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Quasi-self-dual, right LCD and ACD codes over a noncommutative non-unital ring

Research Article

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Abstract: This paper presents the study of QSD (quasi-self-dual), right-LCD (linear complementary dual), and ACD (additive complementary dual) codes over a noncommutative local ring $R = \langle a, b \mid 3a = 3b = 0,\ a^2 = a,\ b^2 = b,\ ab = b,\ ba = a\rangle$ of order 9. Initially, over this ring R, we introduce QSD codes and characterize their multilevel construction. Then, we delve into the study of right LCD codes over the ring R and demonstrate a method for constructing these codes based on ternary LCD codes. Finally, we introduce the right-ACD codes over this ring and present several criteria for the existence of such codes.

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1. Introduction

The construction of effective error-correcting codes stands out as a highly consequential challenge in the area of coding theory. Algebraic coding theory has developed significantly since the late 1940s when Hamming and Shannon introduced error-correcting codes. This progress encompasses various advancements in linear codes, including BCH codes, Reed-Muller codes, Reed-Solomon codes, quadratic residue codes, self-dual codes, and algebraic geometry codes. Further, the research of self-dual codes has gained considerable attention, primarily due to their associations with quantum error-correcting codes, lattices, and designs. Initially, our primary objective is to examine self-dual codes over the ring R.

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However, it became evident that the conventional relationship between a code's size and its dual size is not always true, as explained in the concept of a "nice code" discussed in Section 4. Recently, Alahmadi et al. [1] introduced QSD codes over a finite non-unital noncommutative ring E of order 4. Further, Kim et al. [6] constructed QSD codes over a non-unital commutative ring with four elements. Later, Kim et al. demonstrated that the ring E possesses the GC-content map and complement map, making it suitable for constructing DNA codes [5]. These aspects motivate us to study QSD codes over a noncommutative non-unital ring $R = \langle a, b \mid 3a = 3b = 0,\ a^2 = a,\ b^2 = b,\ ab = b,\ ba = a \rangle$ of order 9. A QSD code over R refers to a self-orthogonal R-code of length n with size 3^n. Here, we investigate the structure of QSD codes, emphasizing a multilevel construction based on a pair of dual codes.

Subsequently, we move into another class of codes that are characterized by their connection with respective dual codes. More precisely, it constitutes the family of linear codes having complementary dual (LCD). Recall that a linear code C over a finite field $\mathbb{F}_q$ is denoted by $[n, k]$ and refers to a subspace of $\mathbb{F}_q^n$ of dimension k. The dual of the code C, denoted by $C^\perp$, represents the collection of all the orthogonal vectors to C under the Euclidean inner product. Also, a linear code C is called self-orthogonal if $C \subseteq C^\perp$. A linear code C is called an LCD code if there is no common non-zero vector to both C and its dual, i.e., $C \cap C^\perp = \{\mathbf{0}\}$. In 1992, Massey [11] introduced the concept of LCD codes over finite fields and demonstrated the existence of asymptotically good LCD codes. Recently, these codes received considerable attention due to their application in Boolean masking, a robust countermeasure for algorithms in Cryptography [2]. Moreover, Sendrier [14] utilized the hull dimension spectra of linear codes to demonstrate that LCD codes achieve the Gilbert-Varshamov bound. In 2016, Carlet et al. [2] introduced multiple constructions of LCD codes and explored their application in defending against Fault Injection Attacks (FIA) and Side-Channel Attacks (SCA). SCA involves passively recording information leakage to retrieve the key, while FIA involves actively disrupting the computation to obtain exploitable variations at the output. Interestingly, the concept of LCD codes and self-orthogonal codes are both used in our study to develop certain results.

On the other side, in the Boolean masking approach of Carlet-Guilley, the condition $C \oplus C^\perp = \mathbb{F}_q^n$ is necessary, and the minimum distance of C (respectively $C^\perp$) acting as a performance criterion for SCA (and FIA), respectively. Since this model leverages the additivity of C rather than its linearity, it becomes relevant to study ACD codes over finite fields (finite rings). Additionally, ACD codes encompass LCD codes as they include linear codes. Under specific conditions, in 2015, Liu et al. [8] obtained numerous conditions for linear codes over a finite chain ring to be LCD. Li [7] constructed Hermitian LCD cyclic codes over finite fields and investigated their parameters. Later, Liu et al. [9] used a Gray map to extract LCD codes over finite fields from linear codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$. In 2019, Zihui et al. [10] studied LCD codes over finite commutative rings. In 2020, Prakash et al. [13] constructed LCD codes over the ring $\mathcal{R} = \mathbb{F}_q + u\mathbb{F}_q$, where $u^2 = 1$ and q is a power of an odd prime p. Additionally, they have demonstrated the application of LCD codes within multi-secret sharing schemes. Later, in 2021, Islam et al. [3] studied cyclic codes over a non-chain ring $R_{e,q}$ and their application to LCD codes. Recently, there has been extensive research on LCD codes over various structures [4, 12, 15, 17–19].

As per our survey, the investigation of LCD and ACD codes over non-unital rings first emerged in [16], in which authors focused on studying left-LCD and left-ACD codes. This motivates us to study right-LCD, and right-ACD codes over a non-unital noncommutative ring with nine elements. Our research demonstrates that free LCD codes over R encompass ternary LCD codes, and ACD codes over R encompass free LCD codes over R as a specific case. These observations highlight the importance of studying LCD codes over R. In particular, we characterize a free LCD R-code C in terms of a ternary generator matrix G. Additionally, we introduce the concept of an ACD code over R, called the right-ACD code. We present various results for the existence of right-ACD codes over R.

This paper is organized as follows: Section 2 discusses the study of codes over the ring R and the duality of an R-code. Section 3 deals with QSD codes over R while Section 4 studies LCD codes over R. Section 5 focuses on the study of right-ACD codes over R. Finally, Section 6 concludes the work.

2. Codes over a noncommutative non-unital local ring of order 9

Let R be a ring generated by a and b under certain relations as follows:

$$R = \langle a, b \mid 3a = 3b = 0,\ a^2 = a,\ b^2 = b,\ ab = b,\ ba = a \rangle.$$

For example, consider the ring R generated by two matrices a and b over $\mathbb{F}_3$ where

$$a = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Thus, the ring R consists of 9 elements $R = \{0,\ a,\ b,\ 2a,\ 2b,\ a+b,\ 2a+b,\ a+2b,\ 2a+2b\}$ and has characteristics 3. One can easily derive its addition table. The multiplication table is given in Table 1.

Table 1. Multiplication table for R.

$\cdot$	0	a	b	$2a$	$2b$	$a+b$	$2a+b$	$a+2b$	$2a+2b$
0	0	0	0	0	0	0	0	0	0
a	0	a	b	$2a$	$2b$	$a+b$	$2a+b$	$a+2b$	$2a+2b$
b	0	a	b	$2a$	$2b$	$a+b$	$2a+b$	$a+2b$	$2a+2b$
$2a$	0	$2a$	$2b$	a	b	$2a+2b$	$a+2b$	$2a+b$	$a+b$
$2b$	0	$2a$	$2b$	a	b	$2a+2b$	$a+2b$	$2a+b$	$a+b$
$a+b$	0	$2a$	$2b$	a	b	$2a+2b$	$a+2b$	$2a+b$	$a+b$
$2a+b$	0	0	0	0	0	0	0	0	0
$a+2b$	0	0	0	0	0	0	0	0	0
$2a+2b$	0	a	b	$2a$	$2b$	$a+b$	$2a+b$	$a+2b$	$2a+2b$

The above table shows that the ring R is noncommutative and has no identity element with respect to the given multiplication. Also, $ax = x$ and $bx = x$ for all $x \in R$. Further, the ring R has a unique maximal ideal $I = \{0, 2a+b, a+2b\}$, and hence it is a local ring with residue class field $R/I \cong \mathbb{F}_3 = \{0, 1, 2\}$, the finite field of order 3. Let $c = a + 2b$. Then every element $r \in R$ has a c-adic decomposition as follows:

$$r = xa + yc \quad \text{where } x, y \in \mathbb{F}_3.$$

Define a natural action of $\mathbb{F}_3$ on R as

$$r0 = 0r = 0, \quad 1r = r1 = r, \quad 2r = r2 = 2r.$$

Note that this action is distributive, i.e., for all $r \in R$, we have $r(x \oplus_{\mathbb{F}_3} y) = rx + ry$ where $\oplus_{\mathbb{F}_3}$ denotes the addition in $\mathbb{F}_3$.

Next, define a map $\alpha : R \to R/I = \mathbb{F}_3$ by

$$\alpha(0) = \alpha(2a+b) = \alpha(a+2b) = 0,\ \alpha(a) = \alpha(b) = \alpha(2a+2b) = 1,\ \alpha(2a) = \alpha(2b) = \alpha(a+b) = 2.$$

This map can be extended in a natural way from R^n to $\mathbb{F}_3^n$ and is known as the map of reduction modulo I.

Definition 2.1 (Linear and additive codes over R)**.** *A right R-submodule of R^n is called a linear code over R or a linear R-code, while any additive subgroup of R^n is called an additive R-code.*

Definition 2.2 (Permutation-equivalent codes)**.** *A code C_1 is called permutation-equivalent to a code C_2 if and only if there exists a coordinate permutation from C_1 to C_2.*

We define two ternary linear codes associated with a linear code C over R of length n.

(i) **Residue code:** We define the residue code of the code C by

$$Res(C) = \{\alpha(\boldsymbol{x}) \mid \boldsymbol{x} \in C\}.$$

(ii) **Torsion code:** The torsion code of the code C is defined by

$$Tor(C) = \{\boldsymbol{v} \in \mathbb{F}_3^n \mid \boldsymbol{v}c \in C\},$$

where $c = a + 2b$.

Remark 2.3. *Throughout the paper, a and b represent the generators of the ring R satisfying the relations $3a = 3b = 0$, $a^2 = a$, $b^2 = b$, $ab = b$, $ba = a$ and $c = a + 2b$. Also, we denote the dimension of the residue code by k_1 and the dimension of the torsion code by $k_1 + k_2$.*

2.1. Duality

For any $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \dots, y_n) \in R^n$, define an inner product on R^n as follows

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{j=1}^{n} x_j y_j.$$

Under this inner product, we define two duals as follows.

(i) **Right dual:** The right dual of the code C is defined by

$$C^{\perp_R} = \{\boldsymbol{y} \in R^n \mid \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0, \forall\, \boldsymbol{x} \in C\}.$$

Here, the right dual of the code C forms a right R-module.

(ii) **Left dual:** The left dual of the code C is defined by

$$C^{\perp_L} = \{\boldsymbol{y} \in R^n \mid \langle \boldsymbol{y}, \boldsymbol{x} \rangle = 0, \forall\, \boldsymbol{x} \in C\}.$$

Also, the left dual of the code C forms a left R-module.

Definition 2.4 (Self-orthogonal code)**.** *A linear R-code C is called self-orthogonal code if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$ for all $\boldsymbol{x}, \boldsymbol{y} \in C$.*

Definition 2.5 (Self-dual code)**.** *An R-code C is called right self-dual if $C = C^{\perp_R}$, and is called left self-dual if $C = C^{\perp_L}$. Hence, a code C is called self-dual if $C = C^{\perp_R} = C^{\perp_L}$.*

3. Quasi-self-dual (QSD) codes over R

This section studies the QSD codes over the ring R and demonstrates their multilevel construction from ternary linear codes.

Definition 3.1 (Quasi-self-dual code)**.** *An R-linear code C is called a quasi-self-dual code if it is self-orthogonal and has size 3^n.*

Next, we present the multilevel construction of QSD codes by utilizing ternary linear codes.

Theorem 3.2. *Let D be a ternary linear code of length n. If D is self-orthogonal, then the code C constructed through the relationship*

$$C = Da + D^{\perp}c,$$

is a QSD code. Additionally, $Res(C) = D$ and $Tor(C) = D^{\perp}$.

Proof. Let $C = Da + D^{\perp}c$. First, we prove that C is an R-linear code. By using the linearity of D and $D^{\perp}$, we see that the code C is closed under addition. Also, every element $r \in R$ has a c-adic decomposition

$$r = xa + yc \quad \text{where } x, y \in \mathbb{F}_3.$$

Let $\boldsymbol{x}' \in C$. Then $\boldsymbol{x}'$ can be written as $\boldsymbol{x}' = \boldsymbol{d}a + \boldsymbol{d}'c$, where $\boldsymbol{d} \in D$ and $\boldsymbol{d}' \in D^{\perp}$. Hence, we have

$$\begin{aligned} \boldsymbol{x}'r = &\ (\boldsymbol{d}a + \boldsymbol{d}'c)(xa + yc) \\ = &\ \boldsymbol{d}axa + \boldsymbol{d}ayc + \boldsymbol{d}'cxa + \boldsymbol{d}'cyc \\ = &\ \boldsymbol{d}xa + \boldsymbol{d}yc \\ = &\ (\boldsymbol{d}x)a + (\boldsymbol{d}y)c. \end{aligned}$$

Since $\boldsymbol{d} \in D$, $x, y \in \mathbb{F}_3$ and D is a ternary linear code, we see that $\boldsymbol{d}x, \boldsymbol{d}y \in D$. Moreover, $\boldsymbol{d}y \in D^{\perp}$ by using self-orthogonality of D. Hence, $\boldsymbol{x}'r \in C$. Thus, C is a linear code over R. Further, let $\boldsymbol{x} = \boldsymbol{x}_1a + \boldsymbol{x}'c$, $\boldsymbol{y} = \boldsymbol{y}_1a + \boldsymbol{y}'c \in C$ for some $\boldsymbol{x}_1, \boldsymbol{y}_1 \in D$ and $\boldsymbol{x}', \boldsymbol{y}' \in D^{\perp}$. Then, their inner product

$$\begin{aligned} \langle \boldsymbol{x}, \boldsymbol{y} \rangle &= \langle \boldsymbol{x}_1a + \boldsymbol{x}'c,\ \boldsymbol{y}_1a + \boldsymbol{y}'c \rangle \\ &= \langle \boldsymbol{x}_1a, \boldsymbol{y}_1a \rangle + \langle \boldsymbol{x}'c, \boldsymbol{y}_1a \rangle + \langle \boldsymbol{x}_1a, \boldsymbol{y}'c \rangle + \langle \boldsymbol{x}'c, \boldsymbol{y}'c \rangle \\ &= \langle \boldsymbol{x}_1, \boldsymbol{y}_1 \rangle a + \langle \boldsymbol{x}_1, \boldsymbol{y}' \rangle c \\ &= 0, \end{aligned}$$

as D is self-orthogonal. Hence, C is self-orthogonal. Now, $C = Da + D^{\perp}c$ implies that $|C| = |D||D^{\perp}| = 3^n$. Thus, C is a QSD code. Moreover, the torsion and residue codes can be derived easily just by applying their definitions. □

The next result shows that Theorem 3.2 does not hold if D is not self-orthogonal.

Theorem 3.3. *Let D be a ternary linear code and C be a code over R defined by the relationship $C = Da + D^{\perp}c$. Then C is always an additive code over R but never be a linear R-code unless and until D is self-orthogonal.*

Proof. Let D be a ternary linear code and $C = Da + D^{\perp}c$. Then C is an additive code over R by using the linearity of D and $D^{\perp}$. Now, suppose C is a linear R-code and let $\boldsymbol{d} \in D$. Then there exists some $\boldsymbol{d}' \in D^{\perp}$ such that $\boldsymbol{x}' = \boldsymbol{d}a + \boldsymbol{d}'c \in C$. Moreover, every element $r \in R$ has a c-adic decomposition form $r = xa + yc$ where $x, y \in \mathbb{F}_3$. Next,

$$\boldsymbol{x}'r = (\boldsymbol{d}a + \boldsymbol{d}'c)(xa + yc) = \boldsymbol{d}axa + \boldsymbol{d}ayc + \boldsymbol{d}'cxa + \boldsymbol{d}'cyc = \boldsymbol{d}xa + \boldsymbol{d}yc = (\boldsymbol{d}x)a + (\boldsymbol{d}y)c.$$

Since C is linear, $\boldsymbol{x}'r \in C$, we have $\boldsymbol{d}y \in D^{\perp}$. This implies that $\boldsymbol{d}yy^{-1} = \boldsymbol{d} \in D^{\perp}$ where y^{-1} denotes the multiplicative inverse of $y \in \mathbb{F}_3$. Therefore, $D \subseteq D^{\perp}$. Thus, D is self-orthogonal. □

The following two results are true without the QSD requirement.

Lemma 3.4. *Let C be a linear code over R and $\boldsymbol{x}a + \boldsymbol{y}c$ be an arbitrary codeword of C where $\boldsymbol{x}$ and $\boldsymbol{y}$ are ternary vectors. Then $\boldsymbol{x} \in Res(C)$, $Res(C)a \subseteq C$ and $\boldsymbol{y} \in Tor(C)$.*

Proof. Let $\boldsymbol{x}a + \boldsymbol{y}c$ be an arbitrary codeword of C. Then $\alpha(\boldsymbol{x}a + \boldsymbol{y}c) = \boldsymbol{x}$ implies that $\boldsymbol{x} \in Res(C)$. Also, for any $\boldsymbol{x} \in Res(C)$, there exists $\boldsymbol{x}a + \boldsymbol{y}c \in C$ such that $\alpha(\boldsymbol{x}a + \boldsymbol{y}c) = \boldsymbol{x}$. Since C is linear, $(\boldsymbol{x}a + \boldsymbol{y}c)a = \boldsymbol{x}a \in C$. Hence, $Res(C)a \subseteq C$. Further, by using linearity of C, $(\boldsymbol{x}a + \boldsymbol{y}c) + 2\boldsymbol{x}a = \boldsymbol{y}c \in C$. Thus, $\boldsymbol{y} \in Tor(C)$. □

Theorem 3.5. *If C is a linear R-code, then $C = Res(C)a \oplus Tor(C)c$ as modules.*

Proof. Any arbitrary codeword $\boldsymbol{x}$ of C has a c-adic decomposition form $\boldsymbol{x} = \boldsymbol{u}a + \boldsymbol{v}c$ for some ternary vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Since $\alpha(\boldsymbol{x}) = \alpha(\boldsymbol{u}a + \boldsymbol{v}c) = \boldsymbol{u}$, $\boldsymbol{u} \in Res(C)$. Then, by Lemma 3.4, $\boldsymbol{u}a \in C$. Also, linearity of C implies that $(\boldsymbol{u}a + \boldsymbol{v}c) + 2\boldsymbol{u}a = \boldsymbol{v}c \in C$. This shows that $\boldsymbol{v} \in Tor(C)$. Therefore, $C \subseteq Res(C)a + Tor(C)c$. The converse inclusion holds by using linearity of C and $Res(C)a \subseteq C$, $Tor(C)c \subseteq C$. Hence, $C = Res(C)a + Tor(C)c$. Now, it remains to show that this sum is direct. Let $\boldsymbol{x} \in Res(C)a \cap Tor(C)c$. Then $\boldsymbol{x} = \boldsymbol{u}a = \boldsymbol{v}c$ for some $\boldsymbol{u} \in Res(C)$ and $\boldsymbol{v} \in Tor(C)$. Hence, $\boldsymbol{u}a = \boldsymbol{v}c$ implies that $(\boldsymbol{u}a)a = (\boldsymbol{v}c)a$. Therefore, $\boldsymbol{u}a = \boldsymbol{0}$. That is, $\boldsymbol{x} = \boldsymbol{0}$. Thus, the given sum is direct. □

Lemma 3.6. *A linear R-code C is self-orthogonal if and only if $Res(C) \subseteq Tor(C)^{\perp}$. Moreover, C is QSD if and only if $Res(C) = Tor(C)^{\perp}$.*

Proof. Let C be a self-orthogonal code over R. For any $\boldsymbol{u} \in Res(C)$, there exists $\boldsymbol{u}a + \boldsymbol{u'}c \in C$ such that $\alpha(\boldsymbol{u}a + \boldsymbol{u'}c) = \boldsymbol{u}$. Further, for any $v \in Tor(C)$, we have $\boldsymbol{v}c \in C$. Since C is self-orthogonal, $\langle \boldsymbol{u}a + \boldsymbol{u'}c, \boldsymbol{v}c\rangle = \langle \boldsymbol{u}, \boldsymbol{v}\rangle c = 0$. It implies that $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = 0$, and so $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal. Hence, $v \in Res(C)^{\perp}$. Then, $Tor(C) \subseteq Res(C)^{\perp}$ implies that $Res(C) \subseteq Tor(C)^{\perp}$. Conversely, suppose $Res(C) \subseteq Tor(C)^{\perp}$ and $\boldsymbol{x}, \boldsymbol{y} \in C$. Then, by Theorem 3.5, $\boldsymbol{x} = \boldsymbol{u}a + \boldsymbol{v}c$ and $\boldsymbol{y} = \boldsymbol{u}'a + \boldsymbol{v}'c$ for some $\boldsymbol{u}, \boldsymbol{u}' \in Res(C)$ and $\boldsymbol{v}, \boldsymbol{v}' \in Tor(C)$. Following a similar procedure given in [1], we can show that if C is a linear R-code, then $Res(C) \subseteq Tor(C)$. This implies that $Tor(C)^{\perp} \subseteq Res(C)^{\perp}$. Then $Res(C) \subseteq Tor(C)^{\perp} \subseteq Res(C)^{\perp}$ implies that

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \langle \boldsymbol{u}a + \boldsymbol{v}c, \boldsymbol{u}'a + \boldsymbol{v}'c\rangle = \langle \boldsymbol{u}, \boldsymbol{u}'\rangle a + \langle \boldsymbol{u}, \boldsymbol{v}'\rangle c = 0.$$

Thus, C is self-orthogonal.

On the other hand, suppose C is QSD. Then $|C| = 3^{2k_1+k_2} = 3^n$ implies that $2k_1 + k_2 = n$. This shows that $k_1 = n - (k_1 + k_2)$, and so $Res(C)$ and $Tor(C)^{\perp}$ have equal dimensions. Since $Res(C)$ and $Tor(C)^{\perp}$ have equal dimensions and $Res(C) \subseteq Tor(C)^{\perp}$, $Res(C) = Tor(C)^{\perp}$. Conversely, suppose that $Res(C) = Tor(C)^{\perp}$. Since $Res(C) \subseteq Tor(C)$, $Tor(C)^{\perp} \subseteq Res(C)^{\perp}$. Hence, $Res(C) = Tor(C)^{\perp} \subseteq Res(C)^{\perp}$ implies that $Res(C)$ is self-orthogonal. Thus, by Theorems 3.2 and 3.5, C is QSD. □

4. LCD codes over the ring R

This section deals with the LCD codes over the ring R and characterizes free LCD R-codes in terms of a ternary generator matrix.

Definition 4.1 (Generating set)**.** *Let $X = \{\boldsymbol{x}_1,\ \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m\}$ be a subset of an R-linear code C. Then (right) R-span of X is defined by*

$$\langle X\rangle_R = \{\boldsymbol{x}_1\alpha_1 + \boldsymbol{x}_2\alpha_2 + \cdots + \boldsymbol{x}_m\alpha_m \mid \alpha_i \in R,\ \forall i\ \}.$$

We define the additive span of X by

$$\langle X\rangle_{\mathbb{F}_3} = \{\boldsymbol{x}_1\gamma_1 + \boldsymbol{x}_2\gamma_2 + \cdots + \boldsymbol{x}_m\gamma_m \mid \gamma_i \in \mathbb{F}_3,\ \forall i\}.$$

Note that $\langle X\rangle_R$ does not always contain $\langle X\rangle_{\mathbb{F}_3}$ since the ring R has no unity element.

A subset $X = \{\boldsymbol{x}_1,\ \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m\}$ of C is called a generating set for the code C if

$$\langle X\rangle_R \cup \langle X\rangle_{\mathbb{F}_3} = C.$$

Definition 4.2 (Generator matrix)**.** *Consider a set* $X = \{\boldsymbol{x}_1,\ \boldsymbol{x}_2, \dots, \boldsymbol{x}_m\} \subset C$ *which generates the linear code* C*. Then, we define its generator matrix* G_R *as an* $m \times n$ *matrix whose rows are* $\boldsymbol{x}_1,\ \boldsymbol{x}_2, \dots, \boldsymbol{x}_m$ *such that* $\langle G \rangle_R = \langle X \rangle_R \cup \langle X \rangle_{\mathbb{F}_3}$*.*

Definition 4.3 (Right-nice code)**.** *An* R*-linear code* C *is called right-nice if*

$$|C| \cdot |C^{\perp_R}| = 9^n.$$

Remark 4.4. *The relation* $|C| \cdot |C^{\perp_R}| = 9^n$ *does not hold in general. For example, let us consider an additive* R*-code* C *of length* 3 *with an additive generator matrix*

$$G = \begin{bmatrix} a & b & 0 \\ 0 & 0 & b \end{bmatrix}.$$

Let $(x,\ y,\ z) \in C^{\perp_R}$*. Then, we have*

$$\langle (a,\ b,\ 0),\ (x,\ y,\ z) \rangle = 0 \text{ and } \langle (0,\ 0,\ b),\ (x,\ y,\ z) \rangle = 0.$$

Hence, we get two equations: $ax + by = x + y = 0$ *and* $z = 0$*. These imply that* $y = 2x$ *and* $z = 0$*. Therefore,* $C^{\perp_R}$ *can be given by the set*

$$C^{\perp_R} = \{\ (\ x,\ 2x,\ 0) \mid x \in R\ \}.$$

It is easy to see that $|C| = |C^{\perp_R}| = 9$*. Hence,* $|C||C^{\perp_R}| = 9^2 < 9^3$*.*

Definition 4.5 (Left-nice code)**.** *An* R*-linear code* C *is called left-nice if*

$$|C| \cdot |C^{\perp_L}| = 9^n.$$

Definition 4.6 (Right-LCD code)**.** *A right-nice code* C *is called right-LCD code if* $C \cap C^{\perp_R} = \{\mathbf{0}\}$*. Note that* C *is a right-LCD code if and only if* $C \oplus C^{\perp_R} = R^n$*.*

Definition 4.7 (Left-LCD code)**.** *A left-nice code* C *is called left-LCD code if* $C \cap C^{\perp_L} = \{\mathbf{0}\}$*. Note that* C *is a left-LCD code if and only if* $C \oplus C^{\perp_L} = R^n$*.*

Remark 4.8. *There exists no non-trivial left-LCD code over* R*.*

Proof. Let $C \neq \{\mathbf{0}\}$ be an R-linear code. Since C is linear, for any $\boldsymbol{x} \in C$, $\boldsymbol{x}c \in C$. From Table 1, we can see that $\boldsymbol{x}c = \boldsymbol{u}c$ where u is a ternary vector. Since c is a left zero divisor, $\langle \boldsymbol{x}c, \boldsymbol{y} \rangle = 0$ for all $\boldsymbol{y} \in C$. Hence $\boldsymbol{x}c \in C^{\perp_L}$, which shows that $C \cap C^{\perp_L} \neq \{\mathbf{0}\}$. Thus, a left-LCD code over R does not exist. □

Therefore, in our further investigations, we consider the right-LCD codes as LCD codes and hence the right dual only.

Definition 4.9 (Free code)**.** *An* R*-linear code* C *is free if it can be written as a finite direct sum of* R *(*R *as a right* $R-$*module), i.e.,* $C = R \oplus R \oplus \cdots \oplus R$ *where* $R = \langle x_i \rangle_R$ *for some* $x_i \in R$*.*

Following [1], we see that C is free if and only if $k_2 = 0$.

Theorem 4.10. *Let* C *be a free linear code over* R *with generator matrix* G_R*. Then*

(i) $C = \langle Ga \rangle_R$*, where* G *is a ternary matrix.*

(ii) $C^{\perp_R} = \langle Ha \rangle_R$*, where* H *is a ternary parity check matrix.*

Proof. (i) Since C is a free linear code, there exists a generating set $X = \{\boldsymbol{x}_1,\ \boldsymbol{x}_2, \dots, \boldsymbol{x}_m\} \subset C$ such that $C = \langle \boldsymbol{x}_1 \rangle_R \oplus \cdots \oplus \langle \boldsymbol{x}_m \rangle_R$, where $\langle \boldsymbol{x}_i \rangle_R = R$ for each i. Note that $\langle \boldsymbol{x}_i \rangle_R = \langle \boldsymbol{x}_i a \rangle_R$ for $i = 1, 2, \dots, m$, since $ax = x$ for all $x \in R$. Since G_R consists of the rows $\boldsymbol{x}_1,\ \boldsymbol{x}_2, \dots, \boldsymbol{x}_m$, so we can replace these rows by $\boldsymbol{x}_1 a,\ \boldsymbol{x}_2 a, \dots, \boldsymbol{x}_m a$. From Table 1, we can see that $\boldsymbol{x}_i a = \boldsymbol{x}'_i a$, where $\boldsymbol{x}'_i$ is a ternary vector. Hence, $C = \langle Ga \rangle_R$ for some ternary matrix G.

(ii) Note that $(Ga)(Ha)^T = (aG)(Ha)^T = aGH^Ta = \mathbf{0}$, since H is a parity check matrix related to G. Therefore, $\langle Ha\rangle_R \subseteq C^{\perp_R}$. Conversely, we show that $C^{\perp_R} \subseteq \langle Ha\rangle_R$.
Let $G = [I|A]$, and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \in C^{\perp_R}$. Also, suppose

$$Ga = [I|A]a = \begin{bmatrix} \boldsymbol{r}_1 a \\ \boldsymbol{r}_2 a \\ \vdots \\ \boldsymbol{r}_k a \end{bmatrix},$$

where $\boldsymbol{r}_i$'s are the ternary rows of G. Then $\langle \boldsymbol{r}_i a,\ \boldsymbol{\beta}\rangle = 0$ for $1 \leq i \leq k$. That is, for any $1 \leq i \leq k$,

$$m_{i_1} a\beta_{i_1} + m_{i_2} a\beta_{i_2} + \cdots + m_{i_j} a\beta_{i_j} = 0,$$

for some $1 \leq j \leq k$ and m_{i_l}'s are either 1 or 2. This implies that

$$m_{i_1} \beta_{i_1} + m_{i_2} \beta_{i_2} + \cdots + m_{i_j} \beta_{i_j} = 0,$$

since $ax = x$ for all $x \in R$. Moreover, $G = [I|A]$ is in its standard form, we get k equations as follows:

$$\beta_1 + 0 + 0 + \cdots + 0 + m_{k+i_1}\beta_{k+i_1} + \cdots + m_{i_{s_1}}\beta_{i_{s_1}} = 0,$$

$$0 + \beta_2 + 0 + \cdots + 0 + m_{k+i_2}\beta_{k+i_2} + \cdots + m_{i_{s_1}}\beta_{i_{s_2}} = 0,$$

$$\vdots$$

$$0 + 0 + \cdots + 0 + \beta_k + m_{k+i_k}\beta_{k+i_k} + \cdots + m_{i_{s_k}}\beta_{i_{s_k}} = 0,$$

where for any $j = 1, 2, \ldots, k$, the indices $k + i_j, \ldots, i_{s_j}$ of $\boldsymbol{\beta}$ correspond to the non-zero coordinates of $\boldsymbol{r}_j a$. Considering the above k equations, we have maximum $n - k$ independent variables. Hence, there can exist maximum 9^{n-k} solutions for $\boldsymbol{\beta}$. Therefore, $|C^{\perp_R}| \leq 9^{n-k}$. Also, $|\langle Ha\rangle_R| = 9^{n-k}$. Thus, $C^{\perp_R} = \langle Ha\rangle_R$.

□

Corollary 4.11. *Let C be an R-linear code. Then it is permutation-equivalent to an additive R-code with an additive generator matrix*

$$\begin{bmatrix} \boldsymbol{I}_{k_1} a & \boldsymbol{X} a & \boldsymbol{Y} a \\ \boldsymbol{I}_{k_1} b & \boldsymbol{X} b & \boldsymbol{Y} b \\ \mathbf{0} & \boldsymbol{I}_{k_2} c & \boldsymbol{Z} c \end{bmatrix},$$

where $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ are ternary matrices and $\boldsymbol{I}_k$ is the identity matrix of order k.

Proof. Let $C' = \langle C\rangle_R$. Obviously, C' is free. Then, by Theorem 4.10, $C' = \langle G_1 a\rangle$, where G_1 is a ternary matrix. Suppose G_1 is in the standard form $[I_{k_1}|A]$, then for all $y \in C \setminus C'$, we can assume that $\boldsymbol{y} = (\underbrace{0, 0, \ldots, 0}_{k_1},\ \boldsymbol{y'})$. Since $\boldsymbol{y} + 2\boldsymbol{y}a = \boldsymbol{y}_3 c$ for some ternary vector $\boldsymbol{y}_3$ and $2\boldsymbol{y}a \in C'$, $\boldsymbol{y}_3 c \notin C'$, we have $\boldsymbol{y} = \boldsymbol{y}a + \boldsymbol{y}_3 c$. Hence, we can find an $\mathbb{F}_3$-linearly independent set $Y_3 = \{\boldsymbol{y}_3^1 c,\ \boldsymbol{y}_3^2 c, \ldots, \boldsymbol{y}_3^{k_2} c \mid \boldsymbol{y}_3^k$ is a ternary vector for all $k\}$ such that $C = C' \oplus \langle Y_3\rangle_{\mathbb{F}_3}$. Hence the result. □

Next, we define a map $\phi : R \to R$ by

$$\phi(0) = 0,\ \phi(a) = a + 2b = c,\ \phi(b) = 2b,\ \phi(a + b) = a + b,\ \phi(2a) = 2a + b,$$

$$\phi(2b) = b,\ \phi(a + 2b) = a,\ \phi(2a + b) = 2a,\ \phi(2a + 2b) = 2a + 2b.$$

We can easily see that

$$\phi(x + y) = \phi(x) + \phi(y) \text{ and } y\phi(z) = \phi(yz) \text{ for all } x, y, z \in R.$$

We can extend this map naturally on R^n. That is,

$$\phi(\boldsymbol{y}) = (\phi(y_1),\ \phi(y_2)\dots,\phi(y_n)) \text{ for all } \boldsymbol{y} = (y_1,\ y_2\dots,y_n) \in R^n.$$

Further, we have

$$\phi(\boldsymbol{x} + \boldsymbol{y}) = \phi(\boldsymbol{x}) + \phi(\boldsymbol{y}) \text{ and } \boldsymbol{y}\phi(z) = \phi(\boldsymbol{y}z) \text{ for all } z \in R \text{ and } \boldsymbol{x}, \boldsymbol{y} \in R^n.$$

Theorem 4.12. *Let C be a linear R-code. Then it is free if and only if $\boldsymbol{v}c \in C$ implies $\boldsymbol{v}a \in C$ where v is a ternary vector.*

Proof. Let C be free. Then there exists a generating set $X = \{\boldsymbol{x}_1,\ \boldsymbol{x}_2, \dots, \boldsymbol{x}_m\} \subset C$ such that

$$C = \langle \boldsymbol{x}_1 \rangle_R \oplus \langle \boldsymbol{x}_2 \rangle_R \oplus \cdots \oplus \langle \boldsymbol{x}_m \rangle_R.$$

Since $\boldsymbol{v}c \in C$, $\boldsymbol{v}c = \boldsymbol{x}_1\alpha_1 + \boldsymbol{x}_2\alpha_2 + \cdots + \boldsymbol{x}_m\alpha_m$ where each $\alpha_i \in R$.
Now,

$$\begin{aligned} \boldsymbol{v}a = \ \boldsymbol{v}aa = \ & \boldsymbol{v}a\phi(c) = \phi(\boldsymbol{v}ac) = \phi(\boldsymbol{v}c) \\ = \ & \phi(\boldsymbol{x}_1\alpha_1 + \boldsymbol{x}_2\alpha_2 + \cdots + \boldsymbol{x}_m\alpha_m) \\ = \ & \phi(\boldsymbol{x}_1\alpha_1) + \phi(\boldsymbol{x}_2\alpha_2) + \cdots + \phi(\boldsymbol{x}_m\alpha_m) \\ = \ & \boldsymbol{x}_1\phi(\alpha_1) + \boldsymbol{x}_2\phi(\alpha_2) + \cdots + \boldsymbol{x}_m\phi(\alpha_m). \end{aligned}$$

This shows that $\boldsymbol{v}a$ is a linear combination of the vectors of the generating set X. Hence, $\boldsymbol{v}a \in C$. Conversely, Suppose for any $\boldsymbol{v}c \in C$, we have $\boldsymbol{v}a \in C$. Following a similar procedure given in [1], we can show that if C is a linear code, then $Res(C) \subseteq Tor(C)$. For any $\boldsymbol{v} \in Tor(C)$, $\boldsymbol{v}c \in C$. Hence, $\boldsymbol{v}a \in C$. Since $\boldsymbol{v}a \in C$ and $\alpha(\boldsymbol{v}a) = \boldsymbol{v}$, $\boldsymbol{v} \in Res(C)$. This implies that $Tor(C) \subseteq Res(C)$. Hence, $Res(C) = Tor(C)$. Thus, $k_2 = 0$ implies that C is free. □

Proposition 4.13. *Let C be an R-linear code. Then it is free if and only if it is right-nice.*

Proof. Let C be an R-linear code. Then

$$|C||C^{\perp_R}| = \ 9^{k_1}3^{k_2}9^{n-k_1} = 9^n3^{k_2}.$$

Thus, C is free if and only if it is right-nice. □

Corollary 4.14. *An LCD R-code C is free.*

Proof. By definition, an LCD R-code C is right-nice. Hence, by Proposition 4.13, it is free. □

Next, we demonstrate a method for constructing right-LCD codes over R using ternary LCD codes.

Proposition 4.15. *Let D be a ternary LCD code with generator matrix $G_{k\times n}$. Then the $R-$span of Ga, $\langle Ga \rangle_R$ is an LCD R-code.*

Proof. Let $C = \langle Ga \rangle_R$ and H be a ternary parity check matrix for D. Then, by (ii) of Theorem 4.10,

$$C^{\perp_R} = \langle Ha \rangle_R.$$

Hence, $|C||C^{\perp_R}| = |C||\langle Ha \rangle_R| = 9^k 9^{n-k} = 9^n$. This shows that C is right-nice.
Next, we prove $C \cap C^{\perp_R} = \{\mathbf{0}\}$. If possible, let $\boldsymbol{x}(\neq \mathbf{0}) \in C \cap C^{\perp_R}$. Then, we have two possible cases:
Case 1: Suppose $\boldsymbol{x}$ is not a product of a ternary vector by c, then

$$\boldsymbol{x} = \sum \boldsymbol{r}_i a \alpha_i, \text{ where } \boldsymbol{r}_i a \text{ is a row of } Ga \text{ for some distinct } i \text{ and } \alpha_i \in R.$$

Also,

$$\boldsymbol{x} = \sum \boldsymbol{r}_t a \beta_t, \text{ where } \boldsymbol{r}_t a \text{ is a row of } Ha \text{ for some distinct } t \text{ and } \beta_t \in R.$$

Since $ax = x \neq \mathbf{0}$,

$$\boldsymbol{x} = \sum \boldsymbol{r}_i \alpha_i, \text{ where } \boldsymbol{r}_i \text{ is a row of } G \text{ for some distinct } i \text{ and } \alpha_i \in R.$$

Again,

$$\boldsymbol{x} = \sum \boldsymbol{r}_t \beta_t, \text{ where } \boldsymbol{r}_t \text{ is a row of } H \text{ for some distinct } t \text{ and } \beta_t \in R.$$

Now, from Table 1, we have

$$\boldsymbol{x}a = \sum \boldsymbol{r}_{i_j} a, \text{ where } \boldsymbol{r}_{i_j} \text{ is a row of } G \text{ for some distinct } i_j.$$

Also,

$$\boldsymbol{x}a = \sum \boldsymbol{r}_{t_s} a, \text{ where } \boldsymbol{r}_{t_s} \text{ is a row of } H \text{ for some distinct } t_s.$$

Hence, $\mathbf{0} = \sum \boldsymbol{r}_{i_j} a - \sum \boldsymbol{r}_{t_s} a = (\sum \boldsymbol{r}_{i_j} - \sum \boldsymbol{r}_{t_s})a$. Therefore, $\sum \boldsymbol{r}_{i_j} - \sum \boldsymbol{r}_{t_s} = \mathbf{0}$, i.e., $\sum \boldsymbol{r}_{i_j} = \sum \boldsymbol{r}_{t_s} \in D \cap D^{\perp}$, which contradicts that D is LCD.
Case 2: Let $\boldsymbol{x}$ be the product of a ternary vector $\boldsymbol{v}$ by c, i.e., $\boldsymbol{x} = \boldsymbol{v}c$. Let the generator matrix G of D be in the standard form. Then, by investigating generator matrix Ga, it turns out that $\boldsymbol{v}$ can be expressed as a sum of scalar multiples of some rows of G. Hence, $\boldsymbol{v}$ is a codeword of D. Similarly, we can see that $\boldsymbol{v}$ is also a codeword of $D^{\perp}$. Hence, $\boldsymbol{v} \in D \cap D^{\perp}$, which contradicts that D is LCD.

In both cases, we arrive at a contradiction. Hence, we conclude that $C \cap C^{\perp_R} = \{\mathbf{0}\}$. Thus, C is an LCD R-code. □

Next, we give an example of LCD code over R constructed by the method described in Proposition 4.15.

Example 4.16. *Let $C_1 = \{(0,\ 0), (1,\ 2), (2,\ 1)\}$. Then C_1 is a ternary linear code of length 2. Its dual is given by*

$$C_1^{\perp} = \{(0,\ 0), (1,\ 1), (2,\ 2)\}.$$

We see that $C_1 \cap C_1^{\perp} = \{\mathbf{0}\}$ and $|C_1||C_1^{\perp}| = 3^2 = 3^n$. Hence, C_1 is a ternary LCD code of length 2. Consider a generator matrix for C_1 as $G = \begin{bmatrix} 1 & 2 \end{bmatrix}$. Then, by Proposition 4.15, the $R-$span of Ga, $\langle Ga \rangle_R$ is an LCD code over the ring R. Now, we validate all the conditions of the definition of an LCD

R-code to show that $\langle Ga\rangle_R$ is an LCD R-code.
Consider $C = \langle Ga\rangle_R = \langle \begin{bmatrix} a & 2a \end{bmatrix} \rangle_R$. Then the code C can be represented by the set

$$\begin{aligned} C &= \{(a,\ 2a)x \mid x \in R\} \\ &= \{(x,\ 2x) \mid x \in R\}. \end{aligned}$$

We can easily see that $|C| = 9$. Let $(\alpha, \beta) \in C^{\perp_R}$. Then

$$\langle (x, 2x),\ (\alpha, \beta)\rangle = 0 \textit{ implies that } x\alpha + 2x\beta = 0.$$

Now, the following cases arise:

(i) When $\alpha = \beta$. Then

$$x\alpha + 2x\beta = x\alpha + 2x\alpha = 3x\alpha = 0.$$

Hence, we have $\{(\alpha, \alpha) \mid \alpha \in R\} \subseteq C^{\perp_R}$.

(ii) When $\alpha \neq \beta$. Then, from Table 1, we can see that for $x = a \in R$, there does not exist $\alpha, \beta \in R$ such that $x\alpha + 2x\beta = 0$.

From both cases, we conclude that

$$C^{\perp_R} = \{(\alpha, \alpha) \mid \alpha \in R\} \textit{ and } |C^{\perp_R}| = 9.$$

It is easy to see that $C \cap C^{\perp_R} = \{\mathbf{0}\}$ and $|C||C^{\perp_R}| = 9^2 = 9^n$. Hence, C is an LCD R-code.

Next, we see the inverse assertion of Proposition 4.15 holds partially.

Proposition 4.17. *A free LCD R-code C is generated by the matrix G_3a where G_3 is a generator matrix for a ternary LCD code D.*

Proof. By Theorem 4.10, C is generated by the rows of the matrix G_3a where G_3 is a generator matrix of a ternary code D. Also, by Theorem 4.10, $C^{\perp_R} = \langle H_3a\rangle_R$ where H_3 is a ternary parity check matrix of D. Since C is an LCD code, $C \cap C^{\perp_R} = \{\mathbf{0}\}$. Now, we claim that $D \cap D^{\perp} = \{\mathbf{0}\}$. If possible, let $\boldsymbol{v}(\neq \mathbf{0}) \in D \cap D^{\perp}$. Then $\boldsymbol{v}a(\neq \mathbf{0}) \in C \cap C^{\perp_R}$, which contradicts the assumption. Hence, D is LCD. □

Corollary 4.18. *The residue and torsion codes of an LCD R-code C are also LCD.*

Proof. By Corollary 4.14, an LCD R-code C is free. Then, by Proposition 4.17, C is generated by the matrix G_3a where G_3 is a generator matrix of a ternary LCD code. Since C is free, $Res(C) = Tor(C)$ and G_3 will be a generator matrix for both. Hence, both are ternary LCD codes. □

5. ACD codes

This section deals with ACD codes over the considered ring R. Here, we introduce right-ACD codes over R and give several criteria for the existence of such codes.

Definition 5.1 (Right-nice and left-nice additive codes)**.** *An additive R-code C is called right-nice (respectively, left-nice) if $|C||C^{\perp_R}| = 9^n$ (respectively, $|C||C^{\perp_L}| = 9^n$).*

Definition 5.2 (Right-ACD and left-ACD codes)**.** *An additive R-code C is said to be right-ACD (respectively, left-ACD) if it is right-nice (respectively, left-nice) and $C \cap C^{\perp_R} = \{\mathbf{0}\}$ (respectively, $C \cap C^{\perp_L} = \{\mathbf{0}\}$).*

Theorem 5.3. *Let C be an additive R-code of length n. Then its right dual is free, but the left dual is not.*

Proof. First, we prove that both duals are linear codes over R. Let $\boldsymbol{x}, \boldsymbol{y} \in C^{\perp_R}$ and $\boldsymbol{z} \in C$. Then, $\langle \boldsymbol{z}, \boldsymbol{x} \rangle = 0$ and $\langle \boldsymbol{z}, \boldsymbol{y} \rangle = 0$. These imply that

$$\langle \boldsymbol{z}, \boldsymbol{x} + \boldsymbol{y} \rangle = \langle \boldsymbol{z}, \boldsymbol{x} \rangle + \langle \boldsymbol{z}, \boldsymbol{y} \rangle = 0 + 0 = 0.$$

Hence, $\boldsymbol{x} + \boldsymbol{y} \in C^{\perp_R}$. Now, let $\boldsymbol{x} \in C^{\perp_R}$ and $\boldsymbol{z} \in C$. Then, for any $r \in R$, $\langle \boldsymbol{z}, \boldsymbol{x}r \rangle = \langle \boldsymbol{z}, \boldsymbol{x} \rangle r = 0$. This shows that $\boldsymbol{x}r \in C^{\perp_R}$. Thus, $C^{\perp_R}$ is a right submodule of R^n and hence a linear code over R.
Let $\boldsymbol{x}, \boldsymbol{y} \in C^{\perp_L}$ and $\boldsymbol{z} \in C$. Then

$$\langle \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \boldsymbol{z} \rangle + \langle \boldsymbol{y}, \boldsymbol{z} \rangle = 0 + 0 = 0.$$

Therefore, $\boldsymbol{x} + \boldsymbol{y} \in C^{\perp_L}$. Also, $\langle \boldsymbol{x}a, \boldsymbol{z} \rangle = \langle \boldsymbol{x}b, \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \boldsymbol{z} \rangle = 0$. Now, let $r \in R$. Then it has a c-adic decomposition as $ua + vc$ where $u, v \in \mathbb{F}_3$. Then

$$\begin{aligned}
\langle \boldsymbol{x}r,\ \boldsymbol{z} \rangle &= \langle \boldsymbol{x}(ua + vc),\ \boldsymbol{z} \rangle \\
&= \langle \boldsymbol{x}ua,\ \boldsymbol{z} \rangle + \langle \boldsymbol{x}vc,\ \boldsymbol{z} \rangle \\
&= \langle \boldsymbol{x}u, \boldsymbol{z} \rangle + 0 \\
&= \langle \boldsymbol{x}, \boldsymbol{z} \rangle u \\
&= 0 \cdot u \\
&= 0.
\end{aligned}$$

This shows that $\boldsymbol{x}r \in C^{\perp_L}$. Hence, $C^{\perp_L}$ is also an R-linear code.

Next, we show that the right dual is free but not the left one.

(i) Let $\boldsymbol{x} = \boldsymbol{v}c \in C^{\perp_R}$, where $\boldsymbol{v}$ is a ternary vector. Then, from Table 1, for any $\boldsymbol{y} \in C$, $\langle \boldsymbol{y}, \boldsymbol{x} \rangle = \langle \boldsymbol{y}, \boldsymbol{v}c \rangle = lc = 0$. So l will be a multiple of 3. Now, let $\boldsymbol{z} = \boldsymbol{v}a$, then by seeing Table 1, $\langle \boldsymbol{y}, \boldsymbol{z} \rangle = \langle \boldsymbol{y}, \boldsymbol{v}a \rangle = la = 0$. This shows that $\boldsymbol{z} = \boldsymbol{v}a \in C^{\perp_R}$. Hence, by Theorem 4.12, $C^{\perp_R}$ is free.

(ii) Suppose $C^{\perp_L}$ is free. Since c is a left zero divisor, $\langle cI_n \rangle_{\mathbb{F}_3} \subseteq C^{\perp_L}$. Then, by Theorem 4.12, $\langle aI_n \rangle_{\mathbb{F}_3} \subseteq C^{\perp_L}$ and so $\langle bI_n \rangle_{\mathbb{F}_3} \subseteq C^{\perp_L}$. These imply that $|C^{\perp_L}| \geq 3^n \times 3^n = 9^n$. Again $C^{\perp_L} \subseteq R^n$ implies that $|C^{\perp_L}| \leq 9^n$. Hence, $C^{\perp_L} = R^n$. In this case, $C = \{\boldsymbol{0}\}$, which is not possible. Therefore, $C^{\perp_L}$ is not free.

□

Theorem 5.4. *Let C be an R-linear code. Then $(C^{\perp_R})^{\perp_R} = C$ if and only if C is right-nice.*

Proof. Let $(C^{\perp_R})^{\perp_R} = C$. We have to prove that C is right-nice. From Theorem 5.3, we know that $C^{\perp_R}$ is free, and from Proposition 4.13, a linear code over R is free if and only if it is right-nice. Hence, $|C||C^{\perp_R}| = |C^{\perp_R}||(C^{\perp_R})^{\perp_R}| = 9^n$ implies that C is right-nice.

Conversely, let C be a right-nice code. Since C and $C^{\perp_R}$ are free, by Theorem 4.10, for each $\boldsymbol{x} \in C$ and $\boldsymbol{y} \in C^{\perp_R}$, we assume that $\boldsymbol{x} = \boldsymbol{u}a$ and $\boldsymbol{y} = \boldsymbol{v}a$ for some ternary vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Then, their inner product

$$\begin{aligned}
\langle \boldsymbol{y}, \boldsymbol{x} \rangle &= \langle \boldsymbol{v}a, \boldsymbol{u}a \rangle \\
&= \langle \boldsymbol{v}, \boldsymbol{u} \rangle a \\
&= \langle \boldsymbol{u}, \boldsymbol{v} \rangle a \\
&= \langle \boldsymbol{u}a, \boldsymbol{v}a \rangle \\
&= \langle \boldsymbol{x}, \boldsymbol{y} \rangle \\
&= 0.
\end{aligned}$$

This implies that $\boldsymbol{x} \in (C^{\perp_R})^{\perp_R}$. Hence, $C \subseteq (C^{\perp_R})^{\perp_R}$. Further, since C and $C^{\perp_R}$ are right-nice, $|C^{\perp_R}||(C^{\perp_R})^{\perp_R}| = |C||C^{\perp_R}|$ implies that $|C| = |(C^{\perp_R})^{\perp_R}|$. Therefore, $(C^{\perp_R})^{\perp_R} = C$. Thus, $(C^{\perp_R})^{\perp_R} = C$ if and only if C is right-nice. □

Proposition 5.5. *The right dual of an LCD R-code C is also LCD but not the left one.*

Proof. Let C be an LCD R-code. Then, $C \cap C^{\perp_R} = \{\mathbf{0}\}$ and $|C||C^{\perp_R}| = 9^n$. Also, by Theorem 5.4, $(C^{\perp_R})^{\perp_R} = C$. Hence,

$$C^{\perp_R} \cap (C^{\perp_R})^{\perp_R} = C^{\perp_R} \cap C = \{\mathbf{0}\}$$

and

$$|C^{\perp_R}||(C^{\perp_R})^{\perp_R}| = |C^{\perp_R}||C| = 9^n.$$

This shows that the right dual of an LCD R-code C is also an LCD R-code. Further, by Theorem 5.3, $C^{\perp_L}$ is not free. Thus, by Corollary 4.14, $C^{\perp_L}$ is not an LCD code. □

Proposition 5.6. *If C is a right-ACD code over R with 3^m codewords, then m is even.*

Proof. By Theorem 5.3, $C^{\perp_R}$ is free. Then $|C^{\perp_R}| = 9^{k_1} = 3^{2k_1}$. Also, C is right-ACD code over R implies that $|C||C^{\perp_R}| = 9^n = 3^{2n}$. Therefore, $|C| = \frac{3^{2n}}{3^{2k_1}} = 3^{2n-2k_1}$, i.e., $3^m = 3^{2n-2k_1}$. Hence, $m = 2n - 2k_1$, i.e., m is even. □

Remark 5.7. *Let $G = [aI_n]$ be an additive generator matrix of an additive R-code C of length n. Then $H_L = [cI_n]$ will be an additive generator matrix of $C^{\perp_L}$. In this case, the parameters of the both C and $C^{\perp_L}$ are given by $(n,\ 3^n,\ 1)$. It is easy to see that $C \cap C^{\perp_L} = \{\mathbf{0}\}$ and $|C||C^{\perp_L}| = 3^n \times 3^n = 9^n$. Hence, C is a left-ACD code over R. Thus, for any integer m, it is possible to construct a left-ACD code over R with 3^m codewords.*

Theorem 5.8. *Let C be an additive R-code of length n. Let $C \cap C^{\perp_R} = \{\mathbf{0}\}$, $Cc \cap C^{\perp_R} = \{\mathbf{0}\}$ and $|C|\,|C^{\perp_R}| < 9^n$. Consider $\mathbf{0} \neq \mathbf{z} = \boldsymbol{x} + \boldsymbol{y}c \in C + Cc$. If $z \in C^{\perp_R}$, then $\boldsymbol{x} \neq \mathbf{0}, \boldsymbol{y}c \neq \mathbf{0}$ and $\boldsymbol{x}$ is a product of a ternary vector by c such that $\boldsymbol{x} \in C \setminus Cc$.*

Proof. Let $\mathbf{0} \neq \mathbf{z} = \mathbf{x} + \mathbf{y}c \in C + Cc$ where $\mathbf{x}, \mathbf{y} \in C$. Assume that $z \in C^{\perp_R}$. Then the following cases arise:

(1) If $\mathbf{x} = \mathbf{0},\ \mathbf{y}c = \mathbf{0}$, then $\mathbf{z} = \mathbf{0}$ which is not true. Therefore, this case is not possible.

(2) If $\mathbf{x} = \mathbf{0}$ and $\mathbf{y}c \neq \mathbf{0}$, then $\mathbf{z} = \mathbf{y}c \in Cc$. In this case, $\mathbf{z} = \mathbf{y}c \notin C^{\perp_R}$ as $Cc \cap C^{\perp_R} = \{\mathbf{0}\}$. It again leads to a contradiction. So, this case is also not possible.

(3) If $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y}c = \mathbf{0}$, then $\mathbf{z} = \mathbf{x} \in C$. In this case, $\mathbf{z} = \mathbf{x} \notin C^{\perp_R}$ as $C \cap C^{\perp_R} = \{\mathbf{0}\}$. It again leads to a contradiction. So, this case is also not possible.

(4) If $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y}c \neq \mathbf{0}$, then $\mathbf{z} = \mathbf{x} + \mathbf{y}c$. Assume that $\mathbf{x}$ is not a product of a ternary vector by c. Then $\mathbf{x}c \neq \mathbf{0}$. Since $C^{\perp_R}$ is a free linear code over R and $\mathbf{z} \in C^{\perp_R}$, we have $\mathbf{z}c = (\mathbf{x} + \mathbf{y}c)c = \mathbf{x}c \in C^{\perp_R}$. Thus, $\mathbf{x}c \in Cc \cap C^{\perp_R} = \{\mathbf{0}\}$, i.e., $\mathbf{x}c = \mathbf{0}$ which is a contradiction. Hence, $\boldsymbol{x}$ is a product of a ternary vector by c. Now, if $\mathbf{x} \in C \cap Cc$, then $\mathbf{z} \in Cc$, i.e., $\mathbf{z} \in Cc \cap C^{\perp_R} = \{\mathbf{0}\}$. This implies that $\mathbf{z} = \mathbf{0}$, which is a contradiction. Thus, $\boldsymbol{x} \in C \setminus Cc$.

The intended result is obtained by combining all the above cases. □

Proposition 5.9. *Let C be an additive R-code of length n. Let $C \cap C^{\perp_R} = \{\mathbf{0}\}$, $Cc \cap C^{\perp_R} = \{\mathbf{0}\}$ and $|C|\,|C^{\perp_R}| < 9^n$. Then C can be extended to a right-ACD code by adding some codewords of Cc.*

Proof. We prove it in two steps.
Step 1: We first show that $Cc \nsubseteq C$. Let us consider the set

$$D = \langle C \rangle_R = Ca + Cc = \{ \boldsymbol{x}a + \boldsymbol{y}c \mid \boldsymbol{x}, \boldsymbol{y} \in C \setminus Cc\},$$

where $\boldsymbol{x}, \boldsymbol{y}$ are not product of a ternary vector by c, since c is a left zero divisor. It is easy to see that D is a free linear R-code and $C^{\perp_R} = D^{\perp_R}$. By Theorem 4.10 and Proposition 4.15, we have $|D||D^{\perp_R}| = 9^n$. If possible, let $Cc \subseteq C$. Then, we define a map $f : C \to D$ by

$$f(\boldsymbol{x}) = \boldsymbol{x}a,\ f(\boldsymbol{x}c) = \boldsymbol{x}c,\ f(\boldsymbol{0}) = \boldsymbol{0},\ \text{for any } \boldsymbol{x} \in C \setminus Cc.$$

Let $f(\boldsymbol{z}_1 + \boldsymbol{z}_2) = f(\boldsymbol{z}_1) + f(\boldsymbol{z}_2)$, for all $\boldsymbol{z}_1, \boldsymbol{z}_2 \in C$. It is easy to see that f is a surjective linear map over $\mathbb{F}_3$. Since f is surjective, $|C| \geq |D|$. Then

$$|C||C^{\perp_R}| \geq |D||C^{\perp_R}| \ = \ |D||D^{\perp_R}| \ = \ 9^n,$$

which is a contradiction. Hence, $Cc \nsubseteq C$. Therefore, C can be extended to a right-nice code by adding some codewords of Cc.
Step 2: We prove that there exists $\boldsymbol{y}c \in Cc \setminus C$ such that $\boldsymbol{x} + \boldsymbol{y}c \notin C^{\perp_R}$ for all $\boldsymbol{x} \in C$. Suppose this is not true. Then, there exists $\boldsymbol{x} \in C$ such that $\boldsymbol{x} + \boldsymbol{y}c \in C^{\perp_R}$ for all $\boldsymbol{y}c \in Cc \setminus C$. Hence, by Theorem 5.8, $\boldsymbol{x}$ is a product of a ternary vector by c and $\boldsymbol{x} \in C \setminus Cc$. For $\boldsymbol{y}_1c, \boldsymbol{y}_2c \in Cc \setminus C$ and $\boldsymbol{y}_1c \neq \boldsymbol{y}_2c$, there exist $\boldsymbol{x}_1, \boldsymbol{x}_2 \in C$ such that $\boldsymbol{x}_1 + \boldsymbol{y}_1c, \boldsymbol{x}_2 + \boldsymbol{y}_2c \in C^{\perp_R}$. We have $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$, otherwise if $\boldsymbol{x}_1 = \boldsymbol{x}_2$, then $(\boldsymbol{x}_1 + \boldsymbol{y}_1c) + 2(\boldsymbol{x}_2 + \boldsymbol{y}_2c) \in C^{\perp_R}$. This implies that $(\boldsymbol{y}_1 + 2\boldsymbol{y}_2)c \in C^{\perp_R}$. Since $\boldsymbol{y}_1, \boldsymbol{y}_2 \in C$, $\boldsymbol{y}_1 + 2\boldsymbol{y}_2 \in C$ and $(\boldsymbol{y}_1 + 2\boldsymbol{y}_2)c \in Cc$. Hence, $(\boldsymbol{y}_1 + 2\boldsymbol{y}_2)c \in Cc \cap C^{\perp_R}$, which is a contradiction. Thus, $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$. Now, we define a map $g : D \to C$ and discuss the below given two cases:

(i) For any $\boldsymbol{y} \in C \setminus Cc$, if $\boldsymbol{y}c \in Cc \setminus C$, then define g by

$$g(\boldsymbol{y}a) = \boldsymbol{y},\ g(\boldsymbol{y}c) = \boldsymbol{x},\ g(\boldsymbol{0}) = \boldsymbol{0},$$

where $\boldsymbol{x}$ is the aforementioned vector such that $\boldsymbol{x} + \boldsymbol{y}c \in C^{\perp_R}$. Let $g(\boldsymbol{z}_1 + \boldsymbol{z}_2) = g(\boldsymbol{z}_1) + g(\boldsymbol{z}_2)$, for all $\boldsymbol{z}_1, \boldsymbol{z}_2 \in D$. Then, obviously g is an $\mathbb{F}_3$-linear map. We can see that g is injective as $g(\boldsymbol{y}_1a) = g(\boldsymbol{y}_2a)$ implies that $\boldsymbol{y}_1 = \boldsymbol{y}_2$ and hence $\boldsymbol{y}_1a = \boldsymbol{y}_2a$. Also, if $g(\boldsymbol{y}_1c) \neq g(\boldsymbol{y}_2c)$, then we must have $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$, since for $\boldsymbol{y}_1c, \boldsymbol{y}_2c \in Cc$, there exist $\boldsymbol{x}_1, \boldsymbol{x}_2 \in C$ such that $\boldsymbol{x}_1 + \boldsymbol{y}_1c,\ \boldsymbol{x}_2 + \boldsymbol{y}_2c \in C^{\perp_R}$ and $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$.

(ii) For any $\boldsymbol{y} \in C \setminus Cc$, if $\boldsymbol{y}c \in C$, then define g by

$$g(\boldsymbol{y}a) = \boldsymbol{y},\ g(\boldsymbol{y}c) = \boldsymbol{y}c,\ g(\boldsymbol{0}) = \boldsymbol{0}.$$

Let $g(\boldsymbol{z}_1 + \boldsymbol{z}_2) = g(\boldsymbol{z}_1) + g(\boldsymbol{z}_2)$, for all $\boldsymbol{z}_1, \boldsymbol{z}_2 \in D$. Then, obviously g is an $\mathbb{F}_3$-linear map. We can see that g is injective as $g(\boldsymbol{y}_1a) = g(\boldsymbol{y}_2a)$ implies that $\boldsymbol{y}_1 = \boldsymbol{y}_2$ and hence $\boldsymbol{y}_1a = \boldsymbol{y}_2a$.

In both cases, we see that g is an $\mathbb{F}_3$-linear map, and it is injective. Therefore, $|C| \geq |D|$. This implies that $|C||C^{\perp_R}| \geq |D||D^{\perp_R}| = 9^n$, which is not possible. Hence, there exists $\boldsymbol{y}c \in Cc \setminus C$ such that $\boldsymbol{x} + \boldsymbol{y}c \notin C^{\perp_R}$, for all $\boldsymbol{x} \in C$.

After these two steps, we add $\boldsymbol{y}c$ to C to get a new code C_1. Obviously, $C_1 \cap C_1^{\perp_R} = \{\boldsymbol{0}\}$. If C_1 is right-nice, then C_1 is right-ACD. Otherwise, repeat both steps for C_1. □

Now, we extend the definitions of torsion and residue codes of an R-linear code to an additive R-code. Clearly, these codes are ternary linear codes. Towards this, we consider $dim(Res(C)) = m_1$ and $dim(Tor(C)) = m_2$.

Let C be a linear code over R with the generator matrix G as given in Corollary 4.11. Then $dim(Res(C)) = k_1 = m_1$ and $dim(Tor(C)) = k_1 + k_2 = m_1 + k_2 = m_2$ where k_2 is the $\mathbb{F}_3$-dimension of the set of elements of C which are scalar multiples of $c \in R$ and can not be generated by the upper two blocks of G.

Any arbitrary codeword of the code C can be written in c-adic decomposition form as $\boldsymbol{x}a + \boldsymbol{y}c$ where $\boldsymbol{x}, \boldsymbol{y}$ are ternary vectors so that $\alpha(\boldsymbol{x}a + \boldsymbol{y}c) = \boldsymbol{x}$.

Theorem 5.10. *If C is an additive $R-$code of length n, then $C^{\perp_R} = \langle Res(C)^{\perp}\rangle_R$, and*

$$|C^{\perp_R}| = 9^{n-dim(Res(C))} = 9^{n-m_1}.$$

Proof. Let $\boldsymbol{x} \in C$. Then there exists $\boldsymbol{u} \in Res(C)$ such that $\boldsymbol{x}a = \boldsymbol{u}a$. Let $\boldsymbol{v} \in Res(C)^{\perp}$. Then

$$\langle \boldsymbol{x}, \boldsymbol{v}a\rangle = \langle \boldsymbol{x}a, \boldsymbol{v}a\rangle = \langle \boldsymbol{u}a, \boldsymbol{v}a\rangle = \langle \boldsymbol{u}, \boldsymbol{v}\rangle a = 0,$$

and

$$\langle \boldsymbol{x}, \boldsymbol{v}b\rangle = \langle \boldsymbol{x}a, \boldsymbol{v}b\rangle = \langle \boldsymbol{u}a, \boldsymbol{v}b\rangle = \langle \boldsymbol{u}b, \boldsymbol{v}b\rangle = \langle \boldsymbol{u}, \boldsymbol{v}\rangle b = 0.$$

This shows that $\boldsymbol{v}a, \boldsymbol{v}b \in C^{\perp_R}$. Since a and b are generators of R and $\boldsymbol{u}a, \boldsymbol{v}a \in C^{\perp_R}$, we see that $\langle Res(C)^{\perp}\rangle_R \subseteq C^{\perp_R}$. It is easy to prove that $\langle Res(C)^{\perp}\rangle_R = \langle Res(C)\rangle_R^{\perp_R}$. Let $\boldsymbol{u} \in Res(C)$. Then, there exists $\boldsymbol{u}a + \boldsymbol{v}c \in C$ such that $\alpha(\boldsymbol{u}a + \boldsymbol{v}c) = \boldsymbol{u}$. Since c is a left zero-divisor, for any $\boldsymbol{y} \in C^{\perp_R}$, we have

$$\langle \boldsymbol{u}a, \boldsymbol{y}\rangle = \langle \boldsymbol{u}a + \boldsymbol{v}c, \boldsymbol{y}\rangle = 0,$$

and

$$\langle \boldsymbol{u}b,\ \boldsymbol{y}\rangle = \langle \boldsymbol{u}a,\ \boldsymbol{y}\rangle = 0.$$

This shows that $\boldsymbol{y} \in \langle Res(C)\rangle_R^{\perp_R}$. Hence, $C^{\perp_R} \subseteq \langle Res(C)\rangle_R^{\perp_R} = \langle Res(C)^{\perp}\rangle_R$. Therefore,

$$C^{\perp_R} = \langle Res(C)^{\perp}\rangle_R, \text{ and } |C^{\perp_R}| = 9^{n-dim(Res(C))} = 9^{n-m_1}.$$

□

Next, we derive a sufficient condition for an additive R-code to be right-ACD.

Proposition 5.11. *An additive R-code C of length n is right ACD if it satisfies the following conditions:*

(i) $Res(C)$ *is ternary LCD,*

(ii) $Tor(C)c \cap C^{\perp_R} = \{\mathbf{0}\}$,

(iii) $3^{m_1} \times 3^{m_2} \times 9^{n-m_1} = 9^n$ *(that is,* $m_1 = m_2$*).*

Proof. It is easy to see that $|C| = |Res(C)||Tor(C)| = 3^{m_1} \times 3^{m_2}$ and $|C^{\perp_R}| = 9^{n-m_1}$. Then, $|C||C^{\perp_R}| = 9^n$ implies that C is right-nice. Let $\boldsymbol{x}(\neq \mathbf{0}) = (x_1, \ldots, x_n) \in C \cap C^{\perp_R}$. Then, $Tor(C)c \cap C^{\perp_R} = \{\mathbf{0}\}$ implies that $\boldsymbol{x}$ is not a product of a ternary vector by c. Hence, $\alpha(\boldsymbol{x})$ is a non-zero ternary vector. Also, for any $\boldsymbol{z} = (z_1,\ z_2, \ldots, z_n) \in C$, $\langle \boldsymbol{z}, \boldsymbol{x}\rangle = 0$. Since α is a ring homomorphism, we have

$$\begin{aligned}\alpha(\boldsymbol{z}) \cdot \alpha(\boldsymbol{x}) &= \sum_{i=1}^{n} \alpha(z_i)\alpha(x_i)\\ &= \alpha\left(\sum_{i=1}^{n} z_i x_i\right)\\ &= \alpha(\langle \boldsymbol{z}, \boldsymbol{x}\rangle)\\ &= \alpha(0)\\ &= 0.\end{aligned}$$

This implies that $\alpha(\boldsymbol{x}) \in Res(C)^{\perp}$, which contradicts that $Res(C)$ is LCD. Hence, $\boldsymbol{x} = \mathbf{0}$. Therefore, C is right-ACD. □

Next, we see that the inverse assertion of Corollary 4.18 is true.

Corollary 5.12. *A free linear R-code C is LCD if Res(C) is a ternary LCD code.*

Proof. Since $Res(C)$ is a ternary LCD code, proving C is an LCD code is sufficient to prove two conditions of Proposition 5.11. By Theorem 5.10,

$$\begin{aligned} Tor(C)c \cap C^{\perp_R} &= Tor(C)c \cap \langle Res(C)^{\perp} \rangle_R \\ &= Res(C)c \cap \langle Res(C)^{\perp} \rangle_R \\ &= Res(C)c \cap Res(C)^{\perp}c \\ &= (Res(C) \cap Res(C)^{\perp})c \\ &= \{\mathbf{0}\}. \end{aligned}$$

Since C is free, $Res(C) = Tor(C)$ implies that $m_1 = m_2$. Hence, C is right-ACD. Thus, C is an LCD code over R. □

Theorem 5.13. *Let C be an additive R-code of length n. Then* $Res(C^{\perp_R}) = Tor(C^{\perp_R}) = Res(C)^{\perp}$.

Proof. By Theorem 5.3, $C^{\perp_R}$ is free. Hence, $Res(C^{\perp_R}) = Tor(C^{\perp_R})$. Let $\boldsymbol{u}a + \boldsymbol{v}c \in C^{\perp_R}$ and $\boldsymbol{u'}a + \boldsymbol{v'}c \in C$ where $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{u'}$ and $\boldsymbol{v'}$ are ternary vectors. Then $\alpha(\boldsymbol{u}a + \boldsymbol{v}c) = \boldsymbol{u}$ is an arbitrary element of $Res(C^{\perp_R})$ and $\alpha(\boldsymbol{u'}a + \boldsymbol{v'}c) = \boldsymbol{u'}$ is an arbitrary element of $Res(C)$. Since $\langle \boldsymbol{u'}a + \boldsymbol{v'}c,\ \boldsymbol{u}a + \boldsymbol{v}c \rangle = 0$ and α is a ring homomorphism,

$$\alpha(\langle \boldsymbol{u'}a + \boldsymbol{v'}c,\ \boldsymbol{u}a + \boldsymbol{v}c \rangle) = \alpha(\boldsymbol{u'}a + \boldsymbol{v'}c) \cdot \alpha(\boldsymbol{u}a + \boldsymbol{v}c) = \boldsymbol{u'} \cdot \boldsymbol{u} = 0.$$

This shows that $\boldsymbol{u} \in Res(C)^{\perp}$. Hence, $Res(C^{\perp_R}) \subseteq Res(C)^{\perp}$. For converse inclusion, let $\boldsymbol{u} \in Res(C)^{\perp}$. Also, let $\boldsymbol{x} = \boldsymbol{v}a + \boldsymbol{w}c \in C$ be an arbitrary codeword of C for some ternary vectors $\boldsymbol{v}$ and $\boldsymbol{w}$. Since $\alpha(\boldsymbol{x}) = \alpha(\boldsymbol{v}a + \boldsymbol{w}c) = \boldsymbol{v}$, $\boldsymbol{v} \in Res(C)$. Then

$$\langle \boldsymbol{x}, \boldsymbol{u}a \rangle = \langle \boldsymbol{v}a + \boldsymbol{w}c, \boldsymbol{u}a \rangle = \langle \boldsymbol{v}a, \boldsymbol{u}a \rangle + \langle \boldsymbol{w}c, \boldsymbol{u}a \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle a + 0 = 0.$$

This implies that $\boldsymbol{u}a \in C^{\perp_R}$. Since $\alpha(\boldsymbol{u}a) = \boldsymbol{u}$, $\boldsymbol{u} \in Res(C^{\perp_R})$. Therefore, $Res(C)^{\perp} \subseteq Res(C^{\perp_R})$. Thus, $Res(C^{\perp_R}) = Res(C)^{\perp}$. Hence, the result. □

Corollary 5.14. *Let C be an R-linear code. Then* $Tor(C)^{\perp} \subseteq Tor(C^{\perp_R})$. *Equality holds if C is free.*

Proof. For a linear code C, we know that $Res(C) \subseteq Tor(C)$ and $Res(C^{\perp_R}) \subseteq Tor(C^{\perp_R})$. Then, $Tor(C)^{\perp} \subseteq Res(C)^{\perp}$. By Theorem 5.13, $Res(C^{\perp_R}) = Res(C)^{\perp}$. Therefore, $Tor(C)^{\perp} \subseteq Res(C^{\perp_R}) \subseteq Tor(C^{\perp_R})$. Now, suppose C is free. By Theorem 5.3, $C^{\perp_R}$ is also free. Hence, $Tor(C^{\perp_R}) = Res(C^{\perp_R}) = Res(C)^{\perp}$. Thus, $Tor(C^{\perp_R}) = Tor(C)^{\perp}$ as C is free. □

Next, we see that the inverse assertion of Proposition 5.11 holds partially.

Proposition 5.15. *Let C be a right-ACD R-code of length n and* $Cc \cap C^{\perp_R} = \{\mathbf{0}\}$. *Then* $3^{m_1} \times 3^{m_2} \times 9^{n-m_1} = 9^n$ *(i.e.,* $m_1 = m_2$*),* $Tor(C)c \cap C^{\perp_R} = \{\mathbf{0}\}$*, and Res(C) is a ternary LCD code.*

Proof. We know that $|C| = |Res(C)||Tor(C)| = 3^{m_1} \times 3^{m_2}$, $|C^{\perp_R}| = 9^{n-m_1}$. Since C is right-ACD, $3^{m_1} \times 3^{m_2} \times 9^{n-m_1} = 9^n$, i.e., $m_1 = m_2$. Let $\boldsymbol{x} \in Tor(C)c \cap C^{\perp_R}$. Then $\boldsymbol{x} \in Tor(C)c$ and $\boldsymbol{x} \in C^{\perp_R}$. Since $\boldsymbol{x} \in Tor(C)c$, we have $\boldsymbol{x} = \boldsymbol{u}c$ for some $\boldsymbol{u} \in Tor(c)$. Then, by definition of the torsion code, $\boldsymbol{x} \in C$. Hence, $\boldsymbol{x} \in C \cap C^{\perp_R}$ implies that $\boldsymbol{x} = \mathbf{0}$. Therefore, $Tor(C)c \cap C^{\perp_R} = \{\mathbf{0}\}$. Let $\boldsymbol{u} \in Res(C) \cap Res(C)^{\perp}$. Then $\boldsymbol{u} \in Res(C)^{\perp} = Res(C^{\perp_R})$. Hence, by Theorem 5.13, there are $\boldsymbol{u}a + \boldsymbol{v}c \in C$ and $\boldsymbol{u}a + \boldsymbol{v'}c \in C^{\perp_R}$ such that $\alpha(\boldsymbol{u}a + \boldsymbol{v}c) = \alpha(\boldsymbol{u}a + \boldsymbol{v'}c) = \boldsymbol{u}$. Since $C^{\perp_R}$ is a linear R-code,

$$(\boldsymbol{u}a + \boldsymbol{v'}c)c = \boldsymbol{u}ac + \boldsymbol{v'}cc = \boldsymbol{u}c \in C^{\perp_R}.$$

Additionally, $(\boldsymbol{u}a + \boldsymbol{v}c)c = \boldsymbol{u}c \in Cc$. Hence, $\boldsymbol{u}c \in Cc \cap C^{\perp_R}$ implies that $\boldsymbol{u}c = \mathbf{0}$. Then, $\boldsymbol{u} = \mathbf{0}$. Therefore, $Res(C)$ is a ternary LCD code. □

Proposition 5.16. *Let C be an additive R-code. Then $Res(C^{\perp_L}) \subseteq Tor(C)^\perp$. Equality holds if C is linear.*

Proof. Let C be an additive R-code and $\boldsymbol{x} = \boldsymbol{u}a + \boldsymbol{v}c \in C^{\perp_L}$ be an arbitrary codeword of $C^{\perp_L}$ for some ternary vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Then $\alpha(\boldsymbol{x}) = \alpha(\boldsymbol{u}a + \boldsymbol{v}c) = \boldsymbol{u}$ is an arbitrary element of $Res(C^{\perp_L})$. Now, let $\boldsymbol{y} \in Tor(C)$. Then $\boldsymbol{y}c \in C$. Further, $\boldsymbol{x} \in C^{\perp_L}$ and $\boldsymbol{y}c \in C$ imply that

$$0 = \langle \boldsymbol{x}, \boldsymbol{y}c\rangle = \langle \boldsymbol{u}a + \boldsymbol{v}c, \boldsymbol{y}c\rangle = \langle \boldsymbol{u}a, \boldsymbol{y}c\rangle + \langle \boldsymbol{v}c, \boldsymbol{y}c\rangle = \langle \boldsymbol{u}, \boldsymbol{y}\rangle c + 0 = \langle \boldsymbol{u}, \boldsymbol{y}\rangle c.$$

Hence, $\langle \boldsymbol{u}, \boldsymbol{y}\rangle = 0$ and so $\boldsymbol{u} \in Tor(C)^\perp$. Thus, $Res(C^{\perp_L}) \subseteq Tor(C)^\perp$.

On the other hand, suppose C is linear and $\boldsymbol{x} \in C$. Then, by Theorem 3.5, $\boldsymbol{x} = \boldsymbol{u}a + \boldsymbol{v}\boldsymbol{c}$ for some $\boldsymbol{u} \in Res(C) \subseteq Tor(C)$ and $\boldsymbol{v} \in Tor(C)$. Let $\boldsymbol{w} \in Tor(C)^\perp$. Then, the inner product

$$\langle \boldsymbol{w}a, \boldsymbol{x}\rangle = \langle \boldsymbol{w}a, \boldsymbol{u}a + \boldsymbol{v}c\rangle = \langle \boldsymbol{w}a, \boldsymbol{u}a\rangle + \langle \boldsymbol{w}a, \boldsymbol{v}c\rangle = \langle \boldsymbol{w}, \boldsymbol{u}\rangle a + \langle \boldsymbol{w}, \boldsymbol{v}\rangle c = 0 + 0 = 0.$$

This shows that $\boldsymbol{w}a \in C^{\perp_L}$. Since $\alpha(\boldsymbol{w}a) = \boldsymbol{w}$, $\boldsymbol{w} \in Res(C^{\perp_L})$. Therefore, $Tor(C)^\perp \subseteq Res(C^{\perp_L})$. Thus, $Res(C^{\perp_L}) = Tor(C)^\perp$. □

Proposition 5.17. *Let C be an R-linear code. Then it is free if and only if $Res(C^{\perp_L}) = Res(C^{\perp_R})$.*

Proof. Let C be a free R-linear code. Then, by Theorem 5.13, $Res(C^{\perp_R}) = Res(C)^\perp$ and by Proposition 5.16, $Res(C^{\perp_L}) = Tor(C)^\perp$. Since C is free, $Res(C) = Tor(C)$ and so $Res(C)^\perp = Tor(C)^\perp$. Therefore, $Res(C^{\perp_L}) = Res(C^{\perp_R})$. Conversely, suppose C is an R-linear code such that $Res(C^{\perp_L}) = Res(C^{\perp_R})$. Then, by Theorem 5.13 and Proposition 5.16, $Tor(C)^\perp = Res(C)^\perp$. This implies that $Tor(C) = Res(C)$. Thus, C is free. □

Next, we give an example to illustrate Propositions 5.9 and 5.11.

Example 5.18. *Consider an additive R-code C with an additive generator matrix*

$$G = \begin{bmatrix} a & b & 0 \\ 0 & 0 & b \end{bmatrix}.$$

For any $(x,\ y,\ z) \in C^{\perp_R}$, we have

$$\langle (a,\ b,\ 0),\ (x,\ y,\ z)\rangle = 0 \text{ and } \langle (0,\ 0,\ b),\ (x,\ y,\ z)\rangle = 0.$$

Hence, we get two equations: $ax + by = x + y = 0$ and $z = 0$. These imply that $y = 2x$ and $z = 0$. Therefore, $C^{\perp_R}$ can be given by the set

$$C^{\perp_R} = \{\ (x,\ 2x,\ 0) \mid x \in R\ \}.$$

It is easy to see that $C \cap C^{\perp_R} = \{\mathbf{0}\}$, $Cc \cap C^{\perp_R} = \{\mathbf{0}\}$, and $|C| = |C^{\perp_R}| = 9$. Then $|C||C^{\perp_R}| = 9^2 < 9^3$. Hence, we can add $(a, b, 0)c = (c, c, 0)$ and $(0, 0, b)c = (0, 0, c)$ in G to get

$$G_1 = \begin{bmatrix} a & b & 0 \\ 0 & 0 & b \\ c & c & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Consider an additive R-code C_1 with an additive generator matrix G_1. Then, by Proposition 5.9, C_1 is a right-ACD code over the ring R. In order to prove C_1 is really a right-ACD code, we check all the conditions of Proposition 5.11. It is easy to see that the generator matrices for $Res(C_1)$ and $Tor(C_1)$ can be given respectively by

$$Res(G_1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Tor(G_1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly, $m_1 = m_2 = 2$*. Now, let* $(x, y, z) \in Res(C_1)^{\perp}$*. Then*

$$\langle (x, y, z),\ (1, 1, 0)\rangle = 0 \text{ and } \langle (x, y, z),\ (0, 0, 1)\rangle = 0.$$

These give us $x + y = 0$ *and* $z = 0$*. These imply that* $y = 2x$ *and* $z = 0$*. Hence,* $Res(C_1)^{\perp}$ *can be given by the set*

$$Res(C_1)^{\perp} = \{\ (x, 2x, 0) \mid x \in \mathbb{F}_3\ \}.$$

Clearly,

$$Res(C_1) \cap Res(C_1)^{\perp} = \{\mathbf{0}\}.$$

Hence, $Res(C_1)$ *is a ternary LCD code. Further,* $Tor(C_1)c$ *has generator matrix*

$$Tor(G_1)c = \begin{bmatrix} c & c & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Clearly,

$$Tor(C_1)c \cap C_1^{\perp_R} = \{\mathbf{0}\}.$$

Hence, by Proposition 5.11, C_1 *is a right-ACD code.*

6. Conclusion

We have introduced QSD, right-LCD and ACD codes over a non-unital noncommutative local ring R of order 9. The multilevel construction of QSD codes has been accomplished by utilizing self-orthogonal ternary linear codes. Further, the connection between ternary LCD codes and LCD codes over the ring R has been demonstrated. We have introduced right-ACD codes over R and given several conditions for the existence of such codes. Finally, the characterization of right-ACD codes over R has also been presented, incorporating their torsion and residue codes. However, it is still open to see the application of these codes to construct the quantum codes.

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Declarations

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