J. Algebra Comb. Discrete Appl.
12(2) • 153–162

Received: 9 November 2023
Accepted: 13 June 2024

Journal of Algebra Combinatorics Discrete Structures and Applications

On submodule spectrum in multiplication le-modules*

Research Article

Sachin Ballal, Mayur Kshirsagar, Sadashiv Puranik

Abstract: In this article, we have studied the Zariski topology related to a submodule element of a le-module. Obtained a base for the complement of the submodule spectrum and topological features, along with some characterizations of the radical of a submodule element, are established. Several algebraic conditions are obtained for an open subset concerning the Zariski topology to become compact, dense, Noetherian, etc.

2020 MSC: 06E10, 06E99, 06F99,06B23, 06F25

Keywords: Prime submodule element, Radical element, Zariski topology, Complete lattices, le-modules

1. Introduction

A novel approach to deal with abstract submodule theory has been introduced by Kumbhakar and Bhuniya [4, 7, 8] as a le-module theory, which has some overlap with many other mathematical ideas which deal with lattices.

In [3], Behboodi and Haddadi introduced and studied those modules whose classical Zariski topology is respectively T_1, Hausdorff or cofinite and several characterizations of such modules are obtained.

In [4], Kumbhakar and Bhuniya discussed algebraic properties of le-module and its relation with the Zariski topology. They have introduced the concepts of pseudo-prime submodule element and pseudo-prime spectrum and obtained equivalent characterizations of pseudo-prime spectrum to be irreducible.

Kumbhakar and Bhuniya [8] introduced and studied the prime spectrum, $Spec(M)$ and extended some important results from modules to le-modules. Also, characterized the connectedness of $Spec(M)$ in terms of quotient ring without idempotent elements other than 0 and 1.

* *This research work is an outcome of the project supported by the Institute of Eminence (UoH-I0E-RC5-22-021), University of Hyderabad.*
Sachin Ballal (Corresponding Author); School of Mathematics and Statistics, University of Hyderabad, Hyderabad-500 046, India (email: sachinballal@uohyd.ac.in).
Mayur Kshirsagar, Sadashiv Puranik; Department of Mathematics, Savitribai Phule Pune University, Pune-411 007, India (email: kshirsagarmb@gmail.com, srpuranik28@gmail.com).

https://doi.org/10.13069/jacodesmath.v12i2.309 ISSN 2148-838X

Several papers have recently addressed the idea of prime submodules of a module M and topology on the set of all prime submodules of M, few of them are listed as [5, 6, 9–13]. Also, the analogous study has been done for multiplicative lattices [15, 16], in [1, 2, 14] for lattice modules and in [4, 8] for le-modules, a structure inspired by the study of lattice modules as well as multiplicative lattices. For more details on le-modules, one may refer [7]. Here, we have extended several results of modules over ring from [5, 12, 13] to le-modules.

An *le-semigroup* refers to a structure $(M, +, \leq, e)$, where $(M, \leq)$ is a complete lattice and $(M, +)$ is a commutative monoid with the zero element 0_M. For all $m, m_i \in M$ and $i \in I$, it satisfies $m \leq e$ and the condition:

$$m + \left(\bigvee_{i \in I} m_i\right) = \bigvee_{i \in I} (m + m_i).$$

Let R be a commutative ring with unity 1_R and $(M, +, \leq, e)$ be an *le-semigroup*. If there is a mapping $. : R \times M \to M$ which satisfies the following conditions, then ${}_RM$ is called an *le-module* over R.

$$\begin{aligned}
&(M1) \quad r(m_1 + m_2) = rm_1 + rm_2,\\
&(M2) \quad (r_1 + r_2)m \leq r_1 m + r_2 m,\\
&(M3) \quad (r_1 r_2)m = r_1(r_2 m),\\
&(M4) \quad 1_R m = m; \quad 0_R m = r0_M = 0_M,\\
&(M5) \quad r\left(\bigvee_{i \in I} m_i\right) = \bigvee_{i \in I} (rm_i),
\end{aligned}$$

for all $r, r_1, r_2 \in R$, $m, m_i \in M$, and $i \in I$.

An element n of an *le-module* ${}_RM$ is called a *submodule element* if $n + n \leq n$ and $rn \leq n$, for all $r \in R$. The set of all submodule elements of ${}_RM$ is denoted by $\mathrm{Sub}(M)$. Observe that $0_M = 0_R n \leq n$, for every $n \in \mathrm{Sub}(M)$.

A proper element $m \in Sub(M)$ is said to be prime if for $r \in R$ and $x \in_M$, $rx \leq m$ implies $x \leq m$ or $re \leq m$, i.e. $r \in (m : e)$, where $(m : e) = \{r \in R : re \leq m\}$. We denote the set of all prime elements of ${}_RM$ by $\mathrm{Spec}(M)$.

For an ideal I of R, we denote $Ie = \bigvee \{\sum_{i=1}^{r} a_i e : a_i \in I\}$. Also, the radical of I is denoted by $\mathrm{Rad}(I)$ and is defined as $\mathrm{Rad}(I) = \{a \in R : a^n \in I$ for some positive integer $n\}$. For $n \in \mathrm{Sub}(M)$, as $(n : e)$ is an ideal, we set $\mathrm{Rad}(n) = \mathrm{Rad}(n : e)$.

If every $n \in \mathrm{Sub}(M)$ can be expressed as $n = Ie$, for some ideal I of R, then the *le-module* ${}_RM$ is called a *multiplication le-module*.

We use the following definition: A topological space X is irreducible if $X \neq \emptyset$ and the intersection of any two non-empty open sets in X is always non-empty.

2. Topology associated with submodule element

In [8], Kumbhakar and Bhuniya introduced and studied Zariski topology on $Spec(M)$ of an le-module ${}_RM$. They have defined $V^*(n) = \{p \in \mathrm{Spec}(M) | (n : e) \subseteq (p : e)\}$ for $n \in Sub(M)$ and showed that the collection $V^*(M) = \{V^*(n) : n \in \mathrm{Sub}(M)\}$ forms a topology on ${}_RM$ called the Zariski topology and is denoted by $\mathfrak{T}^*(M)$. Also, for $n \in \mathrm{Sub}(M)$, $V(n)$ is defined as $V(n) = \{p \in \mathrm{Spec}(M) | n \leq p\}$ and showed that the collection $V(M) = \{V(n) : n \in \mathrm{Sub}(M)\}$ forms the Zariski topology, denoted by $\mathfrak{T}(M)$ if and only if the finite union of subsets of $V(M)$ is closed.

Essentially, we need the following results to show that the finite union of subsets of $V(M)$ is closed.

For $n \in \mathrm{Sub}(M)$, we denote $\chi_n = \mathrm{Spec}(M) \backslash V(n)$ and for $l \in \mathrm{Sub}(M)$, we notice that $V^*(l) = V(l) \backslash V(n)$.

Theorem 2.1 ([8]). *Let $_RM$ be an le-module. Then for $n_1, n_2, n_i \in$Sub(M) and $i \in S$, an indexed set.*

1. $V(0_M) = Spec(M)$
2. $V(e) = \emptyset$
3. $\bigcap_{i\in S} V(n_i) = V\left(\sum_{i\in S} n_i\right)$
4. $V(n_1) \cup V(n_2) \subseteq V(n_1 \wedge n_2)$.

Theorem 2.2 ([7]). *Let $_RM$ be an le-module. If $\{n_i\}_{i\in S} \subseteq Sub(M)$, then $(\wedge_{i\in S} n_i : e) = \cap_{i\in S}(n_i : e)$.*

The following result characterises multiplication le-modules in terms of submodule elements.

Lemma 2.3. *Let $_RM$ be an le-module. Then $_RM$ is a multiplication le-module if and only if $n = (n : e)e$ for every $n \in Sub(M)$.*

Proof. Suppose that $_RM$ is a multiplication le-module and $n \in Sub(M)$. Then $n = Ie$, for some ideal I of R. For $1 \le i \le r$, where r is a positive integer, let $x_i \in (n : e)$. Then $\sum_{i=1}^{r} x_i e \le n+n+\cdots+n = rn \le n$. Therefore $(n : e)e \le n$. Now, as $n = Ie$, we have $I \subseteq (n : e)$ and therefore $n = Ie \le (n : e)e$. Consequntly, $(n : e)e = n$.

The converse is obvious. □

Theorem 2.4 ([7]). *Let $_RM$ be a multiplication le-module. If $p \in$Spec(M), then $(p : e)$ is a prime ideal of R.*

The following result shows that, the converse of Theorem 2.4 is true if $_RM$ is a multiplication le-module.

Theorem 2.5. *Let $_RM$ be a multiplication le-module and $p \in {_RM}$. Then $p \in$Spec(M) if and only if $(p : e)$ is a prime ideal of R.*

Proof. If part follows from Theorem 2.4.

Conversely, suppose that $(p : e)$ is a prime ideal of R and $p \notin$ Spec(M). Then there exist $r \in R$ and $x \in M$ such that $x \not\le p$ and $r \notin (p : e)$. Now consider $n = Rx$. Then $n \in$ Sub(M) and $x \le n$. Therefore, there exist $r \in R$ and $n \in$ Sub(M) such that $rn \le p$, but $n \not\le p$ and $r \notin (p : e)$.

Now, since $n \not\le p$, by Lemma 2.3, we have $(n : e)e \not\le (p : e)e$. This implies $(n : e) \not\subseteq (p : e)$. Therefore, there exists $r_1 \in R$ such that $r_1 \in (n : e)$ but $r_1 \notin (p : e)$.

Thus $rr_1e \le rn \le p$, i.e., $rr_1 \in (p : e)$, a contradiction to $r, r_1 \notin (p : e)$ and to the hypothesis of the ideal $(p : e)$ to be prime. □

Theorem 2.6. *Let $_RM$ be a multiplication le-module. Then $V(n_1) \cup V(n_2) = V(n_1 \wedge n_2)$ for $n_1, n_2 \in$ Sub(M).*

Proof. By Theorem 2.1(4), we have $V(n_1) \cup V(n_2) \subseteq V(n_1 \wedge n_2)$. Now, let $p \in V(n_1 \wedge n_2)$. This implies $n_1 \wedge n_2 \le p$. Therefore $(n_1 \wedge n_2 : e) \subseteq (p : e)$. (1)

By Theorem 2.2, we have $(n_1 \wedge n_2 : e) = (n_1 : e) \cap (n_2 : e)$ and therefore equation (1) reduces to $(n_1 : e) \cap (n_2 : e) \subseteq (p : e)$. Again, since $(p : e)$ is a prime ideal, by Theorem 2.4, we have, $(n_1 : e) \subseteq (p : e)$ or $(n_2 : e) \subseteq (p : e)$.

Therefore $(n_1 : e)e \le (p : e)e$ or $(n_2 : e)e \le (p : e)e$. Now as $n_1, n_2 \in$ Sub(M), by Lemma 2.3, we have $n_1 = (n_1 : e)e \le (p : e)e = p$ or $n_2 = (n_2 : e)e \le (p : e)e = p$. Hence $p \in V(n_1) \cup V(n_2)$.

Consequently, $V(n_1) \cup V(n_2) = V(n_1 \wedge n_2)$. □

Lemma 2.7 ([8]). *Let $_RM$ be an le-module. Then $V(Ie) \cup V(Je) = V(IJe)$ for ideals I and J of R.*

Theorem 2.8 ([8]). *Let $_RM$ be a multiplication le-module and $n_i \in Sub(M)$, $i \in S$. Then*

1. $V^*(0_M) = \chi_n$
2. $V^*(n) = \emptyset$
3. $\bigcap_{i \in S} V^*(n_i) = V^*\left(\sum_{i \in S} n_i\right)$
4. $V^*(n_1) \cup V^*(n_2) = V^*(IJe) = V^*(n_1 \wedge n_2)$, *where* $n_1 = Ie$ *and* $n_2 = Je$.

From Theorem 2.8, we conclude that for multiplication le-module $_RM$, there exists a topology, denoted by $\mathfrak{T}_{\mathfrak{n}}{}^*(M)$, such that the family of its closed sets is $V^*(M)$.

Theorem 2.9. *Let $_RM$ be a multiplication le-module and I be an ideal of R. Then $V(Ie) = \bigcap_{a \in I} V((a)e)$, where (a) denotes an ideal genereted by $a \in I$.*

Proof. Let $p \in V(Ie)$. Therefore $Ie \leq p$ and which implies $(a)e \leq p$ for all $a \in I$. Thus, $p \in \bigcap_{a \in I} V((a)e)$. Now, suppose that $p \in \bigcap_{a \in I} V((a)e)$. This implies $(a)e \leq p$ for all $a \in I$. Therefore by the definition of Ie, we have $Ie \leq p$ and hence $p \in V(Ie)$. Consequently, $V(Ie) = \bigcap_{a \in I} V((a)e)$. □

Corollary 2.10. *Let $_RM$ be a multiplication le-module and I be an ideal in R. Then $V^*(Ie) = \bigcap_{a \in I} V^*((a)e)$, where (a) denotes an ideal genereted by $a \in I$.*

Theorem 2.11. *Let $_RM$ be a multiplication le-module and $n = Ie \in Sub(M)$, where I is an ideal of R. If $(\chi_n)^r = \chi_n - V^*((r)e)$, where $r \in R$, then $\mathcal{B} = \{(\chi_n)^r : r \in R\}$ forms a basis for the topology $\mathfrak{T}_{\mathfrak{n}}^*(M)$ on χ_n.*

Proof. If $\chi_n = \emptyset$, then $(\chi_n)^r = \emptyset$ and the result holds trivialy.

Now, suppose that $\chi_n \neq \emptyset$. Let U be an open set in χ_n. Therefore $U = \chi_n - V^*(l)$ for some $l \in Sub(M)$ and hence $U = \chi_n - V^*(Je)$, where $l = Je$ for some ideal J of R. Then $U = \chi_n - \bigcap_{a_i \in J} V^*((a_i)e)$ by Lemma 2.9 and hence $\bigcup_{a_i \in J} (\chi_n - V^*((a_i)e)) = \bigcup_{a_i \in J} (\chi_n)^{a_i}$

Consequently, $\mathcal{B} = \{(\chi_n)^r : r \in R\}$ forms a basis for the topology $\mathfrak{T}_{\mathfrak{n}}^*(M)$. □

Theorem 2.12. *Let $_RM$ be a multiplication le-module and $n = Ie \in Sub(M)$, for some ideal I of R. Then*

$$(\chi_n)^l = Spec(M) - V(IJe), \text{ where } l = Je \in Sub(M).$$

Proof.
$$\begin{aligned}(\chi_n)^l = \chi_n - V^*(l) = \chi_n - V^*(Je) &= (\mathrm{Spec}(M) - V(Ie)) - (V(Je) - V(Ie)) \\ &= \mathrm{Spec}(M) - (V(Ie) \cup V(Je)) = \mathrm{Spec}(M) - V(IJe).\end{aligned}$$
□

The following result shows that the finite intersection of basis elements is again a basis element.

Theorem 2.13. *Let $_RM$ be a multiplication le-module. Then for ideals J_1, J_2 of R, $(\chi_n)^{J_1e} \bigcap (\chi_n)^{J_2e} = (\chi_n)^{J_1J_2e}$.*

Proof. Let J_1, J_2 be two ideals of R. Then $(\chi_n)^{J_1e} \bigcap (\chi_n)^{J_2e} = (\chi_n - V^*(J_1e))$ $\bigcap (\chi_n - V^*(J_2e)) = (\chi_n) - (V^*(J_1e) \bigcup V^*(J_2e)) = \chi_n - (V^*(J_1J_2e)) = (\chi_n)^{J_1J_2e}$.
Consequently, $(\chi_n)^{J_1e} \bigcap (\chi_n)^{J_2e} = (\chi_n)^{J_1J_2e}$. □

The following result characterizes the radical of a submodule element in a multiplication le-module.

Theorem 2.14. *Let ${}_RM$ be a multiplication le-module and $n \in Sub(M)$. Then*

$$Rad(n) = \bigcap\{(p : e) | p \in Spec(M), p \geq n\}$$

.

Proof. Let $r \in \mathrm{Rad}(n)$. This implies $re \leq p$ for all $p \in \mathrm{Spec}(M)$ with $p \geq n$.

Therefore, $r \in (p : e)$. Since $p \in \mathrm{Spec}(M)$, by Theorem 2.4, we have $(p : e)$ is a prime ideal and which implies $r \in (p : e)$. Therefore, $\mathrm{Rad}(n) \subseteq \bigcap\{(p : e)|p \in Spec(M), p \geq n\}$.

Now, suppose that $r \notin \mathrm{Rad}(n)$.

Let $\sum = \{(x : e)|x \in Sub(M), x \geq n \text{ and } r^k \notin (x : e), k \geq 1\}$. Note that $\sum \neq \emptyset$, since $(n : e) \in \sum$. By Zorn's lemma, there exists maximal element, say $(q : e) \in \sum$. We now prove that the ideal $(q : e)$ is prime ideal of R. On contrary, suppose that $(q : e)$ is not a prime ideal of R. Therefore their exist $x, y \notin (q : e)$ but $xy \in (q : e)$.

Consider $I = \{z \in R|xz \in (q : e)\}$. Note that, I is an ideal of R. If $a \in (q : e)$, then $xa \in (q : e)$ and this implies $a \in I$. Also $y \notin (q : e)$, but $xy \in (q : e)$ and hence $y \in I$. Therefore, $(q : e) \subsetneq I$. Again, $n \leq q = (q : e)e < Ie$ and we have $I = (Ie : e)$. Therefore, by the maximality of $(q : e), r^s \in I$ for some $s \geq 1$.

Now, let $J = \{z \in R|r^s z \in (q : e)\}$. Note that, J is an ideal and $(q : e) \subsetneq J$ since $x \in J$ and $x \notin (q : e)$. Therefore by the maximality of $(q : e)$, $r^l \in J$ for some $l \geq 1$. Therefore by the definition of an ideal J, we have $r^{s+l} \in (q : e)$, a contradiction. Thus the $(q : e)$ is prime ideal and by Theorem 2.5, $q \in \mathrm{Spec}(M)$. Consequently, $Rad(n) = \bigcap\{(p : e)|p \in Spec(M), p \geq n\}$. □

Corollary 2.15. *Let ${}_RM$ be a multiplication le-module. Then* $Rad(0_M) = \bigcap_{p \in Spec(M)} (p : e)$.

Theorem 2.16. *Let ${}_RM$ be a multiplication le-module and $n, l \in Sub(M)$. Then $(\chi_n)^l = \emptyset$ if and only if $Rad(IJe) \subseteq Rad(0_M)$, where $n = Ie, l = Je$ and I, J are ideals of R.*

Proof. Suppose that $(\chi_n)^l = \emptyset$ for $l = Je \in Sub(M)$ and fix $r \in \mathrm{Rad}(IJe) = \mathrm{Rad}(IJe : e)$. This implies

$$r^k e \leq IJe \text{ for some } k \geq 1. \qquad (1)$$

Since $(\chi_n)^l = \emptyset$, we have, $(\mathrm{Spec}(M) - V(n)) - V^*(l) = \emptyset$. Therefore $(\mathrm{Spec}(M) - (V(l) \cup V(n)) = \emptyset$, i.e., $\mathrm{Spec}(M) = V(l) \cup V(n)$. Thus $p \in V(Ie) \cup V(Je) = V(IJe)$, for all $p \in \mathrm{Spec}(M)$ and which implies, $IJe \leq p$, for all $p \in \mathrm{Spec}(M)$.
Therefore the inequality (1) reduces to, $r^k e \leq IJe \leq p$, i.e., $r^k \in (p : e)$, for all $p \in \mathrm{Spec}(M)$. Now, since $(p : e)$ is a prime ideal, then by Theorem 2.14, we have $r \in \bigcap_{p \in \mathrm{Spec}(M)} (p : e) = \mathrm{Rad}\ (0_M)$. Therefore $\mathrm{Rad}(IJe) \subseteq \mathrm{Rad}(0_M)$.

Conversely, suppose that, $\mathrm{Rad}(IJe) \subseteq \mathrm{Rad}(0_M)$ and $(\chi_n)^l \neq \emptyset$. Let $p \in (\chi_n)^l = (\chi_{Ie})^{Je} = \mathrm{Spec}(M) - V(IJe)$. This implies $p \notin V(IJe)$, i.e., $IJe \not\leq p$ and therefore $IJ \not\subseteq (p : e)$.
Then there exists $r \in IJ$ but $r \notin (p : e)$. As $r \in IJ$, we have, $re \leq IJe$.
Hence $r \in (IJe : e) \subseteq \mathrm{Rad}(IJe : e) = Rad(IJe)$. But as $r \notin (p : e)$, we have,

$$r \notin \bigcap_{p \in \mathrm{Spec}(0_M)} (p : e) = \mathrm{Rad}(0_M),$$

a contradiction to the fact that, $\mathrm{Rad}(IJe) \subseteq \mathrm{Rad}(0_M)$. □

Theorem 2.17. *Let $_RM$ be a multiplication le-module and $l_1, l_2, n \in Sub(M)$. Then the following are equivalent:*

1. $V(l_1) = V(l_2)$
2. $(\chi_n)^{l_1} = (\chi_n)^{l_2}$
3. $Rad(l_1) = Rad(l_2)$.

Proof. (1) $\Rightarrow$ (2) Suppose that $V(l_1) = V(l_2)$. This implies $V(l_1) - V(n) = V(l_2) - V(n)$, i.e., $V^*(l_1) = V^*(l_2)$. Therefore $(\chi_n)^{l_1} = \chi_n - V^*(l_1) = \chi_n - V^*(l_2) = (\chi_n)^{l_2}$.
(2) $\Rightarrow$ (3) Suppose that $(\chi_n)^{l_1} = (\chi_n)^{l_2}$. This implies $\chi_n - V^*(l_1) = \chi_n - V^*(l_2)$.
Thus $(Spec(M) - V(n)) - (V(l_1) - V(n)) = (Spec(M) - V(n)) - (V(l_2) - V(n))$.
Therefore $Spec(M) - (V(l_1) \cup V(n)) = Spec(M) - (V(l_2) \cup V(n))$ and which implies $V(l_1) = V(l_2)$.

$$\text{Now, } \mathrm{Rad}(l_1) = \bigcap_{l_1 \leq p \in \mathrm{Spec}(M)} (p : e), \text{ by Theorem 2.14}$$
$$= \bigcap_{p \in V(l_1)} (p : e) \quad = \bigcap_{p \in V(l_2)} (p : e) = \mathrm{Rad}(l_2).$$

(3) $\Rightarrow$ (1) Suppose that $\mathrm{Rad}(l_1) = \mathrm{Rad}(l_2)$. If $p \in V(l_1)$, then $l_1 \leq p$ and hence $(l_1 : e) \subseteq (p : e)$. Therefore, $\mathrm{Rad}(l_1 : e) \subseteq \mathrm{Rad}(p : e) = (p : e)$, but $\mathrm{Rad}(l_1) = \mathrm{Rad}(l_2)$.
Thus $\mathrm{Rad}(l_2 : e) = \mathrm{Rad}(l_1 : e) \subseteq (p : e)$ and which implies $(l_2 : e) \subseteq \mathrm{Rad}(l_2 : e) \subseteq (p : e)$.
Therefore $l_2 = (l_2 : e)e \leq p = (p : e)e$, hence $p \in V(l_2)$. Thus $V(l_1) \subseteq V(l_2)$.

Similarly, $V(l_2) \subseteq V(l_1)$ and consequently, $V(l_1) = V(l_2)$. □

Corollary 2.18. *Let $_RM$ be a multiplication le-module and $l_1, l_2 \in {_RM}$. If $(l_1 : e) = (l_2 : e)$, then $(\chi_n)^{l_1} = (\chi_n)^{l_2}$. The converse is true if $l_1, l_2 \in Spec(M)$.*

Proof. Suppose that $(l_1 : e) = (l_2 : e)$. By Lemma 2.13, we have $l_1 = (l_1 : e)e = (l_2 : e)e = l_2$ and hence $V(l_1) = V(l_2)$. Therefore by Theorem 2.17, we have $(\chi_n)^{l_1} = (\chi_n)^{l_2}$.
Conversely, let $l_1, l_2 \in \mathrm{Spec}(M)$ and $(\chi_n)^{l_1} = (\chi_n)^{l_2}$. Therefore by Theorem 2.17, $\mathrm{Rad}(l_1) = \mathrm{Rad}(l_2)$. Since $(l_1 : e), (l_2 : e)$ are prime ideals of R and therefore, $(l_1 : e) = \mathrm{Rad}(l_1 : e) = \mathrm{Rad}(l_1) = \mathrm{Rad}(l_2) = \mathrm{Rad}(l_2 : e) = (l_2 : e)$. □

Theorem 2.19. *Let $_RM$ be a multiplication le-module. Then $(\chi_n)^l = \chi_n$ if and only if $Rad(IJe) = Rad(Ie)$, where $l = Je, n = Ie \in Sub(M)$.*

Proof. Note that, $(\chi_n)^l = (\chi_n)$ if and only if $Spec(M) - (V(n) \cup V(l)) = Spec(M) - V(n)$, if and only if $V(n) \cup V(l) = V(n)$, i.e., $V(Ie) \cup V(Je) = V(IJe) = V(Ie)$. Therefore, $(\chi_n)^l = (\chi_n)$ if and only if $V(IJe) = V(Ie)$. Now, by Theorem 2.17, we have, $V(IJe) = V(Ie)$ if and only if $\mathrm{Rad}(IJe) = \mathrm{Rad}(Ie)$. □

Definition 2.20. *Let $_RM$ be an le-module and $n, m_i \in Sub(M), i \in \Lambda$. We say that n satisfies condition $(*)$ if there is a finite subset Δ of Λ such that $Rad\left(\sum_{i \in \Lambda} m_i\right) = Rad\left(\sum_{j \in \Delta} m_j\right)$, whenever $Rad(n) \subseteq Rad\left(\sum_{i \in \Lambda} m_i\right)$.*

Theorem 2.21. *Let $_RM$ be an le-module and $n, m_i \in Sub(M), i \in \Lambda$. Then the following statements hold:*

1. *$(\chi_n)^l$ is compact for every $l \in Sub(M)$.*

2. *If n satisfies condition $(*)$, then χ_n is compact.*

3. *If χ_n is compact, then $Rad(n) \subseteq Rad\left(\sum_{i\in\Delta} m_i\right)$ for a finite subset Δ of Λ.*

Proof. 1. For $l \in Sub(M)$, $(\chi_n)^l$ being a basis element for the topology $\mathfrak{T}_n^*(M)$, it is compact.

2. Suppose that, $n \in \mathrm{Sub}(M)$ satisfies condition $(*)$. Let $\chi_n = \bigcup_{i\in\Lambda} (\chi_n)^{m_i}$. Therefore, $\chi_n = \bigcup_{i\in\Lambda} (\chi_n)^{m_i} = \bigcup_{i\in\Lambda} (\chi_n - V^*(m_i)) = \chi_n - \left(\bigcap_{i\in\Lambda} V^*(m_i)\right) = \chi_n - V^*(\sum_{i\in\Delta} m_i)$.

   Thus, we have, $V(V^*(\sum_{i\in\Delta} m_i) \subseteq V(n)$ and this implies $\mathrm{Rad}(n) \subseteq \mathrm{Rad}\left(\sum_{i\in\Lambda} m_i\right)$.

   But by the condition $(*)$, we have, $\mathrm{Rad}\left(\sum_{i\in\Lambda} m_i\right) = \mathrm{Rad}\left(\sum_{i\in\Delta} m_i\right)$ for some finite subset Δ of Λ.

   Therefore, $\mathrm{Rad}(n) \subseteq \mathrm{Rad}\left(\sum_{i\in\Delta} m_i\right)$ and this implies $V\left(\sum_{i\in\Delta} m_i\right) \subseteq V(n)$.

   Hence $V^*\left(\sum_{i\in\Delta} m_i\right) = V\left(\sum_{i\in\Delta} m_i\right) - V(n) = \emptyset$.

   Thus $\chi_n = \chi_n - V^*\left(\sum_{i\in\Delta} m_i\right) = \chi_n - \left(\bigcap_{i\in\Delta} V^*(m_i)\right) = \left(\bigcup_{i\in\Delta} (\chi_n)^{m_i}\right)$ and consequently, χ_n is compact.

3. Suppose that χ_n is compact and $n = \left(\sum_{i\in\Lambda} m_i\right)$. Then $V(n) = V\left(\sum_{i\in\Lambda} m_i\right)$.

   Hence $V^*\left(\sum_{i\in\Lambda} m_i\right) = V\left(\sum_{i\in\Lambda} m_i\right) - V(n) = \emptyset$. Therefore $\chi_n = \chi_n - \emptyset = \chi_n - V^*\left(\sum_{i\in\Lambda} m_i\right) = \chi_n - \bigcap_{i\in\Lambda} V^*(m_i) = \bigcup_{i\in\Lambda} (\chi_n - V^*(m_i))$. But χ_n is compact, therefore there exists a finite subset $\Delta \subset \Lambda$ such that $\chi_n = \bigcup_{i\in\Delta} (\chi_n - V^*(m_i)) = \bigcup_{i\in\Delta} (\chi_n)^{m_i}$. Therefore, $\chi_n = \bigcup_{i\in\Delta} (\chi_n - V^*(m_i)) = \chi_n - \bigcap_{i\in\Delta} V^*(m_i) = \chi_n - V^*(\sum_{i\in\Delta} m_i)$ and which implies $V\left(\sum_{i\in\Delta} V^*(m_i)\right) \subseteq V(n)$. Consequently, by Theorem 2.14, $\mathrm{Rad}(n) \subseteq \mathrm{Rad}\left(\sum_{i\in\Delta} m_i\right)$.

□

Theorem 2.22. *Let ${}_RM$ be a multiplication le-module. Then the following holds:*

1. *Every open set in $\chi = Spec(M)$ is of the form χ_n for some $n \in Sub(M)$.*
2. *$\chi_n = \chi_l$ if and only if $Rad(n) = Rad(l)$, where $n, l \in Sub(M)$.*
3. *$\chi_n \cap \chi_l = \chi_k$ if and only if $Rad(n \wedge l) = Rad(k)$, where $n, l, k \in Sub(M)$.*

Proof. 1. Let U be an open set in χ. This implies, $U' = \chi - U$ is closed in χ and by definition $U' = V(n)$, for some $n \in \mathrm{Sub}\ (M)$. Therefore, $U = Spec(M) - V(n) = \chi_n$.

2. Suppose that, $\chi_n = \chi_l$. Then $Spec(M) - V(n) = Spec(M) - V(l)$ if and only if $V(n) = V(l)$, therefore $\chi_n = \chi_l$ if and only if $V(n) = V(l)$.
   If $\chi_n = \chi_l$, then $\mathrm{Rad}(n) = \bigcap_{p \in V(n)} (p : e) = \bigcap_{p \in V(l)} (p : e) = \mathrm{Rad}(l)$.
   Now, suppose that, $\mathrm{Rad}(n) = \mathrm{Rad}(l)$ and $p \in V(n)$. This implies $n \leq p$ and therefore $(n : e) \subseteq (p : e)$. Therefore, $\mathrm{Rad}(n) = \mathrm{Rad}(n : e) \subseteq (p : e)$.
   Thus $\mathrm{Rad}(n) = \mathrm{Rad}(l) \subseteq (p : e)$ and which implies that $(l : e) \subseteq (p : e)$. Therefore, by Lemma 2.3, we have, $l = (l : e)e \leq p = (p : e)e$. Thus $p \in V(l)$.
   Similarly, $V(l) \subseteq V(n)$. Consequently, $V(n) = V(l)$, and hence $\chi_n = \chi_l$.

3. For $n, l, k \in \mathrm{Sub}(M)$, $\chi_n \cap \chi_l = \chi_k$ if and only if $(Spec(M) - V(n)) \cap (Spec(M) - V(l)) = Spec(M) - V(k)$ if and only if $Spec(M) - (V(n) \cup V(l)) = Spec(M) - V(k)$ if and only if $V(n) \cup V(l) = V(k)$ if and only if $V(n \wedge l) = V(k)$ if and only if $Rad(n \wedge l) = Rad(k)$.

□

Corollary 2.23. *Let $_RM$ be a multiplication le-module and $n, l \in Sub(M)$. Then $\chi_n \cap \chi_l = \emptyset$ if and only if $Rad(n \wedge l) = Rad(0_M)$.*

Following result characterises the denseness of open set χ_n in $\chi = Spec(M)$.

Theorem 2.24. *Let $_RM$ be a multiplication le-module and $n, l \in Sub(M)$ with $Rad(l) \nsubseteq Rad(0_M)$. Then χ_n is dense in $\chi = Spec(M)$ if and only if $Rad(n \wedge l) \neq Rad(0_M)$.*

Proof. Suppose that, χ_n is dense in $Spec(M)$ and $l \in \mathrm{Sub}(M)$ with $\mathrm{Rad}(l) \nsubseteq \mathrm{Rad}(0_M)$. Therefore by Theorem 2.22(2), we have $\chi_l \nsubseteq \chi_{0_M} = \emptyset$. This implies, χ_l is non-empty, open in the Zariski topology $\mathfrak{T}^*_n(M)$ over $\chi = Spec(M)$. Since χ_n is dense in $Spec(M)$, we have, $\chi_n \cap \chi_l \neq \emptyset$ and therefore, by Corollary 2.23, $\mathrm{Rad}(n \wedge l) \neq Rad(0_M)$.
Conversely, suppose that, $\mathrm{Rad}(n \wedge l) \neq \mathrm{Rad}(0_M)$ for every $l \in \mathrm{Sub}(M)$ with $\mathrm{Rad}(l) \nsubseteq \mathrm{Rad}(0_M)$. By Corollary 2.23, we have, $\chi_n \cap \chi_l \neq \emptyset$ and therefore χ_n is dense in $Spce(M)$. □

Definition 2.25. *Let $_RM$ be a le-module and $n \in Sub(M)$. We define $N_n(k) = \wedge\{p \in Spec(M) : k \leq p, n \nleq p\}$.*

Note that $N_n(k) \in Sub(M)$ and $N_e(k) = Rad(k)$.

Theorem 2.26. *Let $_RM$ be a multiplication le-module and $n \in Sub(M)$. Then $N_n(0_M) \in Spec(M)$ if and only if χ_n is irreducible.*

Proof. Let $N_n(0_M) \in Spec(M)$ and K be any non-empty open subset in χ_n. This implies, $K = \chi_n - V^*(l) = Spec(M) - (V(l) \cup V(n))$ for some $l \in Sub(M)$.
Let $p \in K$. Then $p \notin V(n)$ and $p \notin V(l)$. Hence $p \in \{p_i : p_i \geq 0_M, p_i \ngeq n, p_i \ngeq l\}$. Therefore $l \nleq N_n(0_M)$, otherwise if $l \leq N_n(0_M)$, we have $l \leq p$, a contradiction. Thus $N_n(0_M) \notin V(n) \cup V(l)$ and hence $N_n(0_M) \in K$.
Therefore, any open set of χ_n contains $N_n(0_M)$, hence χ_n is irreducible.

Conversely, Suppose that χ_n is irreducible and $N_n(0_M) \notin Spec$(M). Then there exist $r \in R$ and $m \in M$ with $rm \leq N_n(0_M)$ but $m \nleq N_n(0_M)$ and $re \nleq N_n(0_M)$. Therefore $V^*(Rm) \neq \phi$ and $V^*(Rm) \neq \chi_n$, which implies $(\chi_n)^{Rm} \neq \phi$. Note that $(\chi_n)^{re}$ is a non-empty open subset. Therefore, we have $(\chi_n)^{Rm} \cap (\chi_n)^{re} = (\chi_n)^{Rm \wedge re} \subseteq \chi_n - V^*(rm) \subseteq \chi_n - V^*(N_n(0_M)) = Spec(M) - \{V(N_n(0_M)) \cup - V(N_n(n))\} = \phi$, a contradiction to the irreducibility of χ_n. Consequently, $N_n(0_M) \in Spec(M)$. □

Corollary 2.27. *The meet of all prime submodule elements of $_RM$ is prime if and only if χ_n is irreducible.*

Definition 2.28. *An le-module $_RM$ is said to satisfy condition $\mathcal{T}$ if for every $n \in Sub(M)$ and for any chain $N_n(I_1e) \leq N_n(I_2e) \leq N_n(I_3e) \leq \ldots$, where I_i is an ideal of R, there is integer m such that $N_n(I_me) = N_n(I_{m+i}e)$ for all positive integers i.*

Theorem 2.29. *Let ${}_RM$ be a multiplication le-module and $n \in Sub(M)$. Then the following statements are equivalent:*

1. *${}_RM$ satisfies condition $\mathcal{T}$ for $n \in Sub(M)$.*
2. *χ_n is a Noetherian topological space.*

Proof. (1) $\Rightarrow$ (2) Assume that M satisfies condition $\mathcal{T}$ for $n \in Sub(M)$. Consider the sequence

$$V^*(I_1e) \supseteq V^*(I_2e) \supseteq V^*(I_3e) \supseteq \dots \text{ This implies}$$

$$V(I_1e) - V(n) \supseteq V(I_2e) - V(n) \supseteq V(I_3e) - V(n) \supseteq \dots, \text{ Therefore, we have,}$$

$$\wedge\{p : I_1e \leq p, n \nleq p\} \leq \wedge\{p : I_2e \leq p, n \nleq p\} \leq \wedge\{p : I_3e \leq p, n \nleq p\} \leq \dots$$

Hence by the definition of $N_n(k)$, we have, $N_n(I_1e) \leq N_n(I_2e) \leq N_n(I_3e) \leq \dots$.

Since ${}_RM$ satisfies $\mathcal{T}-$ condition, there exists an integer m such that $N_n(I_me) = N_n(I_{m+i}e)$ for all positive integers i. Let $p \in V^*(I_me)$. This implies $I_me \leq p$, but $n \nleq p$. Therefore, we have, $N_n(I_me) = \wedge\{q \in Spec(M) : I_me \leq q, n \nleq q\} \leq p$.

Since $N_n(I_me) = N_n(I_{m+i}e)$, we have, $N_n(I_{m+1}e) \leq p$ and hence, $I_{m+i}e \leq p$. Therefore, we have, $p \in V^*(I_{m+i}e)$. Consequently, $V^*(I_{m+i}e) = V^*(I_me)$. Hence χ_n is a Noetherian topological space.
(2) $\Rightarrow$ (1) Conversely, suppose χ_n is a Noetherian topological space. Let $N_n(I_1e) \leq N_n(I_2e) \leq N_n(I_3e) \leq \dots$ be a sequence. If $p \in V^*(I_me)$ then $I_me \leq p$ and $n \nleq p$. Therefore, we have, $N_n(I_me) \leq p$ and hence, $N_n(I_{m-1}e) \leq p$. This implies $I_{m-1}e \leq p$ and $n \nleq p$. Therefore , we have, $p \in V(I_{m-1}e)$. Therefore, from above sequence we get,

$$V^*(I_1e) \supseteq V^*(I_2e) \supseteq V^*(I_3e) \supseteq \dots$$

Since χ_n is Noetherian there exists an integer m such that $V^*(I_{m+i}e) = V^*(I_me)$ for all positive integers i.

Therefore, we have, $\{q \in Spec(M) : I_{m+i}e \leq q, n \nleq q\} = \{q \in Spec(M) : I_me \leq q, n \nleq q\}$. This implies, $N_n(I_{m+i}e) = \wedge\{q \in Spec(M) : I_{m+i}e \leq q, n \nleq q\} = \wedge\{q \in Spec(M) : I_me \leq q, n \nleq q\} = N_n(I_me)$ for all positive integers i. Therefore M satisfies condition $\mathcal{T}$. $\square$

References

[1] S. Ballal, V. Kharat, Zariski topology on lattice modules, Asian Eur. J. Math. 8(4) (2015) 1550066.
[2] S. Ballal, V. Kharat, On Minimal Spectrum of multiplication lattice modules, Mathematica Bohemica, 144(1) (2019) 85–97.
[3] M. Behboodi, M. R. Haddadi, Classical Zariski topology of modules and spectral spaces I, Int. Electron. J. Algebra 4(4) (2008) 104–130.
[4] A. K. Bhuniya, M. Kumbhakar, On irreducible pseudo-prime spectrum of topological le-modules, Quasigroups and Related Systems 26(2) (2018) 251–262.
[5] T. Duraivel, Topology on spectrum of modules, J. Ramanujan Math. Soc. 9(1) (1994) 25–34.
[6] D. Hassanzadeh-Lelekaami, H. Roshan-Shekalgourabi, Prime submodules and a sheaf on the prime spectra of modules, Comm. Algebra. 42(7) (2014) 3063–3077.
[7] M. Kumbhakar, A. K. Bhuniya, Uniqueness of primary decomposition in Laskerian le-module, Acta Math. Hunga. 158(1) (2019) 202–215.

[8] M. Kumbhakar, A. K. Bhuniya, On the prime spectrum of an le-module, J. Algebra Appl. 20(12) (2021) 215–220.
[9] C. P. Lu, The Zariski topology on the prime spectrum of a module, Houston J. Math. 25(3) (1999) 417–432.
[10] C. P. Lu, A module whose prime spectrum has the surjective natural map, Houston J. Math. 33(1) (2007) 125–143.
[11] R. McCasland, M. Moore, P. Smith, On the spectrum of a module over a commutative ring, Comm. Algebra. 25(1) (1997) 79–103.
[12] Ortaç Öneş, Mustafa Alkan, Multiplication modules with prime spectrum, Turk. J. Math. 43 (2019) 2000–2009.
[13] Ortaç Öneş, Mustafa Alkan, Zariski subspace topologies on ideals, Hacet. J. Math. Stat. 48(6) (2019) 1667–1674.
[14] N. Phadatare, S. Ballal, V. Kharat, On the second spectrum of lattice modules, Discuss. Math.-Gen. Algebra Appl. 37 (2017) 59–74
[15] N. K. Thakare, C. S. Manjarekar, Abstract spectral theory: Multiplicative lattices in which every character is contained in a unique maximal character, In: Algebra and its applications, Lecture Notes in Pure and Appl.Math. 91, Dekker, New York, (1984) 265–276.
[16] N. K. Thakare, C. S. Manjarekar, S. Maeda, Abstract spectral theory II: Minimal characters and minimal spectrums of multiplicative lattices, Acta Sci. Math. 52(1988) 53–67.