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On the undecidability of Markov properties for Lie superalgebras

Research Article

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Abstract: In this note, we address an algorithmic problem for Lie superalgebras. As a starting point, we use the Theory of Gröbner-Shirshov bases in order to find a normal form for Higman-Neumann-Neumann-extension (HNN-extension, for short) of Lie superalgebras. Then by using HNN-extension we show that there is no algorithm to determine whether a finitely presented Lie superalgebra satisfies Markov properties.

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1. Introduction

A well-known theorem by Aydan [1] and Rabin [29] states that most group-theoretic properties are not recognizable. In 1951, Markov [21] showed that Markov properties are not decidable for finitely presented semigroups. We recall that a property P of a group is called Markov if, firstly it is preserved under isomorphisms; secondly, there is a finitely presented group with property P and finally, not every finitely presented group can be embedded into a group with property P. For instance, the properties of being either trivial or a group or free or abelian as well as finiteness are all Markov. Markov showed that there is no algorithm to decide whether a given presentation defining a semigroup satisfies the property P or not. Markov's result has been studied for Lie algebras by Bokut in [3] based on Shirshov's algorithmic methods. Moreover, Wasserman in [35] used another approach based on the concept of Higman-Neuman-Neuman-extension (HNN-extension, for short) of Lie algebras to show the udecidability of Markov properties for finitely presented Lie algebras. In this note, we develop Wasserman's result to the

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case of Lie superalgebras that are a generalization of Lie algebras. The importance of Lie superalgebras in deformation of algebras described by M. Gerstenhaber in the papers [11], [12]. Today, various applications of some classes of Lie superalgebras and higher dimensional Lie superalgebras have been appeared in the study of super integrable systems, string theory and theoretical physics (see, for instance, [9], [25], [33], [34] and references therein). We employ the concept of HNN-extension of Lie superalgebras and prove that Markov properties are undecidable for finitely presented Lie superalgebras. The concept of HNN-extension is an important construction in combinatorial group theory introduced by Higman, Neumann and Neumann in [15] stating that if A_1 and A_2 are isomorphic subgroups of a group S, then it is possible to find a group H containing S such that A_1 and A_2 are conjugate to each other in H, and S is embeddable in H. Recently, this concept has been adapted to different algebraic structures by several authors in [16–18, 32, 35]. The Gröbner-Shirshov bases Theory (GS-Basis, for short) plays an effective role in terms of finding a normal form for HNN-extension. We recall the HNN-extension of Lie superalgebras introduced by Ladra and his colleagues in [17] and introduce a special case of that via semi-direct products. Then we show that there is not an algorithm to determine whether a finitely presented Lie superalgebra satisfies Markov properties or not.

This paper has been organized as follows. In the first section the Composition-Diamond Lemma for Lie superalgebras and the concept of HNN-extension of Lie superalgebras are recalled. Moreover, a special case of HNN-extension is introduced. In the second section, by using the HNN-extension, it is shown that Markov properties for Lie superalgebras are not decidable.

2. Composition-Diamond Lemma and HNN-extension for Lie superalgebras

A Z_2-graded vector space $\mathfrak{L} = \mathfrak{L}_{\bar{0}} \oplus \mathfrak{L}_{\bar{1}}$ is called a Lie superalgebra with a bilinear superbracket structure $[\ ,\ ] : \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L}$ satisfying the graded anti-symmetry and super Jacobi identity as follows:

$$[x, y] = -(-1)^{|x||y|}[y, x],$$

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]],$$

for all homogeneous elements $x, y, z \in \mathfrak{L}$. Moreover, for $x_{\bar{0}} \in \mathfrak{L}_{\bar{0}}$ and $x_{\bar{1}} \in \mathfrak{L}_{\bar{1}}$ we have $[x_{\bar{0}}, x_{\bar{0}}] = 0$ and $[x_{\bar{1}}, [x_{\bar{1}}, x_{\bar{1}}]] = 0$.

Note that the grading $|\ | : \mathfrak{L} \to \{0, 1\}$ for homogenous elements of $\mathfrak{L}$ is defined by $|x| = 0$, if $x \in \mathfrak{L}_{\bar{0}}$ and $|x| = 1$, if $x \in \mathfrak{L}_{\bar{1}}$.

Given a basis $\{x_i\} \subset \mathfrak{L}_{\bar{0}} \cup \mathfrak{L}_{\bar{1}}$ in $\mathfrak{L}$, the structure coefficients are denoted by $[x_i, x_j] = \alpha_{ij}^k x_k$. The structure coefficients satisfy the super Jacobi identity:

$$(-1)^{i(i+j)}\alpha_{ji}^m\alpha_{ki}^l + \alpha_{il}^m\alpha_{jk}^l + (-1)^{k(i+j)}\alpha_{kl}^m\alpha_{ij}^l = 0, \tag{1}$$

and

$$\alpha_{ij}^k = -(-1)^{ij}\alpha_{ji}^k, \tag{2}$$

and $\alpha_{ij}^k = 0$ whenever $|i| + |j| \neq |k|$.

Definition 2.1. *A map $D\colon \mathfrak{L} \to \mathfrak{L}$ is called a derivation of degree m, for some $m \in Z_2$, if it satisfies the following condition*

$$D([a, b]) = [D(a), b] + (-1)^{|m||a|}[a, D(b)].$$

We set $Der\mathfrak{L} = Der\mathfrak{L}_{\bar{0}} \oplus Der\mathfrak{L}_{\bar{1}}$.

We recall some notions concerning the CD-Lemma for Lie superalgebras and use it to obtain a GS-Basis (GS-bases) for HNN-extension of Lie superalgebras. The Composition-Diamond Lemma (CD-Lemma) is used to deal with questions in the context of free algebras. GS-Basis Theory and CD-Lemma have been established for different algebraic structures and used to prove some interesting embedding theorems, normal forms, word problem and rewriting systems (see, [4–6, 13, 19, 30, 36, 37]). In what follows, we first recall some tools in the theory of GS-Basis of Lie superalgebras in accordance with [6]. Then we describe HNN-extension of Lie superalgebras.

Let $Id(S)$ be an ideal of a free Lie superalgebra $\mathfrak{L}(X)$ that is generated by S. The leading term of any polynomial in the ideal generated by S contains $s \in S$ as a subword. Let $X = X_{\bar{0}} \cup X_{\bar{1}}$ be a Z_2-graded generating set with a linear ordering $<$, and X^* (resp., $X^\star$) be the semigroup of associative words on X (resp., the magma of non-associative words on X). A Z_2-grading is defined on the semigroup X^* and the magma $X^\star$ as $X^* = X^*_{\bar{0}} \oplus X^*_{\bar{1}}$ and $X^\star = X^\star_{\bar{0}} \oplus X^\star_{\bar{1}}$, respectively, where $\bar{0}$ and $\bar{1}$ show the even and odd elements, respectively. The length of any word u is denoted by $\mathrm{len}(u)$. The length of the empty word is denoted by 1. By bracketing on a word u, a canonical bracket removing homomorphism $\pi\colon X^\star \to X^*$ is defined by $\pi((u)) = u$ for $u \in X^*$.

We denote by $<_{\text{lex}}$ and $<_{\text{len-lex}}$ the *lexicographic* and *length-lexicographic* orderings on X^*, respectively. These orderings are denoted as:

1. $u < 1$ for any non-empty word u; and by induction, $u <_{\text{lex}} v$ whenever $u = x_i u'$, $v = x_j v'$, and $x_i \prec x_j$ or $x_i = x_j$ and $u' < v'$.
2. $u <_{\text{len-lex}} v$ if $len(u) <_{\text{lex}} len(v)$ or $\mathrm{len}(u) = \mathrm{len}(v)$ and $u <_{\text{lex}} v$.

Remark 2.2. *The the* lexicographic *and* length -lexicographic *orderings are defined on* $X^\star$ *by*

1. $u <_{lex} v$ *iff* $\pi(u) <_{lex} \pi(v)$
2. $u <_{len\text{-}lex} v$ *iff* $\pi(u) <_{len\text{-}lex} \pi(v)$.

Definition 2.3. *A non-empty word u is called a* Lyndon-Shirshov word *if $u \in X$ or for $u = vw$, $vw > wu$. A* super-Lyndon-Shirshov word *is a non-empty word u, if it is either a Lyndon-Shirshov word or it has the form $u = vv$ with v a Lyndon-Shirshov word in $X^*_{\bar{1}}$.*

Definition 2.4. *A non-empty non-associative word u is called a* Lyndon-Shirshov monomial *if u is either an element of X or*

1. *if $u = u_1 u_2$, then u_1, u_2 are Lyndon-Shirshov monomials with $u_1 > u_2$,*
2. *if $u = (v_1 v_2) w$ then $v_2 \le w$.*

A super-Lyndon-Shirshov monomial *is a non-empty non-associative word u that is either a Lyndon-Shirshov monomial or it has the form $u = vv$ with v a Lyndon-Shirshov monomial in $X^\star_{\bar{1}}$ (see [6]).*

Remark 2.5. *There is a one to one correspondence between the set of super-Lyndon-Shirshov words and the set of super-Lyndon-Shirshov monomials.*

Lemma 2.6. *[6, Lemma 2.3] Let u and v be super-Lyndon-Shirshov words such that v is contained in u as a subword. Write $u = avb$ with $a, b \in X^*$. Then there is an arrangement of brackets $[u] = [(a[v]b)]$ on u such that $[v]$ is a super-Lyndon-Shirshov monomial, $\overline{[u]} = u$ and the leading coefficient of $[u]$ is either 1 or 2.*

The *bracket* on a super-Lyndon-Shirshov word $u = avb$ with v as a subword is defined as follows:

1. $[u]_v = (a[v]b)$ if the leading coefficient of $[u]$ is 1,
2. $[u]_v = \dfrac{1}{2}(a[v]b)$ if the leading coefficient of $[u]$ is 2,

There are two types of composition of polynomials in Lie superalgebras as follows.

1. **Compositions of associative polynomials**.
   (a) If $w = \bar{f}a = b\bar{g}$ with $len(\bar{f}) > len(b)$ and $a, b \in X^*$, then the intersection composition is
   $$(f, g)_w = fa - bg.$$
   (b) If $w = \bar{f} = a\bar{g}b$, then the inclusion composition is defined as
   $$(f, g)_w = f - agb.$$
   Note that $\overline{(f, g)_w} <_{\text{deg-lex}} w$.
2. **Compositions of Lie polynomials**. and $\bar{g}$.
   (a) If there exist some a and b in X^* such that $\bar{f}a = b\bar{g} = w$ with $len(\bar{f}) > len(b)$, then the intersection composition is defined as
   $$\langle f, g\rangle_w = [w]_f - [w]_g.$$
   (b) If there exists $a, b \in X^*$ such that $\bar{f} = a\bar{g}b = w$, then the inclusion composition is defined as
   $$\langle f, g\rangle_w = f - [w]_g.$$
   We note that $\overline{\langle f, g\rangle} <_{\text{deg-lex}} w$.

Let S be a set of monic polynomials in $\mathfrak{L}(X) \subset \mathfrak{A}\langle X\rangle$ and $Id(S)$ be the (Lie) ideal generated by S in the free Lie superalgebra $\mathfrak{L}(X)$.

1. $$f - g = \sum \alpha_i a_i s_i b_i,$$
   for some $\alpha_i \in \mathbb{K}$ and $a_i, b_i \in X^*$, $s_i \in S$ with $a_i s_i b_i <_{\text{len-lex}} w$, then S is said to be *closed under associative composition*.
2. If $f, g \in \mathfrak{L}(X)$ and $w \in X^*$ such that
   $$f - g = \sum \alpha_i(a_i(s_i)b_i),$$
   for some $\alpha_i \in \mathbb{K}$ and $a_i, b_i \in X^*$, $s_i \in S$ with $\overline{(a_i(s_i)b_i)} <_{\text{len-lex}} w$, then S is said to be *closed under Lie composition*.

A GS-Basis for the ideal $Id(S)$ is a set of monic polynomials that are closed under all possible compositions with respect to S. Note that in some literature the term *being closed under compositions* is expressed as *trivial under compositions*. The next lemma is the known CD-Lemma for Lie superalgebras.

Lemma 2.7. *[6, Lemma 2.5] If S is a Gröbner-Shirshov basis for the ideal $Id(S)$, then for any $f \in Id(S)$, the word $\bar{f}$ contains a subword s with $s \in S$.*

Next we recall some notions from [17] related to the HNN-extension of Lie superalgebras. HNN-extension for Lie superalgebras was introduced by Ladra and his colleagues in [17]. They used it to provide an alternative proof for a well-known embedding theorem proved by Shirshov [31]: that every Lie algebra of at most countable dimension is embeddable into a Lie algebra with two generators. It is worth pointing out that this theorem was also proved for Lie superalgebras by Mikhalev [23] and [22]. Assume that the derivation map $d\colon \mathfrak{S} \to \mathfrak{L}$ is defined on subalgebra $\mathfrak{S}$ of Lie superalgebra $\mathfrak{L}$. The HNN-extension is a Lie superalgebra introduced as

$$\mathfrak{H}_d(\mathfrak{L}) := \langle \mathfrak{L}, t : d(a) = [t, a],\ a \in \mathfrak{S} \rangle. \tag{3}$$

The new symbol t which does not belong to $\mathfrak{L}$ along with the new relation $[t, a] = d(a)$ ($a \in \mathfrak{S}$) are added to the presentation of $\mathfrak{L}$. t is called the stable letter. The HNN-extension 3 has an equivalent presentation by structure coefficients. Let X and Y be the linear bases of $\mathfrak{L}$ and the subalgebra $\mathfrak{S}$, respectively, such that $Y < X \setminus Y$. We consider the superbrackets $[x, y] = \sum_v \alpha_{xy}^v v$ and $[x_{\bar{1}}, x_{\bar{1}}] = \sum_v \alpha_{x_{\bar{1}}, x_{\bar{1}}}^v v$ for $x, y \in X$. Since $\mathfrak{S}$ is a subalgebra, $\alpha_{ab} = 0$ for all $a, b \in Y$ such that $v \notin Y$. Moreover, $d(a) = \sum_v \beta_a^v v$, for all $a \in Y$. The following equivalent presentation for HNN-extension of Lie superalgebra $\mathfrak{L}$ has been introduced in [17]:

$$\begin{aligned} \mathfrak{H}_d(L) := \langle X, t\ | f_{xy} &= [x, y] - \sum_v \alpha_{xy}^v v, \\ f_{xx} &= [x_{\bar{1}}, x_{\bar{1}}] - \sum_v \alpha_{x_{\bar{1}}, x_{\bar{1}}}^v v, \\ g_a &= [t, a] - \sum_v \beta_a^v v, \\ & x, x_{\bar{1}}, y \in X,\ x > y,\ |x_{\bar{1}}| = 1,\ |t| = |d|,\ a \in Y \rangle. \end{aligned} \tag{4}$$

In what follows, by computation of composition of polynomials in presentation 4 we get a Gröbner-Shirshov bases for free Lie superalgebra $\mathfrak{H}_d(\mathfrak{L})$. Indeed, there are five possible compositions of polynomials in $\mathfrak{H}_d(\mathfrak{L})$, namely, $f_{xy} \wedge f_{yz}$, $g_a \wedge f_{ab}$, $g_a \wedge f_{aa}$ for $|a| = 1$, $f_{xx} \wedge f_{xy}$ for $|x| = 1$ and $f_{xy} \wedge f_{yy}$, for $|y| = 1$. It has been shown in [17] that the first two compositions are closed modulo $S = \{f_{xy}, g_a, f_{xx}\}$. We show that the rest of compositions are also closed modulo S, therefore, the set S forms a GS-Basis for HNN-extension $\mathfrak{H}_d(L)$. It is worth pointing out that the Theorem 2.8 which has been proved in [17] and [23] remains unchanged; however, an explicit GS-Basis enables us to find a normal form for $\mathfrak{H}_d(\mathfrak{L})$ and use it to prove the Theorems 2.9 and 3.3. We provide the composition $\langle f_{x_0 x_1}, f_{x_1 y} \rangle_{x_0 x_1 y}$ as follows:

$$\begin{aligned} \langle f_{x_0 x_1}, f_{x_1 y} \rangle_{x_0 x_1 y} &= [f_{x_0 x_1}, y] - [x_0, f_{x_1 y}] \\ &= (-1)^{|y|}[[x_0, x_1], y] - \sum_{v \in X} \alpha_{x_0 x_1}^v [v, y] - [x_0, [x_1, y]] + \sum_{v \in X} \alpha_{x_1 y}^v [x_0, v] \end{aligned}$$

By super Jacobi identity: $[x_0, [x_1, y]] + [x_1, [y, x_0]] + (-1)^{|y|}[y, [x_0, x_1]] = 0$, so

$$\begin{aligned} \langle f_{x_0 x_1}, f_{x_1 y} \rangle_{x_0 x_1 y} &= [x_1, [y, x_0]] - \sum_{v \in X} \alpha_{x_0 x_1}^v [v, y] + \sum_{v \in X} \alpha_{x_1 y}^v [x_0, v] \\ &= -[x_1, [x_0, y] - \sum_{v \in X} \alpha_{x_0 y}^v v] + \sum_{v \in X} \alpha_{x_0 y}^v ([x_1, v] - \sum_u \alpha_{x_1 v}^u u + \sum_u \alpha_{x_1 v}^u u) \\ &\quad - \sum_v \alpha_{x_0 x_1}^v ([v, y] - \sum_u \alpha_{vy}^u u + \sum_u \alpha_{vy}^u u) \\ &\quad + \sum_v \alpha_{x_1 y}^v ([x_0, v] - \sum_u \alpha_{x_0 y}^u u + \sum_u \alpha_{x_0 y}^u u) \\ &= -[x_1, f_{x_0 y}] + \sum_{v \in X} \alpha_{x_0 y}^v f_{x_1 v} - \sum_v \alpha_{x_0 x_1}^v f_{vy} + \sum_v \alpha_{x_1 y}^v f_{x_0 v} \\ &\quad + \sum_v \sum_u \alpha_{x_0 y}^v \alpha_{x_1 v}^u u - \sum_v \sum_u \alpha_{x_0 x_1}^v \alpha_{vy}^u + \sum_v \sum_u \alpha_{x_1 y}^v \alpha_{x_0 y}^u u. \end{aligned}$$

Then the relation 1 implies that

$$\langle f_{x_0x_1}, f_{x_1y}\rangle_{x_0x_1y} = -[x_1, f_{x_0y}] + \sum_{v\in X} \alpha^v_{x_0y} f_{x_1v} - \sum_v \alpha^v_{x_0x_1} f_{vy} + \sum_v \alpha^v_{x_1y} f_{x_0v},$$

and by definition of composition of Lie polynomials $\overline{\langle f_{x_0x_1}, f_{x_1y}\rangle_{x_0x_1y}} < x_0x_1y$. As stated in [17], since the composition $\langle f_{xy}, f_{yz}\rangle_{xyz}$ is written as a linear combination of polynomials in S, therefore, it is closed modulo S. However, in the case $|x| = 1$, the $\langle f_{xx}, f_{xy}\rangle_{xxy}$ is also closed modulo S. The difference between our calculation and previous ones in [17] lies in the fact that super Jacobi identity implies $\frac{1}{2}[[x, x], y] = [x, [x, y]]$ which enables us to get a closed composition modulo S. The following computations clarify the composition $\langle f_{xx}, f_{xy}\rangle_{xxy}$, where $|x| = 1$. Note that in the case $|x| = 0$, we have $[x_{\bar{0}}, x_{\bar{0}}] = 0$.

$$\begin{aligned}(f_{xx} \wedge f_{xy})_{xxy} &= \frac{1}{2}[f_{xx}, y] - [x, f_{xy}] \\ &= \frac{1}{2}[[x, x], y] - \frac{1}{2}\sum_{v\in X} \alpha^v_{xx}[v, y] - [x, [x, y]] + \sum_{w\in X} \alpha^v_{xy}[x, w]\end{aligned}$$

by super Jacobi identity we have $\frac{1}{2}[[x, x], y] = [x, [x, y]]$, then

$$\begin{aligned}(f_{xx} \wedge f_{xy})_{xxy} &= -\frac{1}{2}\sum_{v\in X} \alpha^v_{xx}[v, y] + \sum_{u\in X} \alpha^u_{xy}[x, u] \\ &\equiv -\frac{1}{2}\sum_{v\in X} \alpha^v_{xx}(\sum_{z\in X} \alpha^z_{vy} z) + \sum_{u\in X} \alpha^u_{xy}(\sum_{z\in X} \alpha^z_{xu} z) \pmod{S}\end{aligned}$$

with respect to $[x, [x, y]]$, which means that the composition $f_{xx} \wedge f_{xy}$ is closed modulo S with respect to $[x, [x, y]]$. Similarly, the composition $f_{xy} \wedge f_{yy}$ is closed modulo S with respect to $[[x, y], y]$, where $|y| = 1$. Also, for composition $g_a \wedge f_{aa}$, where $|a| = 1$ we have

$$\begin{aligned}(g_a \wedge f_{aa})_{taa} &= [g_a, a] - \frac{1}{2}[t, f_{aa}] \\ &= [[t, a], a] - \sum_{v\in X} \beta^v_a [v, a] - \frac{1}{2}[t, [a, a]] + \frac{1}{2}\sum_{v\in B} \alpha^v_{aa}[t, v]\end{aligned}$$

by super Jacobi identity we have $\frac{1}{2}[t, [a, a]] = [[t, a], a]$, so:

$$\begin{aligned}(g_a \wedge f_{aa})_{taa} &= -\sum_{v\in X} \beta^v_{ta}[v, a] + \sum_{v\in B} \frac{1}{2}\alpha^v_{aa}[t, v] \\ &= -\sum_{v\in X} \beta^v_a(\sum_{w\in X} \alpha^w_{va} w) + \frac{1}{2}\sum_{v\in B}\sum_{v\in B} \alpha^v_{aa}(d(v)) \\ &= -(\sum_{v\in X}[\beta^v_a v, a]) + \frac{1}{2}d(\sum_{v\in B} \alpha^v_{aa} v) = -[d(a), a] + \frac{1}{2}d[a, a] = 0.\end{aligned}$$

In view of the above compositions of relations in $H_d(L)$ along with computations in [17] it is seen that

$$S = \{f_{xy} = [x, y] - \sum_v \alpha^v_{xy} v, f_{xx} = [x_{\bar{1}}, x_{\bar{1}}] - \sum_v \alpha^v_{x_{\bar{1}},x_{\bar{1}}} v, g_a = [t, a] - \sum_v \beta^v_a v\}$$

forms a GS-Basis for HNN-extension $\mathfrak{H}_d(\mathfrak{L})$. Therefore, the leading term of any $f \in Id(S)$ contains the super Lyndon-Shirshov words xy with $x, y \in X$ and $x > y$, or ta with $a \in B$, or super Lyndon-Shirshov words $x_{\bar{1}}x_{\bar{1}}$ from the subspace $X^*_{\bar{1}}$. The next theorem is a known embebedding theorem in the context of Lie superalgebras.

Theorem 2.8. *[17, 23] Every Lie superalgebra $\mathfrak{L}$ over a field of char$(K) \neq 2, 3$ with at most countable dimension can be embedded in a Lie superalgebra with two generators.*

Proof. [17, 23, Theorem 2]. □

In addition, the next result is in line with a theorem stating that every finite rank subalgebra of a free Lie superalgebra is completely determined by its nontrivial even component ([2, Theorem 3.5]).

Theorem 2.9. *Let $\mathfrak{L}$ be a Lie superalgebra and $\mathfrak{S}$ be a subalgebra of $\mathfrak{L}$. If $x_{\bar{0}} \in \mathfrak{L} \setminus \mathfrak{S}$, then t and $x_{\bar{0}}$ freely generate a subalgebra.*

Proof. All the super Lyndon-Shirshov words on the letters $x_{\bar{0}}$ and t are in normal form, hence the subalgebra of $\mathfrak{H}_d(L)$ generated by t and $x_{\bar{0}}$ is freely generated by them. □

We observe that a special case of HNN-extension of Lie superalgebras appears when the subalgebra is the whole algebra, that is, $\mathfrak{S} = \mathfrak{L}$. To describe this special case, we recall the semi-direct product of Lie superalgebras introduced in [10].

Given Lie superalgebras $\mathfrak{L}$ and $\mathfrak{M}$, the action of $\mathfrak{M}$ on $\mathfrak{L}$ is a K-linear map of even degree $\mathfrak{M} \times \mathfrak{L} \to \mathfrak{L}$ defined by $(m, l) \mapsto {}^m l$ such that for all homogeneous $m, m' \in \mathfrak{M}$ and $l, l' \in \mathfrak{L}$ we have

1. ${}^{[m,m']}l = {}^{m}({}^{m'}l) - (-1)^{|m||m'|}\,{}^{m'}({}^{m}l)$;
2. ${}^{m}[l, l'] = [{}^{m}l, l'] + (-1)^{|m||l|}[l, {}^{m}l']$.

Given two Lie superalgebras $\mathfrak{L}$ and $\mathfrak{M}$ with an action of $\mathfrak{M}$ on $\mathfrak{L}$, the semidirect product $\mathfrak{L} \rtimes \mathfrak{M}$ is defined by the following bracket on the underlying supermodule $\mathfrak{L} \oplus \mathfrak{M}$:

$$[(l, m), (l', m')] = ([l, l'] + {}^{m}l' - (-1)^{|m'||l|}({}^{m'}l), [m, m']).$$

The next theorem describes the special case of HNN-extensions of Lie superalgebras through the semi-direct product.

Theorem 2.10. *Let $\mathfrak{L}$ be a Lie superalgebra, and $d : \mathfrak{L} \to \mathfrak{L}$ is a derivation map. Then the HNN-extension $\mathfrak{H}_d(\mathfrak{L})$ is isomorphic to the semi-direct product $\mathfrak{L} \rtimes \mathfrak{M}\langle t \rangle$, where $\mathfrak{M}\langle t \rangle$ is a one dimensional Lie superalgebra which acts on $\mathfrak{L}$ via derivation d.*

Proof. Let $B = \{e_i\}_{i \in I} \subset \mathfrak{L}_{\bar{0}} \cup \mathfrak{L}_{\bar{1}}$ be a basis of $\mathfrak{L}$. Let $\mathfrak{T}$ be the subspace of HNN-extension $\mathfrak{H}_d(\mathfrak{L})$ generated by $B \cup \{t\}$. Since $[e_i, t] = d(e_i) \in \mathfrak{L}$, $\mathfrak{T}$ is a subalgebra of $\mathfrak{H}_d(\mathfrak{L})$. Now, in view of the fact that $B \cup \{t\}$ generates $\mathfrak{H}_d(\mathfrak{L})$ as an algebra, we get $\mathfrak{T} = \mathfrak{H}_d(\mathfrak{L})$. Therefore $B \cup \{t\}$ is a generating set for $\mathfrak{H}_d(\mathfrak{L})$. We conclude that there exists a linear isomorphism $\phi : \mathfrak{H}_d(\mathfrak{L}) \to \mathfrak{L} \rtimes \mathfrak{M}\langle t \rangle$ defined by $\phi(e_i) = (e_i, 0), \phi(t) = (0, t)$. Therefore, we get $\phi([t, x]) = (d(x), 0) = [(0, t), (x, 0)] = [\phi(t), \phi(x)]$. This shows that ϕ is a homomorphism of Lie superalgebras. □

3. Undecidability of Markov properties

In this section, we use the HNN-extension to prove the main theorem regarding undecidability of Markov properties for finitely presented Lie superalgebras. It is worth noting that Decision Problem is an important topic in theoretical computer science and various aspects of mathematics. If there exists no algorithm with termination with the correct answer for a given problem in a family, it called an undecidable problem [28]. In the context of groups and automaton (semi)groups, several problems such as word problem and group isomorphism problem have been studied as a decision problem. For instance, D'Angeli et al., [8] studied decision problems for automaton semigroups and automaton groups related to freeness and finiteness. Moreover, Harju [14] investigated Post Correspondence Problem as a decision problem for free groups.

Interested reader in getting a general idea on decision problems with some applications is encouraged to study [24], [28], [26], [27] and references therein. First, we recall the definition of Markov properties for the case of Lie algebras.

Definition 3.1. *[7] Let P be a property of a Lie algebra that is invariant. P is called a* Markov *property if it satisfies the following conditions:*

I. There exist a finitely presented Lie algebra satisfying the property P.

II. There exists a finitely presented Lie algebra which is not embeddable into a finitely presented Lie algebra satisfying the property P.

Example 3.2. *The properties of being abelian, being a free Lie algebra, the solvable word problem are Markov properties.*

In the sequel, we prove the main theorem. Our proof is in the same fashion of a group-theoretic approach used in [20]. There exists another approach based on CD-Lemma to show that Markov properties for finitely presented Lie algebras are unrecognizable ([7, Theorem 3.6.2]).

Theorem 3.3. *Given a Markov property P of a Lie superalgebra, the problem whether a finitely presented Lie superalgebra satisfies P is undecidable.*

Proof. Let $\mathfrak{L}$ be a finitely presented Lie superalgebra with unsolvable word problem. We construct a family of finitely presented Lie superalgebras through HNN-extension. Then by considering two possible cases for a Lie superalgebra expression w, $w = 0$ and $w \neq 0$, we show that our construction satisfies Markov property P if and only if $w = 0$ in $\mathfrak{L}$. Since $\mathfrak{L}$ is a Lie superalgebra with unsolvable word problem, therefore, we conclude that there is no algorithm to decide whether a finitely presented Lie superalgebra satisfies Markov property P.

Let consider a free product $\mathfrak{L}_2 * \mathfrak{L}$, where $\mathfrak{L}_2$ is a finitely presented Lie superalgebra that cannot be embedded in a finitely presented Lie superalgebra satisfying Markov property P. Since the free product $\mathfrak{L}_2 * \mathfrak{L}$ is finitely presented, so Theorem 2.8 implies that $\mathfrak{L}_2 * \mathfrak{L}$ is embeddable into a Lie superalgebra $\mathfrak{H}$ that is generated by two elements u_1 and u_2.

Now, we construct an HNN-extension of a subalgebra that is generated by two elements m, n with stable letter t as

$$\mathfrak{H}_1 = \langle n, m, t : [n, m] = m, [t, n] = n \rangle,$$

where the base subalgebra is generated by n. Theorem 2.9 implies that $m_{\bar{0}}$ and t freely generate a subalgebra of $\mathfrak{H}_1$. We define an HNN-extension of $\mathfrak{H}$ as follows:

$$\mathfrak{H}_2 = \langle \mathfrak{H},\ y, t' \ :\ [y, u_1] = u_1,\ [t', u_2] = u_2 \rangle,$$

where the stable letter is t'. Since y is not in the base subalgebra for the HNN-extension $\mathfrak{H}_2$, therefore, by Theorem 2.9, $y_{\bar{0}}$ and t' freely generate a subalgebra of $\mathfrak{H}_2$. Moreover, by Lemma 3 in [17], there exists a derivation mapping $y_{\bar{0}}$ and t' to themselves, so that an HNN-extension of $\mathfrak{H}_2$ with stable letter t'' can be defined as

$$\mathfrak{H}_3 = \langle \mathfrak{H}_2, t'' \ :\ [t'', y_{\bar{0}}] = y_{\bar{0}},\ [t'', t'] = t' \rangle.$$

Let $w \in \mathfrak{L}$ and $w \neq 0$. We observe that a normal form for HNN-extension $\mathfrak{H}_2$ and Theorem 2.9 imply that $\mathfrak{H}$ intersects the subalgebra generated by y and t' in 0. Therefore, w is not in the base subalgebra for the HNN-extension $\mathfrak{H}_3$. Clearly, t'' and w generate freely a subalgebra of $\mathfrak{H}_3$. Moreover, $r_{\bar{0}}$ and t generate freely a free subalgebra of $\mathfrak{H}$. Having considered the free product with amalgamation of free Lie superalgebras as $\langle \mathfrak{H}_3 * \mathfrak{H}_1 \mid t = t'',\ r_0 = w \rangle$, we conclude that $(\mathfrak{H}_3 * \mathfrak{H}_1) * \mathfrak{L}_1$ contains a copy of $\mathfrak{L}_2$, hence it does not satisfy P.

In the case $w = 0$ in $\mathfrak{L}$, we get $t = 0$ in the free product $\mathfrak{H}_3 * \mathfrak{H}_1$, $m = 0$ in $\mathfrak{H}_1$, $n = 0$ in $\mathfrak{H}_1$, $t'' = 0$ in $\mathfrak{H}_3 * \mathfrak{H}_1$, $y = t' = 0$ in $\mathfrak{H}_2$, and consequently $u_1 = u_2 = 0$ in $\mathfrak{H}_2$. Therefore, all generators of $\mathfrak{H}_3 * \mathfrak{H}_1$ vanish. This implies that $(\mathfrak{H}_3 * \mathfrak{H}_1) * \mathfrak{L}_1$ is isomorphic to $\mathfrak{L}_1$, where $\mathfrak{L}_1$ is a finitely presented Lie superalgebra satisfying Markov property P. Moreover, $(\mathfrak{H}_3 * \mathfrak{H}_1) * \mathfrak{L}_1$ satisfies P. In other words, $(\mathfrak{H}_3 * \mathfrak{H}_1) * \mathfrak{L}_1$ satisfies P if and only if $w = 0$ in $\mathfrak{L}$.

We have shown that $(\mathfrak{H}_3 * \mathfrak{H}_1) * \mathfrak{L}_1$ satisfies Markov property if and only if $w = 0$. Since a finite presentation for $(\mathfrak{H}_3 * \mathfrak{H}_1) * \mathfrak{L}_1$ can be computed given the Lie superalgebra expression for w, therefore, an algorithm for determining whether $\mathfrak{L}$ satisfies P could have been used to solve the word problem. Since $\mathfrak{L}$ is a Lie superalgebra with unsolvable word problem, we conclude that no such algorithm exists. □

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