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Unit regular graphs over finite rings*

Research Article

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Abstract: This paper introduces the concept of a unit regular graph over an arbitrary ring. A unit regular graph is a simple undirected graph where the vertices represent the elements of the ring. Two vertices are adjacent if their sum is a unit regular element within the ring. The study investigates various properties of this graph, including completeness, vertex degrees, Eulerian and Hamiltonian characteristics, girth, matching number, clique number, and independence number. Additionally, the paper explores specific properties of these graphs for particular types of rings.

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Keywords: Ring, Unit regular element, Unit regular graph

1. Introduction

Each ring considered in this paper is finite and possesses an identity element. Let R be a ring with an identity element. An element $a \in R$ is termed idempotent if $a^2 = a$, and is called a Von-Neumann regular element in R if there exists $b \in R$ such that $aba = a$. If there exists a unit element $u \in R$ satisfying $aua = a$, then a is referred to as a unit regular element. The element u is called an inner inverse of a (see [9]). It is evident that if R contains a unit, then the additive identity 0_R is unit regular. Let $U(R)$ and $U_r(R)$ denote the sets of all units and all unit regular elements of R, respectively. Since $x \in U_r(R)$ if and only if $-x \in U_r(R)$, the set $U_r(R)$ is inverse closed. Moreover, $U_r(R)$ clearly contains $U(R)$.

A ring R with identity 1_R is considered a division ring if, for every $a \in R$ with $a \neq 0_R$, there exists $b \in R$ such that $ab = 1_R$, i.e., a has a multiplicative inverse. If every element $x \in R$ is idempotent, then the ring R is called a Boolean ring. A ring R is termed a unit regular ring if every element $x \in R$ is unit regular. The characterization of unit regular rings is provided, for instance, in [9] and [6]. One of the results given in [9] is as follows.

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Theorem 1.1. ([9]) *Given a unit regular ring R. Then, R is either a Boolean ring or a division ring if any nonzero element of R has a unique inner inverse.*

This subsequent theorem gives a requirement for a ring to be unit regular as given in Theorem 1 in [6].

Theorem 1.2. ([6]) *The necessary and sufficient condition for a ring R to be unit regular is that for any $x \in R$, we can find an element $e \in R$ that is idempotent and an element $u \in R$ that is a unit satisfying $x = e + u$ and $xR \cap eR = 0_R$.*

Recall that a ring R is considered to be *clean* if, for every $x \in R$, there exists an idempotent $e \in R$ and a unit $u \in R$ such that $x = e + u$. From Theorem 1.2, we know that whenever a ring R is unit regular, R is also clean.

Algebraic graphs have garnered significant attention in recent research. Following the introduction of the concept of Cayley graphs, various new algebraic graphs have been introduced and thoroughly examined in the context of different algebraic structures, including groups, rings, and semigroups. In addition to Cayley graphs, some authors have shown interest in prime graphs, coprime graphs, commuting graphs, and noncommuting graphs within the domain of groups, as demonstrated in references such as [2, 5, 7, 15].

Another type of algebraic graph over a semigroup is the *bipartite graph*, where the vertex set consists of the elements of the semigroup and all cosets of the semigroup, as introduced in [13]. A similar graph, known as the *bipartite graph over a ring*, is defined as a graph whose vertex set consists of the ring's elements and subrings, as proposed in [11]. The concept of these two graphs is motivated by the idea of a bipartite graph associated with elements and cosets of groups, as presented in [3]. Certain properties of this graph for a particular group are given in [12].

In [14], the authors introduced the concept of the *Von-Neumann regular graph* of rings. For a given ring R, the Von-Neumann regular graph $\Gamma_{vnr}(R)$ is defined as a simple graph whose vertex set is R, and any two vertices a and b in R are adjacent if and only if $a + b$ is Von-Neumann regular. Later, in [10], some properties of the graph were presented, including domination parameters. Another type of algebraic graph associated with rings is the *unit graph* $\Gamma_u(R)$, proposed in [4] as a generalization of the work presented in [8]. This unit graph is indeed a subgraph of the Von-Neumann regular graph $\Gamma_{vnr}(R)$. In [1], some properties of the unit regular graph related to the eigensharp property are discussed.

In this manuscript, we propose a new notion of the *unit regular graph* of rings. For any ring R, the unit regular graph of R, denoted by $\Gamma_{ur}(R)$, is defined as the simple graph with R as the vertex set, where any two vertices a and b in the graph are connected by an edge if and only if $a + b$ is unit regular. The graph $\Gamma_{ur}(R)$ is obviously a spanning subgraph of $\Gamma_{vnr}(R)$ and is a supergraph of $\Gamma_u(R)$.

In this research, we explore various aspects of the graph, including vertex degrees, Eulerian and Hamiltonian properties, girth, matching number, clique number, and independence number. Moreover, for some particular rings, we present specific properties of these graphs, including chromatic number, planarity, domination number, and some topological indices such as the first Zagreb and the Wiener indices, as well as the subgraph induced by the unit regular elements.

Some terminologies used in this paper are adopted from [16]. For any graph G, we use the symbols $V(G)$ and $E(G)$ to indicate the vertex set and the edge set of G, respectively. The degree of any vertex a in G is denoted by $\deg_G(a)$. A $|V(G)|$-tuple of all vertex degrees in graph G, such that each term is greater than or equal to the previous term, is known as the *degree sequence* of G. Any subgraph K of G is said to be *induced* if $E(K)$ is equal to the set of all edges of G connecting two vertices of K. The *distance* between a vertex a and a vertex b in G, denoted by $d_G(a, b)$, is the number of edges in the shortest path connecting a and b. The *diameter* of G is defined as the maximum distance between any two vertices of G, and the *girth* of G is defined as the number of edges in the shortest cycle in G. A graph G is called *Eulerian* if it contains an Eulerian trail, i.e., a closed trail that contains every edge of G. It is well-known that G is Eulerian if and only if the degree of each vertex in G is even. If there exists a cycle containing all vertices of G, then G is considered to be *Hamiltonian*. For arbitrary graphs G and H, any bijection $f : V(G) \to V(H)$ satisfying $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$ is called an *isomorphism* from

G to H. Specifically, whenever $G = H$, f is known as an *automorphism*. A graph G is considered *vertex transitive* if, for any vertices x and y within G, there exists an automorphism f on G such that $y = f(x)$. A subset $A \subseteq V(G)$ such that for each $x \in V(G) \setminus A$, there exists $y \in A$ that is adjacent to x, is called a *dominating set*. The size of the minimum dominating set of G is called the *domination number* of G, denoted by $\gamma(G)$. On the other hand, a subset $B \subseteq V(G)$ such that for any $a, b \in B$, a is not connected by any edge to b, is called an *independent set*. The size of the maximum independent set of G, denoted by $\alpha(G)$, is termed the *independence number* of G. Analogous to the independent set, a subset C of $E(G)$ is called a *matching* if C contains no two adjacent edges. The *matching number* of G is identified as the size of a maximum matching, denoted by $\nu(G)$. Any complete subgraph of G is known as a *clique*, and the size of the maximum clique of G is termed the *clique number* of G, denoted by $\omega(G)$. The minimum natural number k such that we can assign numbers in $\{1, 2, \ldots, k\}$ to label the elements of $V(G)$ in such a way that any $x \in E(G)$ connects two vertices with different labels, denoted by $\chi(G)$, is called the *chromatic number* of G. A *planar graph* is a graph that can be drawn on a plane such that no edges intersect except at their endpoints. According to Kuratowski's theorem, a graph is planar if and only if it does not contain a subgraph that is a subdivision of the complete graph K_5 or the complete bipartite graph $K_{3,3}$. Clearly, any graph that contains K_5 or $K_{3,3}$ as a subgraph is not planar. The *Wiener index* of a connected graph G is defined as

$$W(G) = \sum_{a,b \in V(G)} d_G(a, b).$$

The *first Zagreb index* of G is defined as the sum

$$M_1(G) = \sum_{a \in V(G)} (\deg_G(a))^2.$$

2. Results

Let us start with this subsequent description.

Definition 2.1. *Given a ring R. The unit regular graph $\Gamma_{ur}(R)$ of R, is a simple undirected graph which vertex set is R and any two elements a and b in R are connected by an edge exactly when $a + b$ is a unit regular.*

Remark 2.2. *For any ring R, 0_R is adjacent to any $x \in U_r(R)$.*

Remark 2.3. *For any ring R, $\Gamma_{ur}(R)$ is a subgraph of $\Gamma_{vnr}(R)$.*

Theorem 2.4. *If R is a division ring or a Boolean ring, then $U_r(R)$ and R coincide, so that $\Gamma_{ur}(R)$ is a complete graph.*

Proof. It is obvious. □

Nevertheless, in order $R = U_r(R)$, R is not necessarily a division ring nor a Boolean ring, as for instance, we have $\mathbb{Z}_6$ that is not division ring nor Boolean ring, but $U_r(R) = R$.

The following theorem presents a requirement for $\Gamma_{ur}(R)$ being complete.

Theorem 2.5. *Given a ring R having an identity element. These three statements are equivalent.*

(i) $\Gamma_{ur}(R)$ *is complete.*

(ii) $U_r(R) = R$ *(ring R is unit regular).*

(iii) *For each $x \in R$, we have an idempotent e and a unit u in R such that x can be expressed as $x = e + u$ and $xR \cap eR = 0_R$.*

Proof. (i)⇔(ii) Let $a \in R$. As $\Gamma_{ur}(R)$ is complete, x is adjacent to 0_R. Thus, $x + 0_R = x \in U_r(R)$. Hence, $R \subseteq U_r(R)$ implying $R = U_r(R)$. Conversely, it is obvious that for any $a, b \in R$ we have $a + b \in U_r(R)$, meaning that $\Gamma_{ur}(R)$ is complete.
(ii)⇔(iii) By Theorem 1.2 as given in [6]. □

Based on Theorem 2.5 and the property that every ring that is unit regular is clean, we can derive the following corollary.

Corollary 2.6. *For arbitrary ring R, in order $\Gamma_{ur}(R)$ to be complete, R is necessarily clean.*

Remark 2.7. *Let a finite ring R be unit regular. The necessary and sufficient for $\Gamma_{ur}(R)$ to be Eulerian is R has odd number elements.*

Theorem 2.8. *Let R be a division ring and $M_2(R)$ be the ring of all matrices of size 2×2 which entries are in R. Then $U_r(M_2(R)) = M_2(R)$, and hence $\Gamma_{ur}(M_2(R))$ is a complete graph.*

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is not a unit. Hence, $det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc = 0$ or $ad = bc$. If $a \neq 0, d \neq 0$, then $bc \neq 0$, so $c \neq 0$. As R is a division ring, we have $\begin{pmatrix} a^{-1} & c^{-1} \\ 0 & -d^{-1} \end{pmatrix} \in M_2(R)$. Therefore, $det\left(\begin{pmatrix} a^{-1} & c^{-1} \\ 0 & -d^{-1} \end{pmatrix}\right) = -(ad)^{-1} \neq 0$, so that $\begin{pmatrix} a^{-1} & c^{-1} \\ 0 & -d^{-1} \end{pmatrix}$ is a unit. Thus,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^{-1} & c^{-1} \\ 0 & -d^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Therefore, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a unit regular. If $a = 0, d \neq 0$, we can find $\begin{pmatrix} 1 & 0 \\ 0 & d^{-1} \end{pmatrix}$ satisfying

$$\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & d^{-1} \end{pmatrix}\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & bd^{-1} \\ c & 1 \end{pmatrix}\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}.$$

Since $det\left(\begin{pmatrix} 1 & 0 \\ 0 & d^{-1} \end{pmatrix}\right) = d^{-1} \neq 0$, so $\begin{pmatrix} 1 & 0 \\ 0 & d^{-1} \end{pmatrix}$ is a unit. Hence, $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$ is unit regular. If $a \neq 0, d = 0$, there exists $\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ such that

$$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} 1 & b \\ ca^{-1} & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}` = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}.$$

As $det\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) = a^{-1} \neq 0$, so $\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ is a unit. Hence, $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ is a unit regular. If $a = 0, d = 0$, there exists $\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$, where $x = \begin{cases} c^{-1}, & c \neq 0 \\ 1, & c = 0 \end{cases}$ and $y = \begin{cases} b^{-1}, & b \neq 0 \\ 1, & b = 0 \end{cases}$. It is obvious that $\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$ a unit in $M_2(R)$. Thus, we get

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} by & 0 \\ 0 & cx \end{pmatrix}\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

Thus, $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ is a unit reguler. As all of non unit elements in $M_2(R)$ are unit regular, we come to a conclusion that $U_r(M_2(R)) = M_2(R)$. By Theorem 2.5, $\Gamma_{ur}(M_2(R))$ is complete. □

In the following figures we give several examples of unit regular graphs.

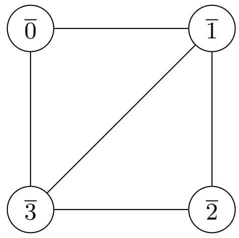

Figure 1. $\Gamma_{ur}(\mathbb{Z}_4)$

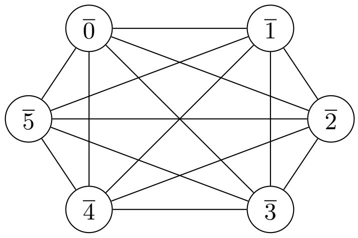

Figure 2. $\Gamma_{ur}(\mathbb{Z}_6)$

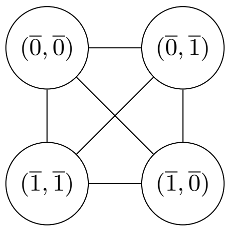

Figure 3. $\Gamma_{ur}(\mathbb{Z}_2 \times \mathbb{Z}_2)$

The following theorem describes all possible degrees of the unit regular graph vertices.

Theorem 2.9. *Let ring R be commutative and with a unit element. For every $a \in R$, we have*

(i) $deg_{\Gamma_{ur}(R)}(a) = |U_r(R)|$ *if* $2a \notin U_r(R)$.

(ii) $deg_{\Gamma_{ur}(R)}(a) = |U_r(R)| - 1$ *if* $2a \in U_r(R)$.

Proof. Take any $a \in R$. Note that $(R, +)$ is a group, so there exists a unique $a_u \in R$ such that $a_u + a = u$, for any $u \in U_r(R)$. We consider these two possibilities.

(i) Suppose that $2a \notin U_r(R)$. For any $u \in U_r(R)$, it follows that $a_u \neq a$. Clearly, a and a_u are adjacent in $\Gamma_{ur}(R)$. Hence, $deg_{\Gamma_{ur}(R)}(a) = |U_r(R)|$.

(ii) Suppose that $2a \in U_r(R)$. Then, $a_{2a} = a$. This implies a and a_{2a} are not adjacent in $\Gamma_{ur}(R)$. For every $u \in U_r(R) \setminus \{2a\}$, we have $a_u \neq a$. We have that a and a_u are connected by an edge in $\Gamma_{ur}(R)$. Hence, $deg_{\Gamma_{ur}(R)}(a) = |U_r(R)| - 1$.

□

Theorem 2.10. *Given a finite ring R with identity. Then $\Gamma_{ur}(R)$ is Eulerian if and only if $|U_r(R)|$ is odd and $\Gamma_{ur}(R)$ is $(|U_r(R)|-1)$-regular, that is all elements of $\Gamma_{ur}(R)$ have degree $|U_r(R)|-1$.*

Proof. Let $\Gamma_{ur}(R)$ be Eulerian. Then each vertex of $\Gamma_{ur}(R)$ has even degree. By Theorem 2.9, all vertices in $\Gamma_{ur}(R)$ are of degree $|U_r(R)|$ or all are of degree $|U_r(R)|-1$, as for otherwise there will a vertex which degree is odd and a vertex which degree is even. Now, assume that all vertices of $\Gamma_{ur}(R)$ have degree $|U_r(R)|$. For any $x \in R$, let $N_{\Gamma_{ur}(R)}(x)$ be the set of all vertices of $\Gamma_{ur}(R)$ that are adjacent to x. Then, by definition of unit regular graph, $|N_{\Gamma_{ur}(R)}(0_R) \cap U_r(R)| = |U_r(R)|-1$. Thus, there exists an element $y \in R \setminus U_r(R)$ such that y is adjacent to 0_R, and hence $y = y + 0_R \in U_r(R)$. Hence, we have a contradiction as $y \notin U_r(R)$. Therefore, all vertices of $\Gamma_{ur}(R)$ have degree $|U_r(R)|-1$. As $\Gamma_{ur}(R)$ is Eulerian, $|U_r(R)|$ must be odd. Conversely, it is clear as all vertices of $\Gamma_{ur}(R)$ have even degree. □

Recall that Ore's theorem states that if for any two different non-adjacent vertices u and v in graph G, the sum of the degree of u and of the degree of v is less than or equal to the number of vertices in G, then G is Hamiltonian. As a consequence of Ore's theorem, we have the following result.

Theorem 2.11. *Let finite ring R have at least 3 elements. If $\Gamma_{ur}(R)$ is connected and $|U_r(R)| \geq \frac{|R|+2}{2}$, then $\Gamma_{ur}(R)$ is Hamiltonian.*

Proof. Let finite ring R has at least 3 elements and $|U_r(R)| \geq \frac{|R|+2}{2}$. By Theorem 2.9, $deg_{\Gamma_{ur}(R)}(x) = |U_r(R)|$ or $deg_{\Gamma_{ur}(R)}(x) = |U_r(R)|-1$. Thus, for arbitrary $x, y \in R$, $deg_{\Gamma_{ur}(R)}(x) + deg_{\Gamma_{ur}(R)}(y)$ is at least $2|U_r(R)|-2$. As $2|U_r(R)|-2 \geq 2\left(\frac{|R|+2}{2}\right)-2 = |R|$, then by Ore's theorem, $\Gamma_{ur}(R)$ is Hamiltonian. □

In the next property, we show that the unit regular element graph of a ring with identity is contain a cycle of length 3.

Theorem 2.12. *For any ring R containing nonzero unit regular element, the graph $\Gamma_{ur}(R)$ has girth 3.*

Proof. Clearly, 0_R is unit regular. Let u be a nonzero unit regular element. We have $-u$ is also a unit regular element. Thus, we have a cycle of length 3 connecting $u, 0_R$ and $-u$. □

As a consequence, we have this subsequently theorem.

Theorem 2.13. *Let R be any ring containing nonzero unit regular element. The Von Neumann regular graph $\Gamma_{vnr}(R)$ associated with R has girth 3.*

Proof. This is as $\Gamma_{ur}(R)$ is a spanning subgraph of $\Gamma_{vnr}(R)$ and by Theorem 2.12. □

Theorem 2.14. *If ring R is not a field and has a nonzero unit regular element and $x + y \in U_r(R)$, for any $x \in R \setminus U_r(R)$ and $0_R \neq y \in U_r(R)$, then $diam(\Gamma_{ur}(R)) \leq 3$.*

Proof. As R is not a field, consequently $diam(\Gamma_{ur}(R)) > 1$. Now, let take arbitrary two vertices u and v in R. If $u, v \in U_r(R)$, then we obtain a path $u - 0_R - v$ of length 2. If $u, v \in R \setminus U_r(R)$, then take a nonzero unit regular element $a \in R$. Then we obtain a path $u - a - v$ of length 2. If $u \in U_r(R)$ and $v \in R \setminus U_r(R)$, then u and v are connected by an edge or we have a path $u - 0_R - a - v$ of length 3 for some nonzero unit regular element $a \in R$. Hence, $diam(\Gamma_{ur}(R)) \leq 3$. □

The next result shows the lower bound of the matching number of $\Gamma_{ur}(R)$.

Theorem 2.15. *Let R be a finite ring and $R_{2,+} = \{x \in R | 2x \neq 0_R\}$. The matching number of $\Gamma_{ur}(R)$ is greater or equal to $\frac{|R_{2,+}|}{2}$. Particularly, if ring R is unit regular with even number elements, then the matching number of $\Gamma_{ur}(R)$ is $\frac{|R|}{2}$.*

Proof. Observe that $x \in R_{2,+}$ is equivalent to $-x \in R_{2,+}$, implying that $|R_{2,+}|$ is even. Let $x \in R_{2,+}$, that is $x \neq -x$. Let $-x = y$. Hence, $x + y = 0_R \in U_r(R)$. Thus, we have independent edge set $\{\{x, -x\}|x \in R_{2,+}\}$ comprising precisely $\frac{|R_{2,+}|}{2}$ elements. Hence, the matching number of $\Gamma_{ur}(R)$ must be greater or equal to $\frac{|R_{2,+}|}{2}$. Particularly, if R is a unit regular ring such that $|R|$ is even, then by Theorem 2.5, $\Gamma_{ur}(R)$ is complete. Thus, the matching number of $\Gamma_{ur}(R)$ is $\frac{|R|}{2}$. □

Recall that any graph that is isomorphic to a graph with vertex set $\{c, x_i, y_i|i = 1, 2, \ldots, r\}$ and edge set $\{cx_i, cy_i, x_iy_i|i = 1, 2, \ldots, r\}$ is called as a friendship graph F_r. The following theorem gives a sufficient condition for the graph $\Gamma_{ur}(R)$ to contain a friendship graph.

Theorem 2.16. *Let R be a finite ring. If $R_{2,+} = \{x \in R|2x \neq 0_R\}$ and $U_r(R) \cap R_{2,+} = \{0_R\}$, then the subgraph of $\Gamma_{ur}(R)$ induced by $U_r(R)$ contains the friendship graph $F_{\frac{|U_r(R)|-1}{2}}$.*

Proof. Obviously, $U_r(R)$ is inverse closed, i.e. $-x \in U_r(R)$ if $x \in U_r(R)$. Let $U_r(R) \cap R_{2,+} = \{0_R\}$. Then $|U_r(R)| = 2k + 1$ for some $k \geq 1$. Moreover, for all nonzero $x \in U_r(R)$, x is adjacent to $-x$, and also to 0_R. Therefore, if $E = \{\{x, -x\}, |0_R \neq x \in U_r(R)\} \cup \{0_R, x\}|0_R \neq x \in U_r(R)\}$, then $U_r(R)$ and E defines the subgraph of $\Gamma_{ur}(R)$ induced by $U_r(R)$ and is a friendship graph F_k with $k = \frac{|U_r(R)|-1}{2}$. □

Theorem 2.17. *Let R^i be a finite division ring for any $i = 1, 2, \ldots, n$. If $R = R^1 \times R^2 \times \ldots \times R^n$, then $U_r(R) = R$ and thus $\Gamma_{ur}(R)$ is complete. If there is i such that $|R^i|$ is even, then $\Gamma_{ur}(R)$ is not Eulerian.*

Proof. Let $R = R^1 \times R^2 \times \ldots \times R^n$ for some division rings R^i's, $i = 1, 2, \ldots, n$. Then $\bar{a} = (a_1, a_2, \ldots, a_n) \in R$ is unit regular exactly when a_i is unit regular for all $i = 1, 2, \ldots, n$. As $U_r(R^i) = R^i$ by Theorem 2.4, hence $U_r(R) = R$. And thus by Theorem 2.5, $\Gamma_{ur}(R)$ is complete and moreover each vertex of $\Gamma_{ur}(R)$ has degree $|R| - 1$. Therefore, if $|R^i|$ is even for some i, then $|R| - 1$ is odd implying $\Gamma_{ur}(R)$ is not Eulerian. □

Theorem 2.18. *For any ring R, if $U_r(R)$ is closed under addition, then the independence number $\alpha(\Gamma_{ur}(R))$ is at most $|R \setminus U_r(R)| + 1$.*

Proof. Clearly, if $a+b \in U_r(R)$ for any $a, b \in U_r(R)$, then any independent set in $\Gamma_{ur}(R)$ cannot include two elements from $U_r(R)$. Additionally, if $a + b \in R \setminus U_r(R)$ for any $a, b \in R \setminus U_r(R)$, then $R \setminus U_r(R)$ is an independent set. And hence to maintain its independence, adding k elements to $R \setminus U_r(R)$ is only possible for $k = 1$. Thus, $\alpha(\Gamma_{ur}(R)) \leq |R \setminus U_r(R)| + 1$. □

Theorem 2.19. *Let R be any finite ring. If $U_r(R)$ is closed under addition, then the clique number of $\Gamma_{ur}(R)$ is at least $|U_r(R)|$.*

Proof. For any ring R, let $U_r(R)$ be additively closed. Then, any elements x and y in $U_r(R)$ will be adjacent. Thus $U_r(R)$ induces a complete subgraph meaning that $U_r(R)$ is a clique. Therefore, the clique number is greater than or equal to $|U_r(R)|$. □

Remember that for any ring R, the characteristic of R is the smallest positive integer k such that $kx = 0_R$ for arbitrary $x \in R$. In the next theorem, we give a sufficient condition for the unit graph $\Gamma_{ur}(R)$ to be vertex transitive, in connection with the characteristic of the ring.

Theorem 2.20. *Let ring R have an identity and have characteristic 2. Then $\Gamma_{ur}(R)$ is vertex transitive.*

Proof. Let $\alpha, \beta \in R$. Construct a function $\psi : R \to R$ with $\psi(a) = a + \beta - \alpha$ for every $a \in R$. By the definition of ψ, we have $\psi(\alpha) = \beta$ and moreover, ψ is a bijection as $(R, +)$ is a group. Furthermore, for any $a, b \in R$, we have that a is adjacent to b, i.e. $a + b \in U_r(R)$ if and only if $\psi(a) + \psi(b) = a + \beta - \alpha + b + \beta - \alpha = a + b \in U_r(R)$, i.e. $\psi(a)$ and $\psi(b)$ are adjacent. Therefore, ψ is an isomorphism and hence $\Gamma_{ur}(R)$ is vertex transitive. □

3. Unit regular graph over some particular rings

Throughout this section we present some more detail properties of the unit regular graph of ring of integers modulo n for some particular n.

Theorem 3.1. *Let $\mathbb{Z}_n$ be the ring of all integer classes modulo n. If $n = p_1p_2 \dots p_k$, where p_i's, $i = 1, 2, \dots, k$, are distinct prime numbers, then $U_r(\mathbb{Z}_n) = \mathbb{Z}_n$ and hence $\Gamma_{ur}(\mathbb{Z}_n)$ is complete.*

Proof. Let $n = p_1p_2 \dots p_k$ for some different primes p_i's, $i = 1, 2, \dots, k$. By Chinese Remainder theorem, clearly we have $\mathbb{Z}_n \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \times \mathbb{Z}_{p_k}$. As $\mathbb{Z}_{p_i}$ is a field, by Theorem 2.17, $U_r(\mathbb{Z}_n) = \mathbb{Z}_n$ and thus $\Gamma_{ur}(\mathbb{Z}_n)$ is complete. □

Theorem 3.2. *Given ring $\mathbb{Z}_n$ with $n = 4p$ for some prime number $p \geq 3$.*

(i) *$U_r(\mathbb{Z}_n) = \mathbb{Z}_n \setminus \{\overline{4t+2} \mid 0 \leq t < p\}$ and thus $|U_r(\mathbb{Z}_n)| = 3p$.*

(ii) *For any $\overline{x} \in \mathbb{Z}_n$, $deg_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}) = 3p - 1$ if x is even and $deg_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}) = 3p$ if x is odd.*

Proof. (i) Let a be a unit regular element. There exists a unit $u \in \mathbb{Z}_n$ satisfying $aua = a$. As u is a unit, we have $gcd\{u, n\} = 1$ so that $u \equiv 1\ mod 4$ or $u \equiv 3\ mod 4$. Let assume that a is of the form $a = \overline{4t+2}$ for some $0 \leq t < p$. Then $a \equiv 2\ mod 4$. For $u \equiv 1\ mod 4$ or $u \equiv 3\ mod 4$, we have $uau \equiv 0\ mod 4$ and hence is not equal to a. Therefore, $a \in \mathbb{Z}_n \setminus \{\overline{4t+2} \mid 0 \leq t < p\}$. Now, let $\overline{a} \in \mathbb{Z}_n \setminus \{\overline{4t+2} \mid 0 \leq t < p\}$. Then, either $\overline{a} = \overline{0}$ or $a \equiv 1\ mod 4$ or $a \equiv 3\ mod 4$. If $a = 0$, then a is a unit regular element. If $a \equiv\ \ mod 4$ or $a \equiv 3\ mod 4$, then $gcd\{a, n\} = 1$ so that $\overline{a}$ is a unit. Consequently, $\overline{a}$ is a unit regular element.

(ii) Let $\overline{x} \in \mathbb{Z}_n$ be arbitrary. If x is even, then $2x \equiv 0\ mod 4$ and thus $2\overline{x}$ is a unit regular element. By Theorem 2.9, the degree of $\overline{x}$ is $|U_r(\mathbb{Z}_n)| - 1 = 3p - 1$. Let now x be odd, say $x = 2r + 1$ for some r. We have $2x$ is of the form $4t + 2$ for some $0 \leq t < p$. Hence $2\overline{x}$ is not a unit regular element. By Theorem 2.9, the degree of $\overline{x}$ is $|U_r(\mathbb{Z}_n)| = 3p$.

□

Theorem 3.3. *Given ring $\mathbb{Z}_n$ with $n = 4p$ for some prime number $p \geq 3$.*

(i) *The degree sequence of unit regular graph $\Gamma_{ur}(\mathbb{Z}_n)$ is*

$$(\underbrace{|U_r(\mathbb{Z}_n)| - 1, \dots, |U_r(\mathbb{Z}_n)| - 1}_{2p}, \underbrace{|U_r(\mathbb{Z}_n)|, \dots, |U_r(\mathbb{Z}_n)|}_{2p}).$$

(ii) *$\Gamma_{ur}(\mathbb{Z}_n)$ is connected with diameter 2.*

(iii) *$\Gamma_{ur}(\mathbb{Z}_n)$ is Hamiltonian but not Eulerian.*

(iv) *The matching number of $\Gamma_{ur}(\mathbb{Z}_n)$ is $2p$.*

(v) *The clique number of $\Gamma_{ur}(\mathbb{Z}_n)$ is $p + 2$.*

(vi) *The independence number of $\Gamma_{ur}(\mathbb{Z}_n)$ is p.*

(vii) *The domination number of $\Gamma_{ur}(\mathbb{Z}_n)$ is 2.*

(viii) *The chromatic number of $\Gamma_{ur}(\mathbb{Z}_n)$ is $p + 2$.*

(ix) *$\Gamma_{ur}(\mathbb{Z}_n)$ is not planar.*

(x) *The first Zagreb index of $\Gamma_{ur}(\mathbb{Z}_n)$ is $M_1(\Gamma_{ur}(\mathbb{Z}_n)) = 36p^3 - 12p^2 + 2p$.*

(xi) *The Wiener index of $\Gamma_{ur}(\mathbb{Z}_n)$ is $W(\Gamma_{ur}(\mathbb{Z}_n)) = 8p^2 - p$.*

(xii) *The subgraph of* $\Gamma_{ur}(\mathbb{Z}_n)$ *induced by* $U_r(\mathbb{Z}_n)$ *is the union of the complete tripartite graph defined by partition* $\{S_0, S_1, S_3\}$ *and the complete subgraph induced by* S_0*, where* $S_i = \{\overline{4t+i} \in \mathbb{Z}_n | 0 \leq t < p\}$ *for* $i = 0, 1, 3$.

Proof. Consider $S_i = \{\overline{4t+i} \in \mathbb{Z}_n | 0 \leq t < p\}$, for any $i = 0, 1, 2, 3$. By Theorem 3.2, $U_r(\mathbb{Z}_n) = S_0 \cup S_1 \cup S_3$.

(i) As a direct consequence of Theorem 3.2.

(ii) By Theorem 3.2, $\Gamma_{ur}(\mathbb{Z}_n)$ is not complete. Hence, $diam(\Gamma_{ur}(\mathbb{Z}_n)) \geq 2$. Let $\overline{x}, \overline{y} \in \mathbb{Z}_n$. If $\overline{x}, \overline{y} \in U_r(\mathbb{Z}_n)$, then we have path $\overline{x} - \overline{0} - \overline{y}$ of length 2. Now, let $\overline{x} \in U_r(\mathbb{Z}_n)$ and $\overline{y} \notin U_r(\mathbb{Z}_n)$. By Theorem 3.2, $\overline{y}$ is in S_2. If $\overline{x} \in S_1 \cup S_3$, then $\overline{x}$ and $\overline{y}$ are connected by an edge. Hence, $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 1$. If $\overline{x} \in S_0$, then we have a path $\overline{x} - \overline{1} - \overline{y}$ of length 2. Now, let $\overline{x}, \overline{y} \notin U_r(\mathbb{Z}_n)$, i.e. $\overline{x}, \overline{y} \in S_2$. We have a path $\overline{x} - \overline{1} - \overline{y}$. Therefore, we conclude that $\Gamma_{ur}(\mathbb{Z}_n)$ is connected with diameter 2.

(iii) By Theorem 2.11 and Theorem 3.2, the unit regular graph $\Gamma_{ur}(\mathbb{Z}_n)$ is Hamiltonian, but as the parity of the degree of its vertices are not the same, $\Gamma_{ur}(\mathbb{Z}_n)$ is not Eulerian.

(iv) Let for any $i = 0, 1, 2, 3$, $S_i = \{a_{i,1}, a_{i,2}, \ldots, a_{i,p}\}$. We form

$$M = \{\{a_{0,k}, a_{1,k}\} | k = 1, 2, \ldots, p\} \cup \{\{a_{2,k}, a_{3,k}\} | k = 1, 2, \ldots, p\}.$$

Thus M has precisely $2p$ elements. Moreover, for any k, $a_{0,k} + a_{1,k} \notin \{\overline{4t+2} \in \mathbb{Z}_n | 0 \leq t < p\}$, i.e. $a_{0,k} + a_{1,k} \in U_r(\mathbb{Z}_n)$. Similarly, for any k, $a_{2,k} + a_{3,k} \notin \{\overline{4t+2}\mathbb{Z}_n | 0 \leq t < p\}$, i.e. $a_{2,k} + a_{3,k} \in U_r(\mathbb{Z}_n)$. This means that each element in M is an edge in $\Gamma_{ur}(\mathbb{Z}_n)$. Thus M is a perfect matching as it covers all vertices in $\Gamma_{ur}(\mathbb{Z}_n)$. Hence, the matching number of $\Gamma_{ur}(\mathbb{Z}_n)$ is $2p$.

(v) For any $x, y \in S_0$, it follows that $x + y \in S_0 \subseteq U_r(\mathbb{Z}_n)$. Hence the subgraph induced by S_0 is complete. Thus, the clique number of $\Gamma_{ur}(\mathbb{Z}_n)$ is at least $|S_0| = p$. Now, let C be a maximal clique in $\Gamma_{ur}(\mathbb{Z}_n)$. This implies C cannot contains two vertices in S_1, as for otherwise we have two vertices that are not adjacent in C. Thus, $|C \cap S_1| \leq 1$. Similarly, $|C \cap S_3| \leq 1$. Observe that C cannot contain simultaneously a vertex from S_0 and and a vertex from S_2. Thus, $C \subseteq S_0 \cup \{a, b\}$ or $C \subseteq S_2 \cup \{c, d\}$ for some $a, c \in S_1$ and $b, d \in S_3$. Let $C \subseteq S_0 \cup \{a, b\}$. Recall that a and b are adjacent. Moreover, both a and b are adjacent to any $x \in S_0$. By maximality of C, C must be equal to $S_0 \cup \{a, b\}$. If $C \subseteq S_2 \cup \{c, d\}$, by similar argument, C and $S_2 \cup \{c, d\}$ must coincide. Thus, $|C| = p + 2$.

(vi) We have S_1 and S_3 are independent sets, as for $i = 1, 3$, for any $x, y \in S_i$ it follows that $x + y \in S_2$, i.e. $x + y \notin U_r(\mathbb{Z}_n)$. Hence, the independence number of $\Gamma_{ur}(\mathbb{Z}_n)$ is greater or equal to p. Let I be the largest independent set. This implies I cannot include two elements from S_0. As for any $x, y \in S_0$, x and y are adjacent. Thus, $|I \cap S_0| \leq 1$. Similarly, $|I \cap S_2| \leq 1$. Moreover, if $I \cap S_1 \neq \emptyset$ or $I \cap S_3 \neq \emptyset$, then $I \cap S_0 = I \cap S_2 = \emptyset$. Also if $I \cap S_1 \neq \emptyset$, then $I \cap S_3 = \emptyset$. Hence, the largest possible independent set is $I = S_1$ or $I = S_3$, which has p vertices. We conclude that the independence number of $\Gamma_{ur}(\mathbb{Z}_n)$ is p.

(vii) If D is a dominating set, then necessarily D is not a singleton since there are no edges connecting a vertex in S_0 and a vertex in S_2 and also by the fact that S_1 and S_3 are independent sets. Thus, the domination number of $\Gamma_{ur}(\mathbb{Z}_n)$ is greater or equal 2. Let $a \in S_0$ and $b \in S_2$ be arbitrary. For each $x \in (S_0 \setminus \{a\}) \cup S_1 \cup S_3$, a and x are adjacent. For each $y \in S_2 \setminus \{b\}$, y is adjacent to b. Thus $D = \{a, b\}$ is a dominating set. We conclude that the domination number of $\Gamma_{ur}(\mathbb{Z}_n)$ is 2.

(viii) By (v), the chromatic number of $\Gamma_{ur}(\mathbb{Z}_n)$ must be at least $p + 2$. Now, let us color the vertices in S_0 with p colors $c_1, c_2, \ldots, c_p$ such that each color c_i appears once. And let also the vertices in S_2 be colored with $c_1, c_2, \ldots, c_p$ and each color appears once. Put color c_{p+1} for all vertices in S_1 and put color c_{p+2} for all vertices in S_3. Then, we find that all edges in $\Gamma_{ur}(\mathbb{Z}_n)$ connect two vertices of different colors. We conclude that the chromatic number of $\Gamma_{ur}(\mathbb{Z}_n)$ is $p + 2$.

(ix) By (v), $\Gamma_{ur}(\mathbb{Z}_n)$ contains the complete graph K_5 implying $\Gamma_{ur}(\mathbb{Z}_n)$ is not planar by the Kuratowski's theorem.

(x) Let $\mathbb{Z}_n^e = \{\overline{x} \in \mathbb{Z}_n | x \text{ even}\}$ and let $\mathbb{Z}_n^o = \{\overline{x} \in \mathbb{Z}_n | x \text{ odd}\}$. By Theorem 3.2, we have

$$\begin{aligned} M_1(\Gamma_{ur}(\mathbb{Z}_n)) &= \sum_{v \in V(G)} (deg_{\Gamma_{ur}(\mathbb{Z}_n)}(v))^2 \\ &= \sum_{v \in \mathbb{Z}_n^e} (deg_{\Gamma_{ur}(\mathbb{Z}_n)}(v))^2 + \sum_{v \in \mathbb{Z}_n^o} (deg_{\Gamma_{ur}(\mathbb{Z}_n)}(v))^2 \\ &= 2p(3p-1)^2 + 2p(3p)^2 \\ &= 36p^3 - 12p^2 + 2p. \end{aligned}$$

(xi) Let for any $i, j \in \{0, 1, 2, 3\}$, $A_{ij} = \{\{\overline{x}, \overline{y}\} \subseteq \mathbb{Z}_n | \overline{x} \neq \overline{y}, \{\overline{x}, \overline{y}\} \cap S_i \neq \emptyset, \{\overline{x}, \overline{y}\} \cap S_j \neq \emptyset\}$. We have $|A_{ij}| = \frac{p(p-1)}{2}$ if $i = j$ and $|A_{ij}| = p^2$ if $i < j$. Let $\{\overline{x}, \overline{y}\} \in A_{ij}$ be arbitrary. If $i = j = 0$ or $i = j = 2$, then $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 1$. If $i = j = 1$ or $i = j = 3$, then $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 0$. For $i < j$, if $i = 0$ and $j = 2$, then $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 2$ and $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 1$ for the remaining cases. Therefore,

$$\begin{aligned} W(\Gamma_{ur}(\mathbb{Z}_n)) &= \sum_{i=0}^{3} \sum_{\{\overline{x},\overline{y}\} \in A_{ii}} d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) + \sum_{0 \le i < j \le 3} \sum_{\{\overline{x},\overline{y}\} \in A_{ij}} d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) \\ &= \frac{p(p-1)}{2}(1+0+1+0) + 2p^2 + 5p^2 \\ &= 8p^2 - p. \end{aligned}$$

(xii) As mentioned in the proof of (v) and (vi), S_0 is a clique and both S_1 and S_3 are independent sets. Hence, S_0 induces a complete subgraph. But for each $i, j \in \{0, 1, 3\}$, with $i \neq j$, the set $S_i \cup S_j$ altogether with all edges connecting a vertex in S_i and a vertex in S_j define a complete bipartite graph with partition $\{S_i, S_j\}$ and thus partition $\{S_0, S_1, S_3\}$ defines a complete tripartite graph. Altogether with the subgraph induced by S_0, the union of the tripartite graph and the complete subgraph induced by S_0 is indeed the subgraph of $\Gamma_{ur}(\mathbb{Z}_n)$ induced by $U_r(\mathbb{Z}_n)$.

□

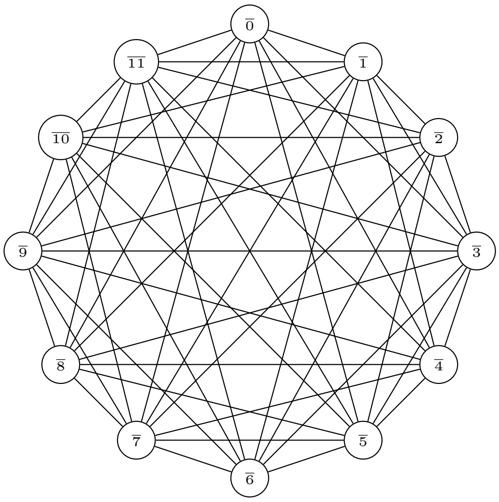

Figure 4. The unit regular graph $\Gamma_{ur}(\mathbb{Z}_{12})$

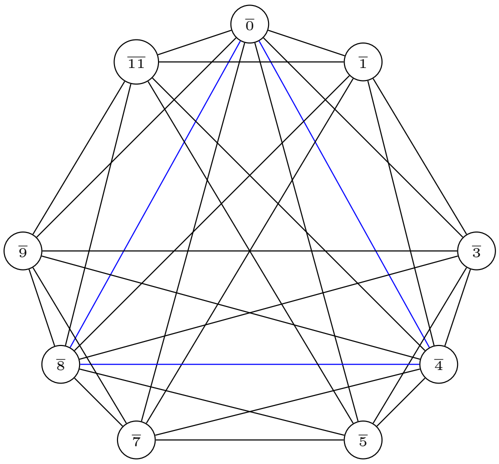

Figure 5. The subgraph of $\Gamma_{ur}(\mathbb{Z}_{12})$ induced by $U_r(\mathbb{Z}_{12})$

Theorem 3.4. *Let $n = 2^k$ for some positive integer k. We have*

(i) $U_r(\mathbb{Z}_n) = \{\overline{x} \in \mathbb{Z}_n \mid x \text{ is odd}\} \cup \{\overline{0}\}$, *so that* $|U_r(\mathbb{Z}_n)| = 2^{k-1} + 1$.

(ii) $deg_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}) = 2^{k-1} + 1$ *if* $\overline{x} \in \mathbb{Z}_n \setminus \{\overline{0}, \overline{2^{k-1}}\}$ *and* $deg_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}) = 2^{k-1}$ *if* $\overline{x} \in \{\overline{0}, \overline{2^{k-1}}\}$.

Proof. (i) We have $U_r(\mathbb{Z}_n) = \{\overline{x} \in \mathbb{Z}_n | gcd(x, n) = 1\} \cup \{\overline{0}\}$. But, as $n = 2^k$, then it follows that $gcd(x, n) = 1$ if and only if $\overline{x}$ is odd. Therefore, $|U_r(\mathbb{Z}_n)| = 2^{k-1} + 1$.

(ii) Let $\overline{x} \in \mathbb{Z}_n \setminus \{\overline{0}, \overline{2^{k-1}}\}$. For all $\overline{u} \in U_r(\mathbb{Z}_n)$, there exists a unique $\overline{y}_u \in \mathbb{Z}_n$ with $\overline{y}_u \neq \overline{x}$ such that $\overline{x} + \overline{y}_u = \overline{u}$. Hence, $\deg_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}) = |U_r(\mathbb{Z}_n)| = 2^{k-1} + 1$. Let $\overline{x} \in \{\overline{0}, \overline{2^{k-1}}\}$. For all $\overline{u} \in U_r(\mathbb{Z}_n)$ with $\overline{u} \neq \overline{0}$, there exists a unique $\overline{y}_u \in \mathbb{Z}_n$ such that $\overline{x} + \overline{y}_u = \overline{u}$. For $\overline{u} = \overline{0}$, we have $\overline{y}_u = \overline{x}$. Hence $\deg_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}) = |U_r(\mathbb{Z}_n)| - 1 = 2^{k-1}$.

□

Theorem 3.4 implies the following several results.

Theorem 3.5. *Given ring $\mathbb{Z}_n$ with $n = 2^k$ for some positive integer k.*

(i) *The degree sequence of $\Gamma(\mathbb{Z}_n)$ is*
$$(|U_r(\mathbb{Z}_n)| - 1, |U_r(\mathbb{Z}_n)| - 1, \underbrace{|U_r(\mathbb{Z}_n)|, \dots, |U_r(\mathbb{Z}_n)|}_{2^k-2}).$$

(ii) *$\Gamma_{ur}(\mathbb{Z}_n)$ is connected with diameter 2.*

(iii) *$\Gamma_{ur}(\mathbb{Z}_n)$ is Hamiltonian but not Eulerian.*

(iv) *$\Gamma_{ur}(\mathbb{Z}_n)$ is planar if and only if $k \leq 2$.*

(v) *The clique number of $\Gamma_{ur}(\mathbb{Z}_n)$ is 4.*

(vi) *The independence number of $\Gamma_{ur}(\mathbb{Z}_n)$ is $2^{k-2} + 1$.*

(vii) *The domination number of $\Gamma_{ur}(\mathbb{Z}_n)$ is 2.*

(viii) *The matching number of $\Gamma_{ur}(\mathbb{Z}_n)$ is 2^{k-1}.*

(ix) *The chromatic number of $\Gamma_{ur}(\mathbb{Z}_n)$ is 4.*

(x) *The first Zagreb index of $\Gamma_{ur}(\mathbb{Z}_n)$ is $M_1(\Gamma_{ur}(\mathbb{Z}_n)) = 2^{3k-2} + 2^{2k} - 2^k - 2$.*

(xi) *The Wiener index of $\Gamma_{ur}(\mathbb{Z}_n)$ is $W(\Gamma_{ur}(\mathbb{Z}_n)) = 2^{2k-1} - 2^{k-1} + 1$.*

(xii) *The subgraph of $\Gamma_{ur}(\mathbb{Z}_n)$ induced by $U_r(\mathbb{Z}_n)$ is the friendship graph $F_{2^{k-2}}$.*

Proof. Let $n = 2^k$ where k is a positive integer. Let $\mathbb{Z}_n^e = \{\overline{x} \in \mathbb{Z}_n | x \text{ even}\}$ and let $\mathbb{Z}_n^o = \{\overline{x} \in \mathbb{Z}_n | x \text{ odd}\}$.

(i) It is a direct consequence of Theorem 3.4.

(ii) Let $x, y \in \mathbb{Z}_n$. If $\overline{x} \in \mathbb{Z}_n^e$ and $\overline{y} \in \mathbb{Z}_n^o$, then $\overline{x}$ and $\overline{y}$ are adjacent, and hence $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 1$. Let $\overline{x}, \overline{y} \in \mathbb{Z}_n^e$. If $\overline{y} = -\overline{x}$, then $\overline{x} + \overline{y} = \overline{0}$ and thus $\overline{x}$ and $\overline{y}$ are adjacent. If $\overline{y} \neq -\overline{x}$ then for any $\overline{z} \in \mathbb{Z}_n^o$, both $\overline{x}$ and $\overline{y}$ are adjacent to $\overline{z}$ and thus we have a path $\overline{x} - \overline{z} - \overline{y}$. Therefore, $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 2$. Similarly, if $\overline{x}, \overline{y} \in \mathbb{Z}_n^o$, then $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y})$ is 1 or 2. We conclude that the diameter of $\Gamma_{ur}(\mathbb{Z}_n)$ is 2.

(iii) By Theorem 3.4 and Ore's theorem on Hamiltonicity of graphs, $\Gamma_{ur}(\mathbb{Z}_n)$ is Hamiltonian. By Theorem 3.4, $\Gamma_{ur}(\mathbb{Z}_n)$ have some vertices of different parity. Hence, $\Gamma_{ur}(\mathbb{Z}_n)$ is not Eulerian.

(iv) If $k = 1$, then $\Gamma_{ur}(\mathbb{Z}_n)$ is isomorphic to the path graph P_2 with two vertices, which is planar. For $k = 2$, then $\Gamma_{ur}(\mathbb{Z}_n)$ is also planar as illustrated on Figure 2. If $k \geq 3$, then $\Gamma_{ur}(\mathbb{Z}_n)$ contains complete bipartite graph $K_{4,4}$ as each vertex $\overline{x} \in \mathbb{Z}_n^e$ is adjacent to any vertex $\overline{y} \in \mathbb{Z}_n^o$. By the Kuratowski's theorem, $\Gamma_{ur}(\mathbb{Z}_n)$ is not planar.

(v) Let C be the maximal clique of $\Gamma_{ur}(\mathbb{Z}_n)$. For any $\overline{x}, \overline{y} \in \mathbb{Z}_n^e$, $\overline{x}$ and $\overline{y}$ are adjacent if only if $\overline{y} = -\overline{x}$. Also, for any $\overline{x}, \overline{y} \in \mathbb{Z}_n^o$, $\overline{x}$ and $\overline{y}$ are adjacent if only if $\overline{y} = -\overline{x}$. Therefore, $|C \cap \mathbb{Z}_n^e| \leq 2$ and $|C \cap \mathbb{Z}_n^e| \leq 2$ impliying that the clique number of $\Gamma_{ur}(\mathbb{Z}_n)$ is at most 4. Consider $C' = \{\overline{1}, \overline{2^k - 1}, \overline{2}, \overline{2^k - 2}$. It is easy to check that C' induces a complete subgraph of $\Gamma_{ur}(\mathbb{Z}_n)$. Hence, the clique number of $\Gamma_{ur}(\mathbb{Z}_n)$ is 4.

(vi) Obviously, $|\mathbb{Z}_n^e| = |\mathbb{Z}_n^o| = 2^{k-1}$. Put $I = \{\overline{2a} | 0 \leq a \leq 2^{k-2}\}$. Then, I is an independent set. Hence, the independence number of $\Gamma_{ur}(\mathbb{Z}_n)$ is at least $2^{k-2}+1$. Let I be the largest independence set in $\Gamma_{ur}(\mathbb{Z}_n)$. It is easy to see that $I \cap \mathbb{Z}_n^o \neq \emptyset$ implies $I \cap \mathbb{Z}_n^e = \emptyset$. Therefore, $I \subseteq \mathbb{Z}_n^e$ or $I \subseteq \mathbb{Z}_n^o$. Let $I \subseteq \mathbb{Z}_n^e$. For any two different vertices $\overline{x}, \overline{y} \in \mathbb{Z}_n^e$, $\overline{x}$ and $\overline{y}$ are adjacent if and only if $\overline{y} = -\overline{x}$. Consider $\mathbb{Z}_n^e \setminus \{\overline{0}, \overline{2^{k-1}}\}$. From $\mathbb{Z}_n^e \setminus \{\overline{0}, \overline{2^{k-1}}\}$, we have $2^{k-2}-1$ edges of the form $\{x, y\}$, with $y = -x$. Thus, $|I \cap (\mathbb{Z}_n^e \setminus \{\overline{0}, \overline{2^{k-1}}\})| \leq 2^{k-2}-1$. Hence, we can take a vertex from each of those edges, so that we obtain $2^{k-2}-1$ independent vertices. Let collect those vertices in A. Now put $B = A \cup \{\overline{0}, \overline{2^{k-1}}\}$. Then B is independent and also the largest. Thus, $I = B$ and I has $2^{k-2}+1$ elements. Now, let $I \subseteq \mathbb{Z}_n^o$. In the subgraph induced by $\mathbb{Z}_n^o$, we have 2^{k-2} edges of the form $\{\overline{x}, \overline{y}\}$ with $\overline{y} = -\overline{x}$. Therefore, $|I| \leq 2^{k-2}$. We conclude that the independent set of $\Gamma_{ur}(\mathbb{Z}_n)$ is $2^{k-2}+1$.

(vii) Observe that $\overline{x}$ is not adjacent to $\overline{y}$ whenever $\overline{x}, \overline{y} \in \mathbb{Z}_n^e$ or $\overline{x}, \overline{y} \in \mathbb{Z}_n^o$, and $\overline{y} \neq -\overline{x}$. Hence, the domination number of $\Gamma_{ur}(\mathbb{Z}_n)$ must be greater or equal 2. As $\{\overline{0}, \overline{1}\}$ is a dominating set, we can conclude that the domination number is 2.

(viii) The matching number of $\Gamma_{ur}(\mathbb{Z}_n)$ is at most 2^{k-1}. Let $\{\overline{x}, \overline{y}\}$ be an edge connecting vertex $\overline{x}$ and vertex $\overline{y}$. Consider $M = \{\{\overline{x}, \overline{x+1}\} | \overline{x} \in \mathbb{Z}_n^e\}$. Then, M contains 2^{k-1} independent edges and covers all vertices in $\mathbb{Z}_n$. Therefore, M is maximum so that the matching number of $\Gamma_{ur}(\mathbb{Z}_n)$ is 2^{k-1}.

(ix) From (v), the chromatic number must be greater or equal to 4. We know that all vertices $\mathbb{Z}_n^e$ are adjacent to all vertices in $\mathbb{Z}_n^o$. Therefore, if $\overline{x} \in \mathbb{Z}_n^e$ and $\overline{y} \in \mathbb{Z}_n^o$, then we must put different colors for $\overline{x}$ and for $\overline{y}$. As for any $\overline{x}, \overline{y} \in \mathbb{Z}_n^o$ or $\overline{x}, \overline{y} \in \mathbb{Z}_n^e$, $\overline{x}$ and $\overline{y}$ are adjacent if and only $\overline{y} = -\overline{x}$, then we can use two colors for vertices in $\overline{x} \in \mathbb{Z}_n^e$, let say c_1 and c_2. Use c_1 for vertices in the set $X_1 = \{\overline{2a} | 0 \leq a < 2^{k-2}-1\}$ and use c_2 for vertices in the set $X_2 = \{\overline{2a} | 2^{k-2} \leq a \leq 2^{k-1}-1\}$. Now, let use c_3 for all vertices in $X_3 = \{\overline{2a+1} | 0 \leq a \leq 2^{k-2}-1\}$ and for the remaining vertices which are in $X_4 = \{\overline{2a+1} | 2^{k-2} \leq a \leq 2^{k-1}-1\}$ use the fourth color c_4. Then due to this coloring, all edges will be connecting two vertices of different colors. We conclude that the chromatic number of $\Gamma_{ur}(\mathbb{Z}_n)$ is exactly 4.

(x) By Theorem 3.4, we obtain

$$\begin{aligned}
M_1(\Gamma_{ur}(\mathbb{Z}_n)) &= \sum_{v \in V(G)} (deg_{\Gamma_{ur}(\mathbb{Z}_n)}(v))^2 \\
&= \sum_{v \in \{\overline{0}, \overline{2^{k-1}}\}} (deg_{\Gamma_{ur}(\mathbb{Z}_n)}(v))^2 + \sum_{v \in \mathbb{Z}_n \setminus \{\overline{0}, \overline{2^{k-1}}\}} (deg_{\Gamma_{ur}(\mathbb{Z}_n)}(v))^2 \\
&= 2(2^{k-1})^2 + (2^k - 2)(2^{k-1}+1)^2 \\
&= 2^{3k-2} + 2^{2k} - 2^k - 2.
\end{aligned}$$

(xi) Let $A_1 = \mathbb{Z}_n^o$ and let $A_2 = \mathbb{Z}_n^e$. We have $|A_1| = |A_2| = 2^{k-1}$. For any $i, j \in \{1, 2\}$, let $A_{ij} = \{\{\overline{x}, \overline{y}\} \subseteq \mathbb{Z}_n | x \neq y, \{\overline{x}, \overline{y}\} \cap A_i \neq \emptyset, \{\overline{x}, \overline{y}\} \cap A_j \neq \emptyset\}$. We obtain $|A_{ij}| = \frac{2^{k-1}(2^{k-1}-1)}{2}$ if $i = j$ and $|A_{ij}| = 2^{2k-2}$ if $i < j$. For any $\overline{x} \in A_1$ and $\overline{y} \in A_2$, $\overline{x}$ is adjacent to $\overline{y}$ by Theorem 3.4. For any $\overline{x}, \overline{y} \in A_1$, we have $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 1$ if $\overline{y} = -\overline{x}$. If $\overline{y} \neq -\overline{x}$, then $\overline{x} + \overline{y} \notin U_r(\mathbb{Z}_n)$ and hence $\overline{x}$ is not adjacent to $\overline{y}$. For any $\overline{z} \in A_2$, we have path $\overline{x} - \overline{z} - \overline{y}$. Hence, $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 2$ if $\overline{y} \neq -\overline{x}$. Now, let $\overline{x}, \overline{y} \in A_2$. If $\overline{x}, \overline{y} \notin \{\overline{0}, \overline{2^{k-1}}\}$, then if $\overline{y} = -\overline{x}$, $\overline{x}$ is adjacent to $\overline{y}$ and $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 1$ as a result. If $\overline{y} \neq -\overline{x}$, then $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 2$ as $\overline{x}$ and $\overline{y}$ are not adjacent and $\overline{x} - \overline{z} - \overline{y}$ is a path for any $\overline{z} \in A_1$. Thus, $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 2$. Therefore, among $\frac{2^{k-1}(2^{k-1}-1)}{2}$ elements of A_{11}, we have 2^{k-2} elements of the form $\{\overline{x}, -\overline{x}\}$, which is satisfying $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, -\overline{x}) = 1$. The remaining $\frac{2^{k-1}(2^{k-1}-1)}{2} - 2^{k-2} = 2^{2k-3} - 2^{k-1}$ elements of A_{11} are of the form $\{\overline{x}, \overline{y}\}$ such that $\overline{y} \neq \overline{x}$ satisfying $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 2$. From all $\frac{2^{k-1}(2^{k-1}-1)}{2}$ elements of A_{22}, we have $\frac{2^{k-1}-2}{2} = 2^{k-2}-1$ elements of the form $\{\overline{x}, -\overline{x}\}$, which is satisfying $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, -\overline{x}) = 1$. The remaining $\frac{2^{k-1}(2^{k-1}-1)}{2} - 2^{k-2} + 1$ elements of A_{22} are of the form $\{\overline{x}, \overline{y}\}$ such that $\overline{y} \neq \overline{x}$ satisfying $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 2$. For $i < j$, all

2^{2k-2} elements of A_{ij} are of the form $\{\overline{x}, \overline{y}\}$ such that $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 1$. Hence, we obtain the Wiener index of $\Gamma_{ur}(\mathbb{Z}_n)$ as the following:

$$\begin{aligned} W(\Gamma_{ur}(\mathbb{Z}_n)) &= 2^{k-2} + 2(2^{2k-3} - 2^{k-1}) + 2^{k-2} - 1 + 2(2^{2k-3} - 2^{k-1} + 1) + 2^{2k-2} \\ &= 3.2^{2k-2} + 2^{k-1} - 2^{k+1} + 1. \end{aligned}$$

(xii) Observe that $U_r(\mathbb{Z}_n) = \mathbb{Z}_n^o \cup \{\overline{0}\}$ and for any $\overline{x} \in \mathbb{Z}_n^o$, $\overline{x}$ is adjacent to $\overline{0}$. Moreover, every $\overline{x}, \overline{y} \in \mathbb{Z}_n^o$, $\overline{x}$ and $\overline{y}$ are adjacent in $\Gamma_{ur}(\mathbb{Z}_n)$ if and only if $\overline{y} = -\overline{x}$. As $\mathbb{Z}_n^o$ has 2^{k-1} elements, then there are precisely 2^{k-2} edges which end vertices are in $\mathbb{Z}_n^o$. This means that $\mathbb{Z}_n^o$ induces 2^{k-2} different path graphs of order 2. Since $\overline{0}$ is adjacent to any $\overline{x} \in \mathbb{Z}_n^o$, then there are 2^{k-2} different cycle subgraphs C_3's, each of order 3, which has $\overline{0}$ as a common vertex. Therefore, the subgraph of $\Gamma_{ur}(\mathbb{Z}_n)$ induced by $U_r(\mathbb{Z}_n)$ is a friendship graph $F_{2^{k-2}}$.

□

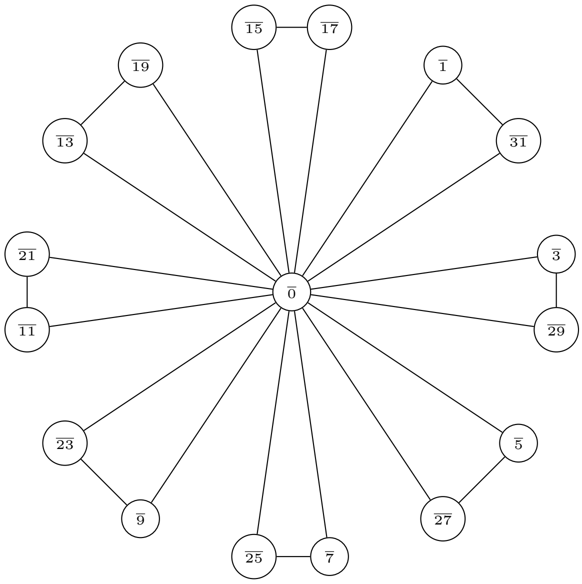

Figure 6. The unit regular graph $\Gamma_{ur}(\mathbb{Z}_{32})$

As a closing of this section, we give characterization on $\Gamma_{ur}(\mathbb{Z}_{p^k})$ for some prime number $p \geq 3$ and positive integer $k \geq 2$.

Theorem 3.6. *Given ring $\mathbb{Z}_n$ with $n = p^k$ for any prime number $p \geq 3$ and for any positive integer $k \geq 2$.*

(i) $U_r(\mathbb{Z}_n) = \mathbb{Z}_n \setminus \{\overline{x} \in \mathbb{Z}_n | gcd(x, n) \neq 1\} \cup \{\overline{0}\}$, *so that* $|U_r(\mathbb{Z}_n)| = p^k - p^{k-1} + 1$.

(ii) $deg_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}) = p^k - p^{k-1} + 1$ *if* $\{\overline{x} | x \equiv 0 \ mod \ p\} \setminus \{\overline{0}\}$ *and* $deg_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}) = p^k - p^{k-1}$ *otherwise.*

Proof. Let $p \geq 3$ be a prime number, $k \geq 2$ be a positive integer and let $n = p^k$.

(i) Observe that $U_r(\mathbb{Z}_n) = \mathbb{Z}_n \setminus \{\overline{x} \in \mathbb{Z}_n | gcd(x, n) \neq 1\} \cup \{\overline{0}\} = \{\overline{x} \in \mathbb{Z}_n | x \not\equiv 0 \ mod \ p\} \cup \{\overline{0}\}$. As $|\{\overline{x} \in \mathbb{Z}_n | x \equiv 0 \ mod \ p\}| = p^{k-1}$, then $|U_r(\mathbb{Z}_n)| = p^k - p^{k-1} + 1$.

(ii) Let $\overline{x} \in \mathbb{Z}_n \setminus U_r(\mathbb{Z}_n)$. Then $\overline{x} = \overline{rp}$ for some $1 \leq r < p^{k-1} - 1$. Then, $2\overline{x} \notin U_r(\mathbb{Z}_n)$, and hence $deg_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}) = p^k - p^{k-1} + 1$. Let $\overline{x} \in U_r(\mathbb{Z}_n)$. If $\overline{x} = \overline{0}$. Then, $2\overline{x} = \overline{0}$. Hence, $deg_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{0}) = p^k - p^{k-1}$. If $\overline{x} = \overline{rp+i}$ for some $0 \leq r < p^{k-1} - 1$ and $i \in \{1, 2, \ldots, p-1\}$, as $p \geq 3$, then $2\overline{x} \in U_r(\mathbb{Z}_n)$. And thus $deg_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}) = p^k - p^{k-1}$.

□

Theorem 3.7. *Given any prime number $p \geq 3$ and any positive integer $k \geq 2$. If $n = p^k$, then the following assertions hold.*

(i) *The degree sequence of unit regular graph $\Gamma(\mathbb{Z}_n)$ is*

$$(\underbrace{|U_r(\mathbb{Z}_n)| - 1, \ldots, |U_r(\mathbb{Z}_n)| - 1}_{p^k - p^{k-1}+1}, \underbrace{|U_r(\mathbb{Z}_n)|, \ldots, |U_r(\mathbb{Z}_n)|}_{p^{k-1}-1})$$

(ii) *$\Gamma_{ur}(\mathbb{Z}_n)$ is connected with diameter 2.*

(iii) *$\Gamma_{ur}(\mathbb{Z}_n)$ is Hamiltonian but not Eulerian.*

(iv) *$\Gamma_{ur}(\mathbb{Z}_n)$ is not planar.*

(v) *The independence number of $\Gamma_{ur}(\mathbb{Z}_n)$ is $\frac{p^{k-1}-1}{2}$.*

(vi) *The clique number of $\Gamma_{ur}(\mathbb{Z}_n)$ is $\frac{p-1}{2}(p^{k-1}) + 2$.*

(vii) *The domination number of $\Gamma_{ur}(\mathbb{Z}_n)$ is 2.*

(viii) *The matching number of $\Gamma_{ur}(\mathbb{Z}_n)$ is $\frac{p^k-1}{2}$.*

(ix) *The chromatic number of $\Gamma_{ur}(\mathbb{Z}_n)$ is $\frac{p-1}{2}(p^{k-1}) + 2$.*

(x) *The first Zagreb index of $\Gamma_{ur}(\mathbb{Z}_n)$ is $M_1(\Gamma_{ur}(\mathbb{Z}_n)) = p^{3k} - 2p^{3k-1} + p^{3k-2} + 2p^{2k-1} - 2p^{2k-2} - 2p^k + 3p^{k-1} - 1$.*

(xi) *The Wiener index of $\Gamma_{ur}(\mathbb{Z}_n)$ is $\frac{1}{2}p^{2k} + \frac{1}{2}p^{2k-1} - p^k - \frac{1}{2}p^{k-1} + \frac{1}{2}$.*

Proof. Let $\overline{\overline{a}} = \{\overline{x} \in \mathbb{Z}_n | x \equiv a \ mod \ p\}$ for any $a = 0, 1, \ldots, p-1$. This set is of size p^{k-1}.

(i) By Theorem 3.6.

(ii) By Theorem 3.6, $\Gamma_{ur}(\mathbb{Z}_n)$ is not complete, and hence its diameter is greater than 1. For any $a = 0, 1, \ldots, p-1$, let $\overline{\overline{a}} = \{\overline{x} \in \mathbb{Z}_n | x \equiv a \ mod \ p\} = \{\overline{ip+a} \in \mathbb{Z}_n | i = 0, 1, \ldots, p^{k-1} - 1\}$. Let $\overline{x} \in \overline{\overline{a}}$ and $\overline{y} \in \overline{\overline{b}}$ for some $a, b = 0, 1, \ldots, p-1$. If $a = b \neq 0$, then $\overline{x}$ and $\overline{y}$ are adjacent and thus $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 1$. If $a = b = 0$, then $\overline{x}$ and $\overline{y}$ are adjacent if and only if $\overline{y} = -\overline{x}$. For $\overline{y} \neq -\overline{x}$, we have path $\overline{y} - \overline{1} - \overline{x}$ of length 2. Thus, $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y})$ is 1 or 2. Let $a \neq b$. If $a + b \neq p$, then $\overline{x}$ and $\overline{y}$ are adjacent. If $a + b = p$, then we always have a path $\overline{x} - \overline{0} - \overline{y}$. Hence, $\Gamma_{ur}(\mathbb{Z}_n)$ is connected with diameter 2.

(iii) Let x and y be connected by an edge in $\Gamma_{ur}(\mathbb{Z}_n)$. The sum of the degrees of x and y is greater than p^k. Therefore, by Ore's theorem, $\Gamma_{ur}(\mathbb{Z}_n)$ is Hamiltonian. By Theorem 3.6, the vertices of $\Gamma_{ur}(\mathbb{Z}_n)$ have different parity degrees. Hence, $\Gamma_{ur}(\mathbb{Z}_n)$ is not Eulerian.

(iv) For $a = 0$ and for any $b = 1, 2, \ldots, p-1$, each vertex in $\overline{\overline{a}}$ is adjacent to any vertex in $\overline{\overline{b}}$. Therefore, $\Gamma_{ur}(\mathbb{Z}_n)$ contains a bipartite graph $K_{p^{k-1}, p^{k-1}}$. As $p \geq 3$ and $k \geq 2$, then by the Kuratowski's theorem, $\Gamma_{ur}(\mathbb{Z}_n)$ is not planar.

(v) If I is a maximal independent set, then I cannot have an element from $U_r(\mathbb{Z}_n)$ and an element from $\mathbb{Z}_n \setminus U_r(\mathbb{Z}_n)$, simultaneously. Moreover, since $\overline{0}$ is adjacent to every $\overline{x} \in U_r(\mathbb{Z}_n)$, necessarily $I \subseteq U_r(\mathbb{Z}_n) \setminus \{\overline{0}\}$ or $I \subseteq \mathbb{Z}_n \setminus U_r(\mathbb{Z}_n)$. Obviously, for any $\overline{x}, \overline{y} \in \mathbb{Z}_n \setminus U_r(\mathbb{Z}_n)$, $\overline{x}$ and $\overline{y}$ are adjacent if and only if $\overline{y} = -\overline{x}$. Thus, if $I \subseteq \mathbb{Z}_n \setminus U_r(\mathbb{Z}_n)$, then $|I| = \frac{p^{k-1}-1}{2}$. Now, consider when $I \subseteq U_r(\mathbb{Z}_n) \setminus \{\overline{0}\}$. Let $\overline{x}, \overline{y} \in U_r(\mathbb{Z}_n) \setminus \{\overline{0}\}$. Let $\overline{x} = \overline{ip+a} \in \overline{\overline{a}}$ and $\overline{y} = \overline{jp+b} \in \overline{\overline{b}}$, $a, b = 1, 2, \ldots, p-1$, $i, j = 0, 1, \ldots, p^{k-1}-1$. Then $\overline{x}$ is not adjacent to $\overline{y}$ if and only if $a+b = p$ and $i+j \neq p^{k-1}-1$. Therefore, the only largest independent set is reached only when $I \subseteq \mathbb{Z}_n \setminus U_r(\mathbb{Z}_n)$ where I has size $\frac{p^{k-1}-1}{2}$.

(vi) We have that $\overline{\overline{a}}$ is a clique for any $a = 1, \ldots, p-1$. Moreover, for $a, b \neq 0$ and $a+b \neq p$, each vertex in $\overline{\overline{a}}$ is adjacent to any vertex in $\overline{\overline{b}}$. Therefore, if $a+b \neq p$, then $\overline{\overline{a}} \cup \overline{\overline{b}}$ is a clique. But the largest subset A of $\{1, 2, \ldots, p-1\}$ such that $a+b \neq p$ for any $a, b \in A$ has size $\frac{p-1}{2}$. Hence, the clique number of $\Gamma_{ur}(\mathbb{Z}_n)$ is greater or equal to $\frac{p-1}{2}(p^{k-1})$. Now, let C be the largest clique in $\Gamma_{ur}(\mathbb{Z}_n)$. If $C \cap \overline{\overline{0}} \neq \emptyset$, then $|C \cap \overline{\overline{0}}| \leq 2$. If $C \cap \overline{\overline{a}} \neq \emptyset$ and $C \cap \overline{\overline{b}} \neq \emptyset$ and $a+b = p$, then $|C \cap \overline{\overline{a}}| \leq 1$ and $|C \cap \overline{\overline{b}}| \leq 1$. If $C \cap \overline{\overline{a}} \neq \emptyset$ and $C \cap \overline{\overline{b}} \neq \emptyset$ and $a+b \neq p$, then $|C \cap \overline{\overline{a}}| \leq p^{k-1}$ and $|C \cap \overline{\overline{b}}| \leq p^{k-1}$. Therefore, the largest C will be reached whenever C contains $\bigcup_{a\in A} \overline{\overline{a}}$ where A is the largest subset of $\{1, 2, \ldots, p-1\}$ such that for any $u, v \in A$, $u+v \neq p$. Thus, $|A| = \frac{p-1}{2}$. Therefore, $\bigcup_{a\in A} \overline{\overline{a}}$ has $\frac{p-1}{2}(p^{k-1})$ vertices. Now, take two elements $\overline{x}, \overline{y} \in \overline{\overline{0}}$ such that $\overline{y} = -\overline{x}$. Then $\overline{x}$ is connected by an edge to $\overline{y}$. Moreover, $\overline{x}$ and $\overline{y}$ are connected to any vertex in $\bigcup_{a\in A} \overline{\overline{a}}$ by some edges. Hence $\bigcup_{a\in A} \overline{\overline{a}} \cup \{\overline{x}, \overline{y}\}$ is of size $\frac{p-1}{2}(p^{k-1}) + 2$ and defines the largest clique. This completes the proof.

(vii) As $\Gamma_{ur}(\mathbb{Z}_n)$ is not complete, then its domination number is greater then 1. Consider $\overline{\overline{a}} = \{\overline{x} \in \mathbb{Z}_n | x \equiv a \ mod \ p\}$, $a = 0, 1, \ldots, p-1$ and put $D = \{\overline{0}, \overline{1}\}$. Then, each $\overline{x} \in \overline{\overline{a}}$, $a \neq 0$, is adjacent to $\overline{0} \in D$. For each $\overline{x} \in \overline{\overline{0}}$, $\overline{x} \neq \overline{0}$, $\overline{x}$ is adjacent to $\overline{1} \in D$. Hence, D is a dominating set. Therefore, the domination number of $\Gamma_{ur}(\mathbb{Z}_n)$ is 2.

(viii) Observe that the order of $\Gamma_{ur}(\mathbb{Z}_n)$ is p^k which is odd. Thus, a perfect matching will be not reached. Thus, the maximum size of a matching in $\Gamma_{ur}(\mathbb{Z}_n)$ is $\frac{p^k-1}{2}$. Let $\overline{\overline{a}} = \{\overline{x} \in \mathbb{Z}_n | x \equiv a \ mod \ p\}$, $a = 0, 1, \ldots, p-1$. Let $A = \{\{\overline{rp}, \overline{(p^{k-1}-r)p}\} | r = 1, 2, \ldots, \frac{p^{k-1}-1}{2}\}$. Hence, A contains $\frac{p^{k-1}-1}{2}$ edges connecting two vertices in $\overline{\overline{0}}$. For any $i = 1, 2, \ldots, \frac{p-1}{2}$, let $B_{i,p-i} = \{\{\overline{rp+i}, \overline{(p^{k-1}-r)p+j} | i+j = p^k\}$. Each $B_{i,p-i}$ contains p^{k-1} edges connecting a vertex in $\overline{\overline{i}}$ to a vertex in $\overline{\overline{p-i}}$. Altogether, $A \cup \bigcup_{i=1}^{\frac{p-1}{2}} B_{i,p-i}$ is a matching of size $\frac{p^k-1}{2}$ covering all nonzero vertices in $\Gamma_{ur}(\mathbb{Z}_n)$. This completes the proof.

(ix) By (vi), the chromatic number of $\Gamma_{ur}(\mathbb{Z}_n)$ is greater or equal $\frac{p-1}{2}p^{k-1} + 2$. Let $\overline{\overline{a}} = \{\overline{x} \in \mathbb{Z}_n | x \equiv a \ mod \ p\}$, $a = 0, 1, \ldots, p-1$. Now, for $i = 1, \ldots, \frac{p-1}{2}$, let color each vertex $\overline{rp+i} \in \overline{\overline{i}}$ with color $c_{r,i}$, $r = 0, 1, \ldots, p^{k-1}-1$. For $i = \frac{p-1}{2}, \ldots, p-1$, let color each vertex $\overline{rp+i} \in \overline{\overline{i}}$ with color $c_{r,(i-\frac{p-1}{2})}$, with $r = 0, 1, \ldots, p^{k-1}-1$. So far, we have used $\frac{p-1}{2}p^{k-1}$ colors. Now, take two more new colors, let say c and c'. Let use c to color vertices $\overline{rp} \in \overline{\overline{0}}$, where $r = 0, \ldots, \frac{p^{k-1}-1}{2}$ and use c' to color vertices $\overline{rp} \in \overline{\overline{0}}$, where $r = \frac{p^{k-1}-1}{2} + 1, \ldots, p^{k-1}-1$. Then by this coloring, we have that each edge in $\Gamma_{ur}(\mathbb{Z}_n)$ connects two vertices of different colors. Thus, we conclude that the chromatic number of $\Gamma_{ur}(\mathbb{Z}_n)$ is $\frac{p-1}{2}p^{k-1} + 2$.

(x) By point (i), we have

$$\begin{aligned} M_1(\Gamma_{ur}(\mathbb{Z}_n)) &= \sum_{\overline{x}\in\mathbb{Z}_n} (deg_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}))^2 \\ &= (p^k - p^{k-1} + 1)((p^{k-1}-1)(p^k - p^{k-1} + 1) + (p^k - p^{k-1})^2) \\ &= p^{3k} - 2p^{3k-1} + p^{3k-2} + 2p^{2k-1} - 2p^{2k-2} - 2p^k + 3p^{k-1} - 1. \end{aligned}$$

(xi) For any $i, j \in \{0, 1, \ldots, p-1\}$, let $A_{ij} = \{\{\overline{x}, \overline{y}\} \subseteq \mathbb{Z}_n | \overline{x} \neq \overline{y}, \{\overline{x}, \overline{y}\} \cap \overline{\overline{i}} \neq \emptyset, \{\overline{x}, \overline{y}\} \cap \overline{\overline{j}} \neq \emptyset\}$. We obtain $|A_{ij}| = \frac{p^{k-1}(p^{k-1}-1)}{2}$ if $i = j$ and $|A_{ij}| = p^{2k-2}$ if $i < j$. For any $\overline{x}, \overline{y} \in \overline{\overline{0}}$, we have $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 1$ if $\overline{y} = -\overline{x}$. If $\overline{y} \neq -\overline{x}$, then $\overline{x} + \overline{y} \notin U_r(\mathbb{Z}_n)$ and hence $\overline{x}$ and $\overline{y}$ are not adjacent. But, for any $\overline{z} \in \overline{\overline{1}}$, we have path $\overline{x} - \overline{z} - \overline{y}$. Hence, $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 2$ if $\overline{y} \neq -\overline{x}$. As there are $\frac{p^{k-1}-1}{2}$ pairs of the form $\{\overline{x}, -\overline{x}\}$, then there are $\frac{p^{k-1}(p^{k-1}-1)}{2} - \frac{p^{k-1}-1}{2} = \frac{(p^{k-1}-1)^2}{2}$ pairs of the form $\{\overline{x}, \overline{y}\}$ where $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 2$. If $\overline{x}, \overline{y} \in \overline{\overline{a}}$, $a \neq 0$, then $\overline{x}$ and $\overline{y}$ are adjacent and thus $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 1$. Now let $\overline{x} \in \overline{\overline{i}}$ and $\overline{y} \in \overline{\overline{j}}$, where $i < j$, and $i, j = 0, 1, \ldots, p-1$. If $i = 0$, then $\overline{x}$ is adjacent to $\overline{y}$. Now, let $i \neq 0$. If $i + j \neq n$, then $\overline{x}$ and $\overline{y}$ are connected by an edge. If $i + j = n$, then $\overline{x}$ and $\overline{y}$ are adjacent if and only if $\overline{x} = \overline{rp + i}$ and $\overline{y} = \overline{(p^{k-1} - r)p + j}$. Thus, for each $\overline{x} \in \overline{\overline{i}}$, $i = 1, 2, \ldots, \frac{p-1}{2}$, there are $1 + (p - 2 - i)p^{k-1}$ $\overline{y}$'s such that $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 1$. The remaining $p^{k-1} - 1$ y's are satisfying $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 2$ as we always have a path $\overline{x} - \overline{0} - \overline{y}$ of length 2. If $i = \frac{p-1}{2}, \ldots, p-1$, then $\overline{x}$ and $\overline{y}$ are adjacent so that $d_{\Gamma_{ur}(\mathbb{Z}_n)}(\overline{x}, \overline{y}) = 1$. If $i = \frac{p-1}{2} + 1, \ldots, p-1$, then $\overline{x}$ is connected by an edge to $\overline{y}$.

Therefore, the Wiener index of $\Gamma_{ur}(\mathbb{Z}_n)$ as the following:

$$\begin{aligned}
W(\Gamma_{ur}(\mathbb{Z}_n)) = & \frac{p^{k-1}-1}{2} + 2\frac{(p^{k-1}-1)^2}{2} + (p-1)\frac{(p^{k-1}-1)p^{k-1}}{2} \\
& + (p-1)(p^{k-1})^2 + \sum_{i=1}^{\frac{p-1}{2}} (p-2-i)(p^{k-1})^2 + \frac{p-1}{2}p^{k-1} \\
& + 2\frac{p-1}{2}(p^{k-1})(p^{k-1}-1) + \sum_{i=\frac{p-1}{2}+1}^{p-1} (p-1-i)(p^{k-1})^2 \\
= & \frac{1}{2}p^{2k} + \frac{1}{2}p^{2k-1} - p^k - \frac{1}{2}p^{k-1} + \frac{1}{2}.
\end{aligned}$$

□

4. Conclusion

In this manuscript we propose the definition of the unit regular graph over rings. We present some fundamental properties of the graph included its completeness, Eulerian and Hamiltonian property, connectivity, girth, matching number, independence number, and clique number. Moreover, we present also some more detail properties of the graph for the ring $\mathbb{Z}_n$ for some particular n. Nevertheless, more general properties of the unit regular graph of the ring $\mathbb{Z}_n$ for arbitrary n have not been obtained. Therefore we give the following open problems.

Open Problem 1 Investigate the characteristic of the unit regular graph over the ring $\mathbb{Z}_n$, for arbitrary n.

Open Problem 2 Investigate the characteristic of the unit regular graph over the ring $M_k(\mathbb{Z}_n)$ of all $k \times k$ matrices over $\mathbb{Z}_n$ for any $k \geq 3$.

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