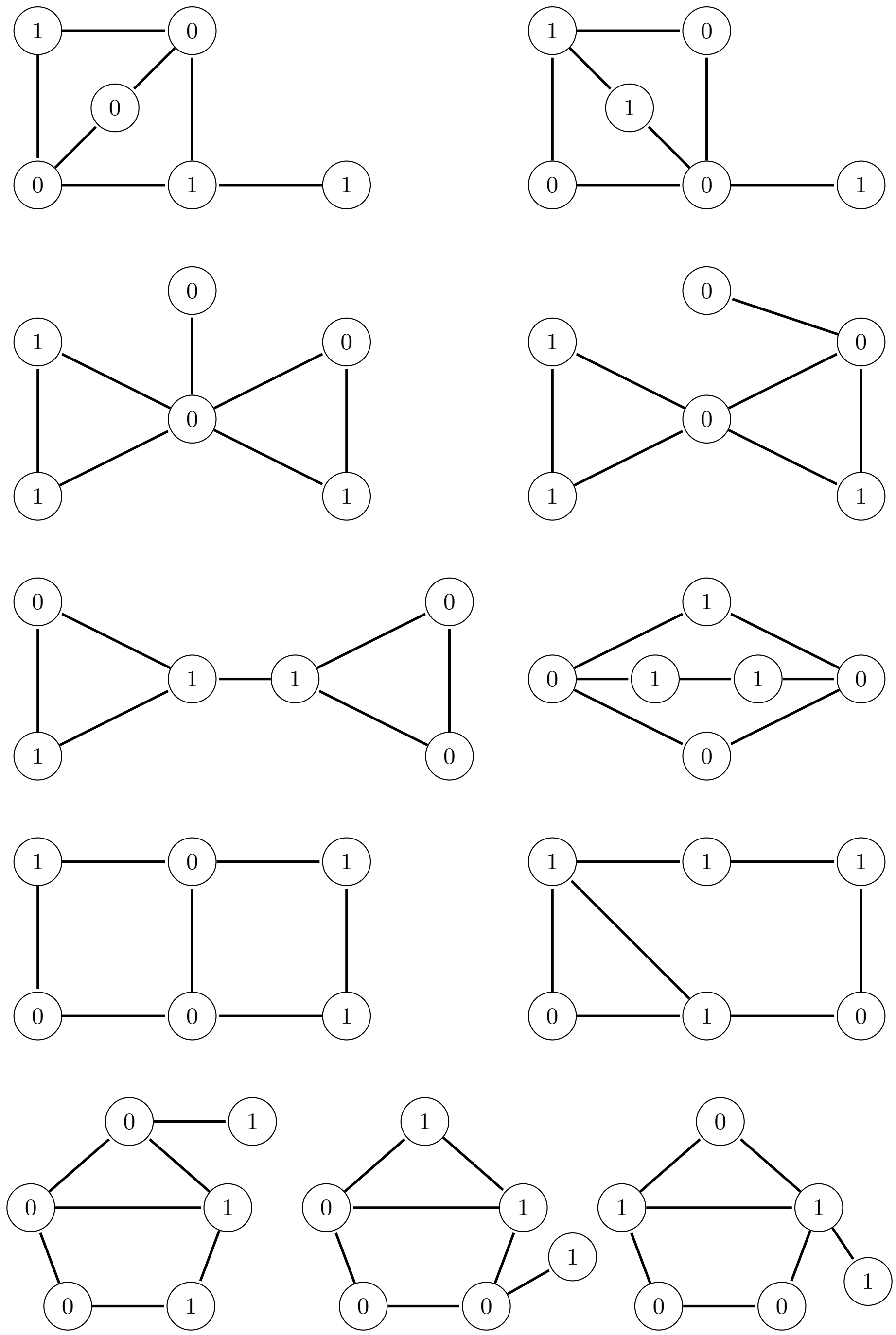
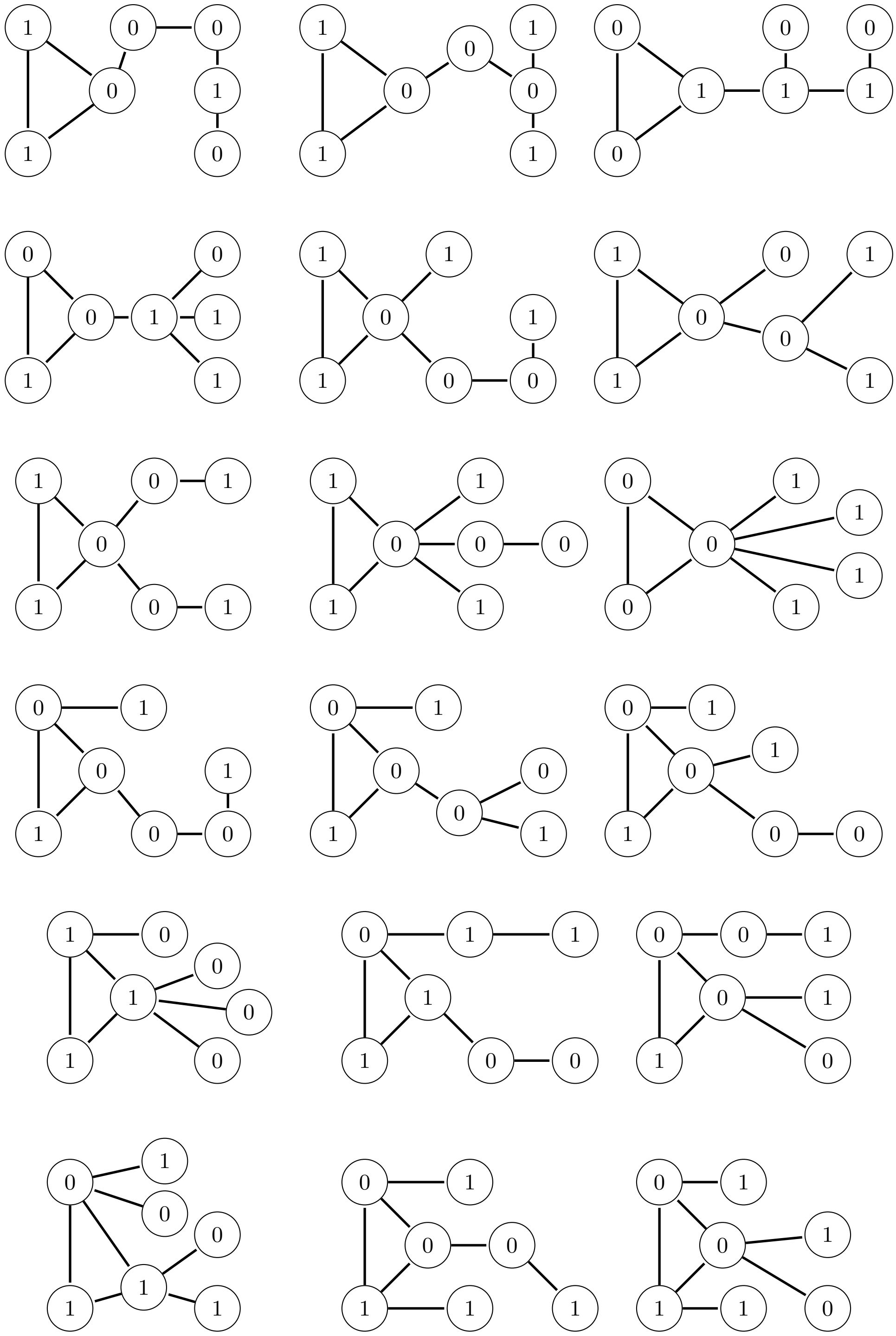


Figure 7. Graphs with seven edges

4. Freyberg-Marr labelings

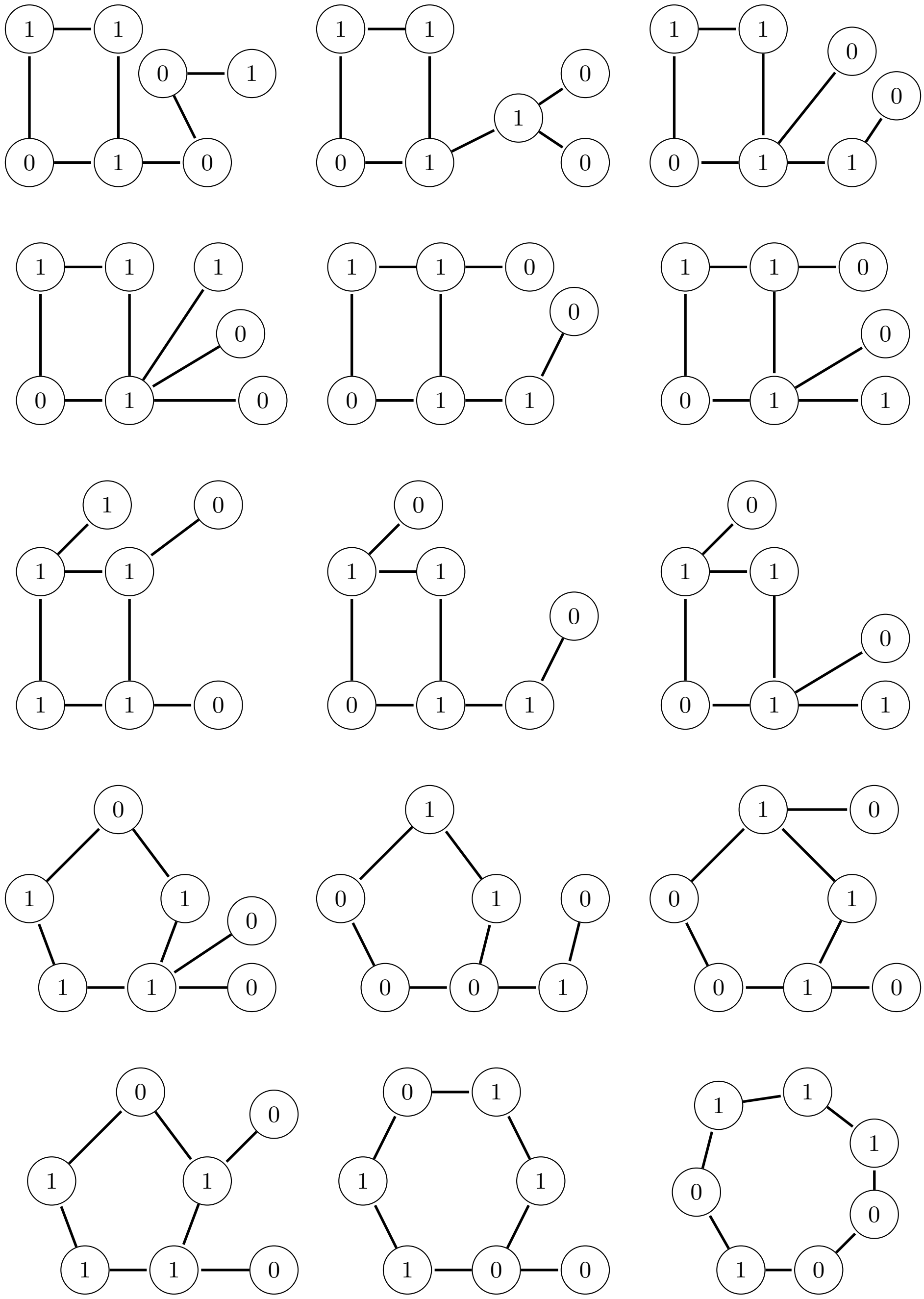
In 2022, at a virtual presentation of the 53rd Southeastern International Conference on Combinatorics, Graph Theory & Computing, Brian Freyberg defined a red-blue coloring as:



Definition 4.1. [5] *Let G be a simple graph with each vertex colored red or blue. If every vertex has an equal number of red and blue neighbors, then the coloring is a red-blue coloring and G is a red-blue graph.*

We now define some graph labelings based on this definition.

Definition 4.2. *Let $G = (V, E)$ be an undirected simple (loopless) graph with vertex set V and edge set*

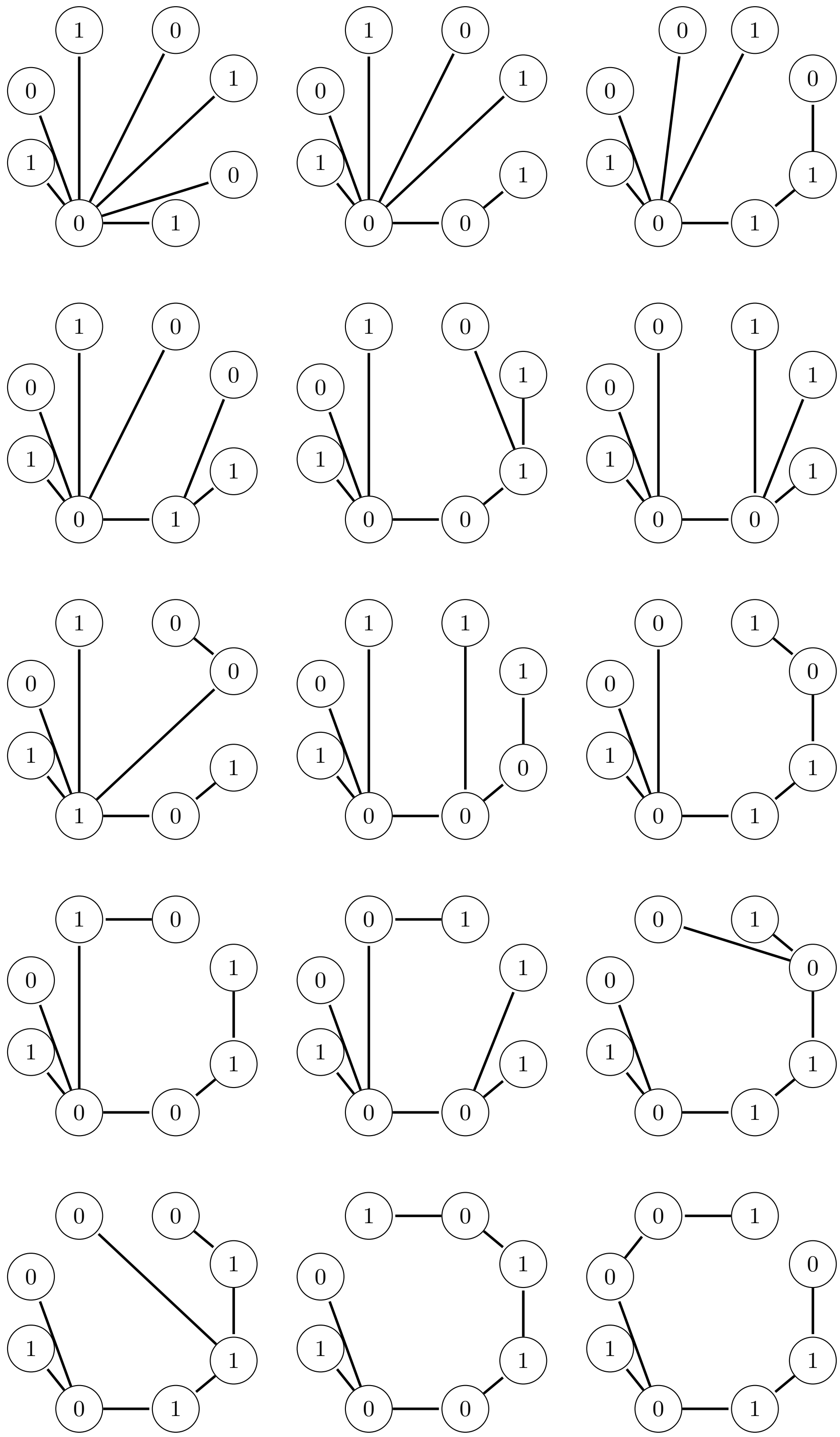


E. *The* open neighborhood *of a vertex* $v \in V$ *is* $N(v) = \{u \in V \mid uv \in E\}$. *The* closed neighborhood *of* v *is* $\overline{N(v)} = N(v) \cup \{v\}$.

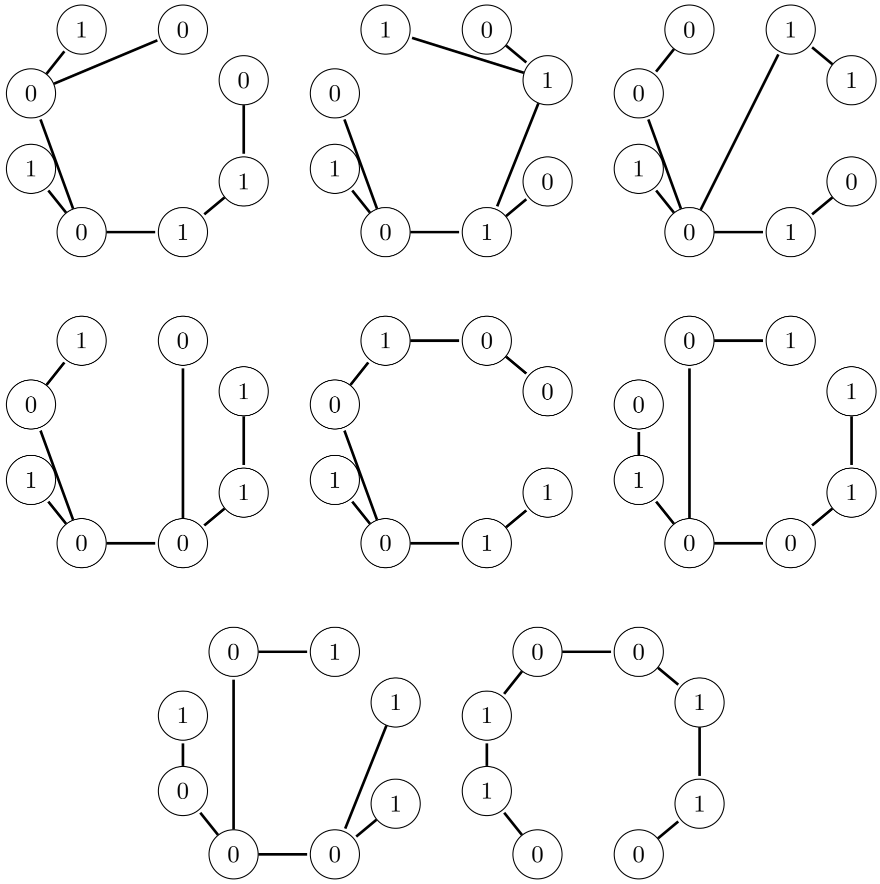
Let $f : V \to \mathcal{X}$ be a labeling of the vertices of G. For $\xi \in \mathcal{X}$ and U a subset of the vertices of G, let $\xi(U)$ denote the vertices of U that are labeled ξ. So that, $\xi(V) = f^{-1}(\xi)$.

Definition 4.3. *Let* $f : V \to \mathcal{X}$ *be labeling of* $G = (V, E) \in \mathcal{G}_n$.

- *The labeling* f *is a* open Freyberg-Marr-labeling, *an* FM_O-labeling, *if for each* $v \in V$, *for any* $\xi, \chi \in \mathcal{X}$, $|\xi(N(v))| = |\chi(N(v))|$.



- *The labeling f is a* closed Freyberg-Marr-labeling*, an* FM_C-labeling*, if for each $v \in V$, for any* $\xi, \chi \in \mathcal{X}$, $|\xi(\overline{N(v)})| = |\chi(\overline{N(v)})|$.



Note that if $\mathcal{X} = \{0, 1\}$, an FM$_O$-labeling is the red-blue labeling introduced by Brian Freyberg [5].

We now define another similar labeling that arises by using open or closed neighborhoods depending on the parity of the vertex degree:

Definition 4.4. *A labeling of* G, $f : V \to \mathcal{X}$ *is an* mixed Freyberg-Marr-labeling*, an* FM$_M$-labeling*, if for each* $v \in V$*, for any* $\xi, \chi \in \mathcal{X}$*,* $|\xi(N(v))| = |\chi(N(v))|$ *if* $\delta(v)$ *is even, and* $|\xi(\overline{N(v)})| = |\chi(\overline{N(v)})|$ *if* $\delta(v)$ *is odd.*

Henceforth, we shall assume that the labeling set is $\{0, 1\}$.

Note that any graph with a vertex of odd degree is not FM$_O$-labelable and any graph with a vertex of even degree is not FM$_C$-labelable. Further every FM$_O$-labelable graph or FM$_C$-labelable graph is an FM$_M$-labelable graph. No graph except the trivial ones ($V = \emptyset$ and $V = \{v_0\}$}) are both FM$_O$-labelable and FM$_C$-labelable, and a graph that is neither FM$_O$-labelable nor FM$_C$-labelable may or may not be FM$_M$-labelable.

Example 4.5.

1. *Every cycle on an even number (≥ 4) of vertices is FM$_O$-labelable.*
2. *The cube (3-dimensional) is FM$_C$-labelable.*
3. *The fan graph on five vertices is neither FM$_O$-labelable graph nor FM$_C$-labelable but is FM$_M$-labelable.*

4. *The Petersen graph (as well as the fan graph on 6 vertices) is neither* FM_O*-labelable graph nor* FM_C*-labelable and is not* FM_M*-labelable.*

The tesseract (4-cube) is FM_O-labelable. Probably every n-cube is either FM_O- or FM_C-labelable. Fan graphs (k-fan for k vertices total) are neither FM_O- nor FM_C-labelable, they have vertices of both even and odd degree. but some are FM_M-labelable, The 5-fan and 8-fan in particular.

The only FM_M-labeling of a graph that is also a cordial labeling is an FM_O-labeling, since the number of edges whose incident vertices are labeled the same and the number of edges whose incident vertices are labeled differently differ precisely by the number of odd vertices, which in any graph is always an even number. Thus, an FM_C-labeling is never a cordial labeling.

Let $G = (V, E) \in \mathcal{G}_n$ be connected with G having m edges and let $f : V \to \{0, 1, \ldots, m-1\}$ be a friendly labeling of the vertices of G. If $g(uv) = |f(u) - f(v)|$ (using real arithmetic) ($g : E \to \{0, 1, \ldots, m-1\}$) is also friendly, then G is said to be *graceful*. If $g(uv) = [f(u) + f(v)] \bmod m$ is friendly then G is said to be *harmonious*. In 1987, Cahit [2] defined a *cordial* graph to be a graph $G = (V, E) \in \mathcal{G}_n$ such that there exists a friendly labeling $f : V \to \{0.1\}$ which induces the labeling $g : E \to \{0.1\}$, defined by $g(uv) = |f(u) - f(v)|$ which is also friendly. Freyberg and Marr [5] called a cordial graph *balanced* if the number of vertices labeled 0 was equal to the number of vertices labeled 1 and the number of edges labeled 0 was equal to the number of edges labeled 1. They then gave an example to show that a FM_O-labelable graph exists which is not balanced cordial. Below, we show that every FM_O-labelable graph is cordial and hence, if an FM_O-labelable graph has an even number of vertices and an even number of edges then it is balanced cordial. We also say a graph is *balanced* $(2,3)$*-orientable* if the vertex labeling is friendly and exactly one third of the edges are incident to vertices labeled the same (the number of edges must be a multiple of 3).

Theorem 4.6. *Let* $G = (V, E) \in \mathcal{G}_n$ *and suppose* $f : V \to \{0, 1\}$ *is an* FM_O *labeling of the vertices. Then* f *is a cordial labeling of* G*. That is if* G *is* FM_O*-labelable then* G *is cordial.*

Proof. Let $f : V \to \{0, 1\}$ be an FM_O labeling of the vertices. Then there are an even number of neighbors of each vertex. Let $2k_i$ be the number of neighbors of v_i. Then, there are k_i neighbors of v_i labeled 0 and k_i labeled 1. Thus the number of edges incident to v_i that are labeled 0 (or 1) is k_i. Thus, the number of edges in G labeled 0 is $\frac{1}{2}\sum_{i=1}^{n} k_i$ and the number of edges in G labeled 1 is $\frac{1}{2}\sum_{i=1}^{n} k_i$. That is, the induced labeling of the edges is friendly, and hence f is a cordial labeling of the vertices of G. ■

The converse of the above theorem is not true, The five vertex wheel graph is cordial but all vertices except the central one has odd degree, so it is not FM_O-labelable. With the added hypothesis that all vertices are of even degree, the Friendship graph on nine vertices is an example of a cordial graph that is not FM_O-labelable. Even with additional hypotheses about the degrees of the vertices, the nember of vertices and the number of edges all being even, the converse is not necessarily true, see Figure 8. The modified $\mathcal{F}_4$ has an even number of vertices, an even number of edges, and every vertex is of even degree.

An obvious question at this point is: Is there a graph G, an orientation $\overrightarrow{G}$, of G, and a vertex labeling $f : V(G) \to \{0, 1\}$ such that f is an FM_O-labeling and f induces a $(2,3)$-cordial labeling of $\overrightarrow{G}$? Actually, since a FM_O-labeling is also a cordial labeling, we only need check the graphs in the appendix to see that we only need to check the 7-cycle and the graph on 5 vertices with degree sequence (4, 4, 2, 2, 2). Neither of these are FM_O-labelable.

We now define a generalization of the Petersen Graph:

Definition 4.7. *Let* $\mathcal{P}_{(n,r)}$ *denote the* (n, r)*-Petersen graph. It has* $2n$ *vertices* $\{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\}$*, (it is in* $\mathcal{G}_{2n}$*) and consists of*

- *an outer cycle: the* n*-cycle* $(v_1, v_2, \ldots, v_n)$*;*
- *an inner star: the edges* $(v'_1, v'_{r+1}), (v'_2, v'_{r+2}), (v'_3, v'_{r+3}), \ldots,$ (v'_n, v'_{r+n})*. where all subscript arithmetic is modulo* n *(some of these may be duplicated);*

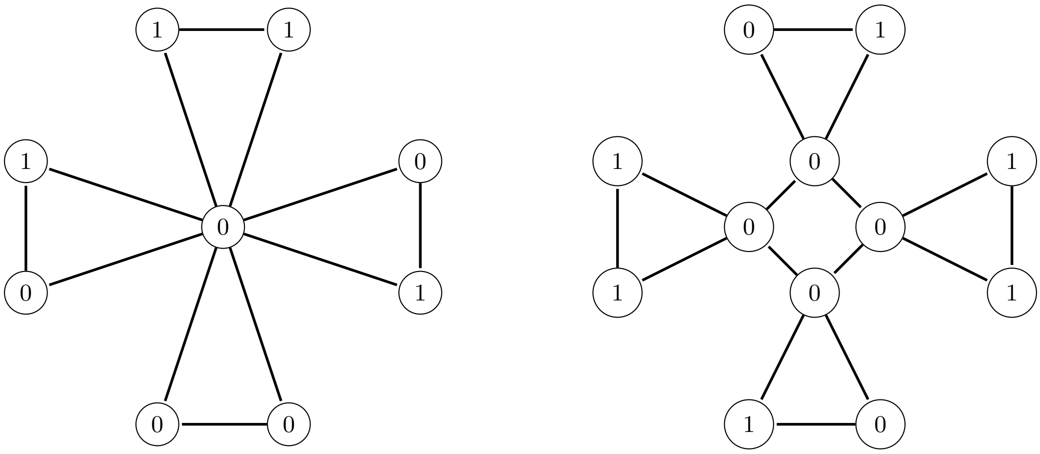


Figure 8. $\mathcal{F}_4$ **and modified** $\mathcal{F}_4$ **are cordial**

- *and the ray edges:* $(v_1, v'_1), (v_2, v'_2), \ldots (v_n, v'_n)$.

The Petersen Graph is $\mathcal{P}_{(5,2)}$. Figure 9 shows $\mathcal{P}_{(8,3)}$, Figure 10 shows $\mathcal{P}_{(6,2)}$, and Figure 11 shows a $\mathcal{P}_{(6,3)}$, graphs illustrating the possible shapes of the inner star graph when n and r are or are not relatively prime.

Fact. There is a vertex labeling f of $\mathcal{P}_{(2k,r)}$ for every $k > 3$ and $r < 2k$ such that r and $2k$ are relatively prime which is an FM_C labeling and which induces a balanced $(2,3)$-cordial labeling of some orientation of $\mathcal{P}_{(2k,r)}$. *Proof.* By labeling the outer and inner vertices 0 and 1 alternately so that the ray edges have the same labeling of their incident vertices, the k "rays" are all labeled 0 while the other $2k$ arcs are labeled ± 1. See Figure 12. ■

5. Conclusions

Above we have listed those graphs that have a cordial labeling that is also a $(2,3)$-orientable labeling of that graph. It would be interesting to investigate the relationships between the Freyberg-Marr labelings and graceful and harmonious labelings of various classes of graphs and the relationships between the binary labelings not investigated above. In Theorem 4.6, we have shown that every FM_O labeling of a graph is a cordial labeling, and it is easily shown that within the set of tree graphs, any graceful labeling of the tree induces a cordial labeling of that tree. (Replace the odd vertex labels with 1 and the even vertex labels with 0.) (All trees have a cordial labeling [2], however, it is conjectured but still unknown if all trees are graceful.)

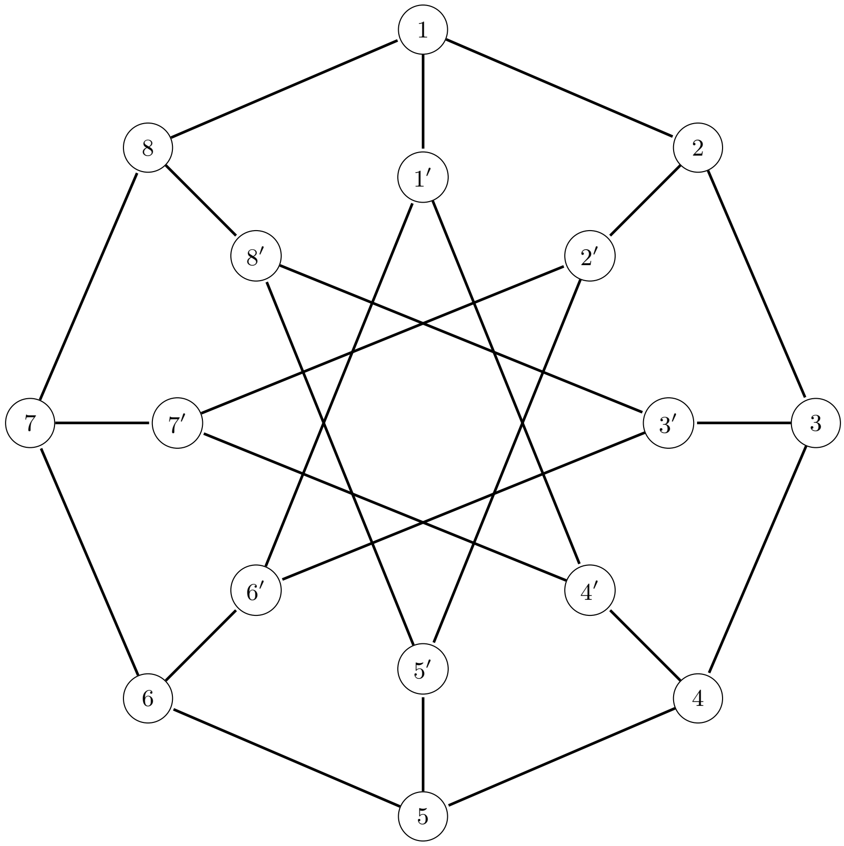


Figure 9. $\mathcal{P}_{(8,3)}$

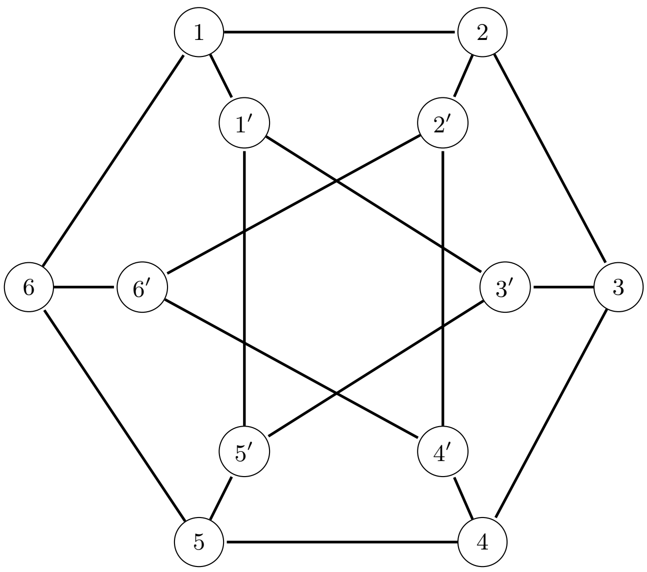


Figure 10. **A $PS_{(6,2)}$ graph.**

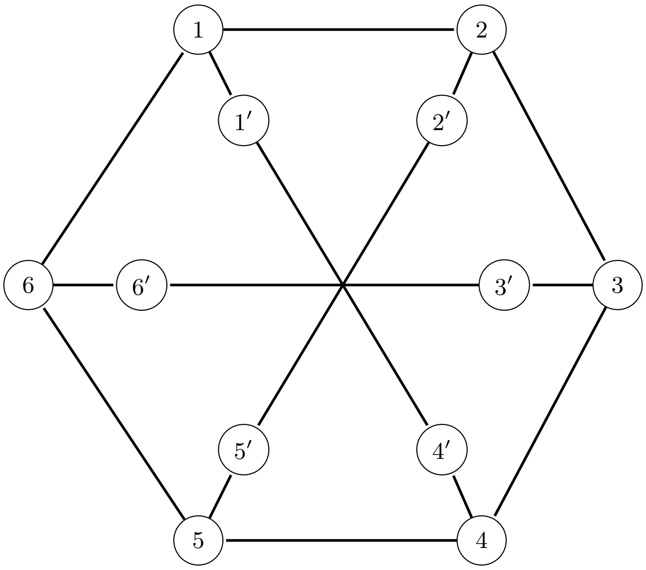


Figure 11. **A $PS_{(6,3)}$ graph.**

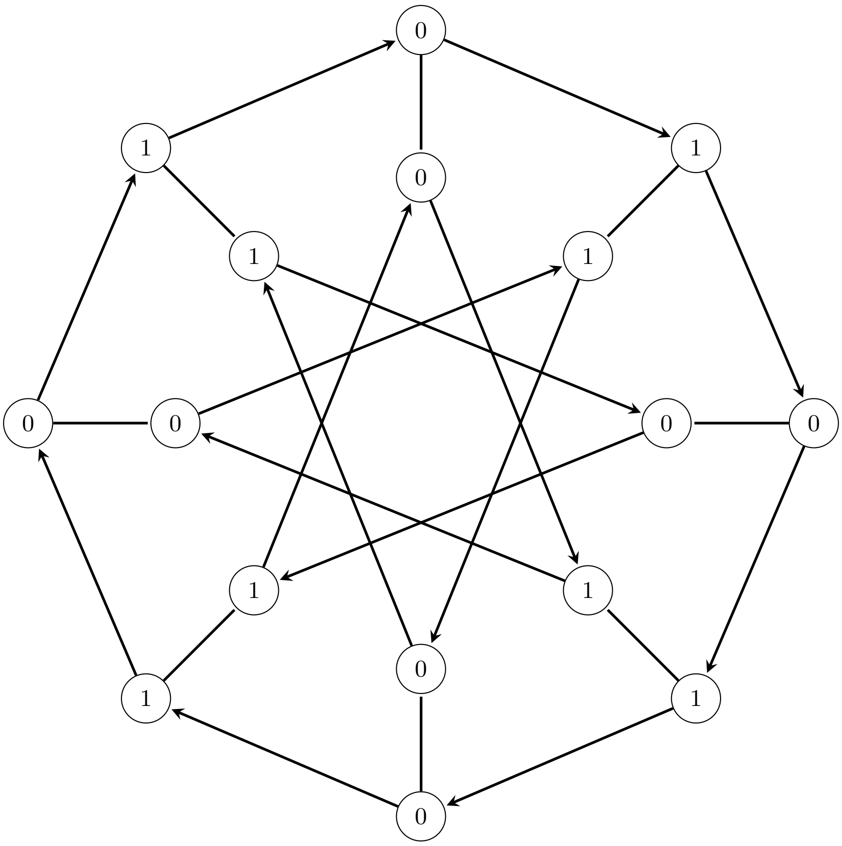


Figure 12. **An oriented $\mathcal{P}_{(8,3)}$ with a vertex labeling which is both an FM$_C$ labeling and which induces a $(2,3)$-cordial labeling. Note: The "ray" edges may be oriented either way.**

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