

Totally projective $QTAG$ -modules and generalizations

Research Article

Ayazul Hasan, Mohd Noman Ali, Vinit Kumar Sharma

Abstract: In this project, we prioritize our study on some types of generalized torsion abelian groups. The torsion abelian group is an important tool in the theory of modules. Analogous to this concept, we study the totally projective modules and discuss its relation with isotype as well as separable submodules. One of the main purposes of the present paper is to give a necessary and sufficient condition for an isotype submodule of a totally projective module to be itself a totally projective module.

2020 MSC: 16K20, 13C12, 13C13

Keywords: $QTAG$ -modules, Totally projective modules, Isotype submodules, Separable submodules

1. Introduction

It was discovered in the 1970s that torsion abelian groups might be included in the theory of modules. The subject known to demonstrate how to achieve this is called torsion abelian group-like-module or for simpleness just a TAG -module. Singh [23] was the first to think about TAG -modules, although he was only interested in TAG -modules M_R having the following conditions relating to uniserial modules while the rings were associated with unity.

(i) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(ii) Given any two uniserial submodules U_1 and U_2 of a homomorphic image of M , for any submodule N of U_1 , any non-zero homomorphism $\phi : N \rightarrow U_2$ can be extended to a homomorphism $\psi : U_1 \rightarrow U_2$, provided the composition length $d(U_1/N) \leq d(U_2/\phi(N))$ holds.

Following this, many researchers have applied fundamental ideas on torsion abelian groups-like-modules from an abelian group theory and improved the concept of module theory (see, for instance,

Ayazul Hasan; College of Applied Industrial Technology, Jazan University, Jazan- P.O. Box 2097, Kingdom of Saudi Arabia (email: ayazulh@jazanu.edu.sa).

Mohd Noman Ali, Vinit Kumar Sharma; Department of Mathematics, Shri Venkateshwara University, Gajraula, Amroha-Uttar Pradesh, India (email: mohdnoman79@rediffmail.com, vksharma.J@gmail.com).

[1, 25]). In 1987 Singh followed this up in his another work [24] and introduced the notion of quasi-torsion abelian groups-like-modules (*QTAG*-modules, for short) by applying condition (i) only. Some authors have presented many important papers on the development types of abelian groups; for example, Danchev [3, 4] presented a study of isotype subgroups in abelian p -groups. Leininger and McReynolds [17] found important results on separable subgroups of mapping class groups. In addition, Walker [26] investigated the influence of *Ulm*'s theorem for totally projective groups.

Furthermore, extensive research has been conducted on the concept of generalized torsion abelian groups in *QTAG*-modules (see, for example [15, 16]). Subsequently, many studies have introduced the use of generalized torsion abelian groups in *QTAG*-modules, such as α -modules [9], Σ -uniserial modules [11], and strongly U -transitive modules [21] etc. Some studies have applied these to the concept of a direct sum of countably generated modules that contribute to enriching this research area too [10, 13].

Finally, the research is organized as follows. The first section, i.e. here, looks at the subject's background and related studies. In Section 2, we briefly discuss several important concepts pertinent to our investigation. The study of totally projective modules is discussed in Section 3, and important theorems and distinctive properties of isotype as well as separable submodules are presented. After that, in Section 4, we pose some unanswered questions that seem to be interesting.

2. Preliminaries

We begin with some terminology. Let all rings into consideration be associative with unity ($1 \neq 0$), and let modules be unital *QTAG*-modules. A uniserial module M is a module over a ring R , whose submodules are totally ordered by inclusion. This means simply that for any two submodules S_1 and S_2 of M , either $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$. An element $u \in M$ is uniform, if uR is a non-zero uniform (hence uniserial) module and for any module M with a unique decomposition series, the symbol $d(M)$ will denote its decomposition length. If u is a uniform element of M (i.e., $u \in M$), then $e(u)$ is called the exponent of u , and $e(u) = d(uR)$. As usual, for such a module M , we set the height of u in M as $H_M(u) = \sup\{d(vR/uR) : v \in M, u \in vR \text{ and } v \text{ uniform}\}$. For every non-negative integer t , $H_t(M) = \{u \in M \mid H_M(u) \geq t\}$ denotes the t -th copies of M which can be viewed as a submodule of M consisting of all elements of height at least t . For a module M , the letter M^1 will always denote in the sequel the submodule of M , containing elements of infinite height. The module M is termed h -divisible if $M = M^1 = \bigcap_{t=0}^{\infty} H_t(M)$, or equivalently, if $H_1(M) = M$. With this in hand, a module M is h -reduced if it does not contain any h -divisible submodule, i.e., it is free from the elements of infinite height. The sum of all simple submodules of M is called the socle of M , denoted by $Soc(M)$. For all $t \geq 0$, $f_M(t)$ is the t -th *Ulm* invariant of M and is equal to $Soc(H_t(M))/Soc(H_{t+1}(M))$.

Next, we add some familiarity as well in terms of infinite height of M . If α is an ordinal, and M is a *QTAG*-module, then the infinite height $H_\alpha(M)$ is defined inductively as follows: $H_0(M) = M$, and if $\alpha > 0$, then $H_\alpha(M) = \bigcap_{\gamma < \alpha} H_\gamma(M)$. Clearly, $H_\alpha(M)$ is a submodule of M , consisting of elements of height at least α . This submodule is also known as α -th *Ulm* submodule.

A submodule N of M is said to be α -pure [20] if, for all ordinal γ , there exists an ordinal α (depending on N) such that $H_\gamma(M) \cap N = H_\gamma(N)$. Besides, a submodule N of M is named isotype, if it is α -pure for every ordinal α . A submodule N of M is said to be nice [18] in M , if the equality $H_\alpha(M/N) = (H_\alpha(M) + N)/N$ holds for all ordinals α , i.e. every coset of M modulo N may be represented by an element of the same height.

An h -reduced module M is totally projective if it has a collection $\mathcal{N}$ of nice submodules such that (i) $0 \in \mathcal{N}$ (ii) if $\{N_i\}_{i \in I}$ is any subset of $\mathcal{N}$, then $\sum_{i \in I} N_i \in \mathcal{N}$ (iii) given any $N \in \mathcal{N}$ and any countable subset X of M , there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated. Call a collection $\mathcal{N}$ of nice submodules of M which satisfies conditions (i), (ii) and (iii) a nice system (see [19]) for M .

We also make use of concepts that can be found, for instance, in [12], and that we briefly review. We say that a submodule S of M is separable if for each $x \in M$ there is a corresponding countably generated

submodule N of S such that

$$\sup\{H_M(x + y) : y \in S\} = \sup\{H_M(x + z) : z \in N\}.$$

Likewise, two submodules S and N of M are compatible, written $S \parallel N$, if for each pair $(x, y) \in S \times N$ there exists $z \in S \cap N$ such that

$$H_M(x + y) \leq H_M(x + z).$$

It is interesting to note that almost all the results that hold for TAG -modules are also valid for $QTAG$ -modules [20]. Many results, stated in the present paper, are generalizations from the reference [14]. For some other rigorous treatment of the topic mentioned here, the readers can see in [5, 8]. In what follows, all notations and notions are standard and will be in agreement with those used in [6, 7]; for the specific ones, we refer the readers to [2].

3. Chief results

Totally projective modules play an important role in the modern theory of modules. We refer to [12, 22] for details and interesting results. The totally projective modules as studied in [13], a generalization of the concept of a direct sum of countably generated h -reduced $QTAG$ -modules. The criterion states as follows: If M is a direct sum of countably generated h -reduced $QTAG$ -modules and S is an isotype submodule of countable length then S must also be a direct sum of countably generated modules. Thus we give here a complete description of certain submodules of the direct sum of countably generated modules for isotype submodules. For the sake of simplicity, we have described our result for the direct sum of countably generated modules, which is an important special case, but everything is done in the more general setting of totally projective modules. Not only are totally projective modules determined by their Ulm invariants but they are one of the most interesting classes of $QTAG$ -modules.

Before proving the major statements that motivate this section, we note in passing the following simple fact.

Proposition 3.1. *Let S be a submodule of a $QTAG$ -module M . Then S is separable in M if and only if, for each $x \in M$, the set $\{H_M(x + y) : y \in S\}$ has a largest element whenever its supremum is not cofinal with ω .*

Proof. Assume that S is separable in M and let $\alpha = \sup\{H_M(x + y) : y \in S\}$. If α is not cofinal with ω , then there is a countably generated submodule N of S such that $\alpha = \sup\{H_M(x + z)\}$ for some $z \in N$. This, in turn, implies that $\alpha = H_M(x + z)$.

Conversely, if there exists $z \in N$ such that $\alpha = H_M(x + z)$. Observe that

$$H_M(x + z) = \alpha = \sup\{H_M(x + y) : y \in S\}$$

where α is not cofinal with ω , then it follows at once that S is separable in M . $\square$

The following theorem establishes a necessary condition for an isotype submodule to be totally projective.

Theorem 3.2. *Let S be an isotype submodule of a $QTAG$ -module M . Then S is separable in M , provided S is totally projective.*

Proof. Suppose that S is totally projective. If S is not separable in M , there exist $x \in M$ and a limit ordinal α such that $H_M(x + y) < \alpha$ for each $y \in S$. Then, if α is not cofinal with ω , we have $\alpha = \sup\{H_M(x + y) : y \in S\}$. Since $S/H_\alpha(S)$ is totally projective and $S/H_\alpha(S) \cong (S, H_\alpha(M))/H_\alpha(M)$ is isotype in $M/H_\alpha(M)$. Without loss of generality, we may assume that $H_\alpha(M) = 0$. Then $H_M(x + y) =$

$H_{M/H_\alpha(M)}(x + y + H_\alpha(M))$ for each $y \in S$. Suppose now that M has length α . It is apparent to see that S is the direct sum of submodules having length less than α . Setting $S_\beta = 0$, we obtain $S = \Sigma_{\beta < \alpha} S_\beta$ such that $H_\beta(S_\beta) = 0$. Now for each $\sigma < \alpha$, there exists $\gamma < \alpha$ such that $\sigma < H_M(x + y) < \alpha$ where $y \in \Sigma_{\beta < \gamma} S_\beta$. Now we choose σ_0 such that $\sigma_0 < \alpha$. Therefore there exists a sequence $\{y_k\}$, $1 \leq k \leq r$ in S , and a strictly increasing sequence of ordinals

$$\sigma_0 < \sigma_1 < \cdots < \sigma_r < \alpha$$

such that the following conditions are satisfied for $k \leq r$.

- (i) $\sigma_{k-1} < H_M(x + y_k) \leq \sigma_k$,
- (ii) $y_k \in \Sigma_{\beta < \sigma_k} S_\beta$,
- (iii) $y_k = \Sigma_{\beta < \sigma_k} y_{k,\beta}$ for $y_{k,\beta} \in S_\beta$ and $H_M(y_{k,\beta}) > \sigma_{n-1}$ such that $y_{k,\beta} = 0$.

Let $\sigma_{r+1} > \sigma_r$, and suppose that $y_{r+1} \in S$ such that the conditions (i) – (ii) hold for $k = r + 1$. After this, let us assume that

$$y_{r+1} = \Sigma_{\beta < \sigma_{r+1}} y_{r+1,\beta}$$

where $y_{r+1,\beta} \in S_\beta$. Observe that y_{r+1} is a set of representatives of the non-zero elements of $y_{r+1,\beta}$ in $H_{\sigma_{r+1}}(S_\beta)$. Thus, we see that condition (iii) from definition of y_{r+1} is satisfied. Therefore, there exists an infinite sequence $\{\sigma_k\}$ of ordinals less than α such that conditions (i) – (iii) are satisfied for all k .

Now, we set $\sigma = \sup\{\sigma_k\}$, one may see that σ is a cofinal with ω and therefore $\sigma < \alpha$. This means that α is not cofinal with ω , and there exists $y \in S$ such that $H_M(x + y) \geq \sigma$. Now we write

$$y = \Sigma_{\beta < \sigma} y_\beta + \Sigma_{\beta \geq \sigma} y_\beta$$

where $y_\beta \in S_\beta$. Then, for each β , we can easily continue along condition (i) that $\sigma_{k-1} < H_M(y - y_k) \leq \sigma_k$, provided that $H_M(x + y) \geq \sigma$. Since S is isotype in M , we have that $\sigma_{k-1} < H_S(y - y_k) \leq \sigma_k$ for all $k \geq 1$. Therefore, since $y_k \in \Sigma_{\beta < \sigma} S_\beta$, for each k , we get that $H_S(y_\alpha) \geq \sigma$. This, in turn, implies that $H_S(y_\alpha) > \sigma_{k-1}$ and $\sigma = \sup\{\sigma_k\}$ for $\beta \geq \sigma$. Without loss of generality, then, assume $\Sigma_{\beta \geq \sigma} y_\beta = 0$. Thus $y = \Sigma_{\beta < \sigma} y_\alpha$, so that we may assume the QTAG-module contains y as well as y_k for each k .

Furthermore, we observe that $y \in \Sigma_{\beta < \sigma_r} S_\beta$ for some $r \geq 1$. Because $k > r$, we deduce that

$$y_k = \Sigma_{\beta < \sigma_r} y_{k,\beta} + \Sigma_{\beta \geq \sigma_r} y_{k,\beta}$$

and thus condition (iii) implies that $y_{k,\beta} = 0$ by setting $H_S(y_{k,\beta}) \geq H_S(y - y_k) > \sigma_{n-1}$ for $\beta \geq \sigma_r$. However, if $y_k \in \Sigma_{\beta < \sigma_r} S_\beta$ for all k with r , which is absurd, then

$$\sigma_{k-1} < H_S(y_{k+1} - y_k) \leq \sigma_k$$

and $H_\beta(S_\beta) = 0$. Therefore, if $k > r$, then $y_k = y_{k+1}$ and $H_S(y_{k+1} - y_k) \leq \sigma_k$, so that S is separable in M . $\square$

The following lemma established an important connection between compatibility and separability.

Lemma 3.3. *Let S and P be submodules of a QTAG-module M such that S is separable in M . Then there exists a submodule Q of M such that*

- (i) $Q \supseteq P$
- (ii) $g(Q) \subseteq g(P) = \aleph_0$
- (iii) $Q \parallel S$.

Proof. In each element $x \in P$, we choose a sequence $\{s_k(x)\}$ in S such that

$$\sup\{H_M(x + y) : y \in S\} = \sup\{H_M(x + s_k(x))\},$$

for $k < \omega$. If we assume, by way of induction, that $Q_0 = P$, and define $Q_1 = \langle Q_0, s_k(x) \rangle$ for $k < \omega$ and $x \in P$. Write $H_M(y + z) = \lambda$, where $y \in S, z \in Q_0$, and λ is an ordinal. Then, there exists a positive integer k such that $H_M(z + s_k(x)) \geq \lambda$ for $x = z \in Q_0 = P$. We choose $s_k(x) \in Q \cap S$ such that $g(Q_1) \leq g(P) = \aleph_0$. Now if we replace Q_1 by $Q_0 = P$, then Q_2 is replaced by $Q_1 = P$ such that $Q_2 \supseteq Q_1 \supseteq Q_0 = P$ where $g(Q_2) \leq g(P) = \aleph_0$. Then for each pair $(u, y) \in Q_1 \times S$, there exists $v \in Q_2 \cap S$ such that $H_M(u + v) \geq H_M(u + y)$. On continuing the same process to Q_k , we get $Q = \cup_{k < \omega} Q_k$. Thus, we see that conditions (i) – (iii) are satisfied for the submodule Q , and the lemma is proved. $\square$

In the light of the last constructions, we obtain the following strengthening of Theorem 3.2.

Theorem 3.4. *Let M be a QTAG-module with an isotype submodule S of M such that M is a totally projective module of cardinality not exceeding $\aleph_1$. Then S is separable in M if and only if S is totally projective.*

Proof. The necessity follows directly from Theorem 3.2.

Next, we deal with the converse implication. Since M is totally projective, M satisfies a nice system of countability (see [19]). In fact, it possesses a collection $\mathcal{N}$ consisting of nice submodules N_i of M for each i .

In order to show that S is totally projective, it suffices to prove that S has an ascending chain

$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_\beta \subseteq \dots$$

of nice submodules N_β of S satisfying the following conditions:

- (i) $N_{\beta+1}/N_\beta$ is countably generated.
- (ii) $N_\gamma = \cup_{\beta < \gamma} N_\beta$ where γ is a limit.
- (iii) $S = \cup_{\beta < \sigma} N_\beta$ where σ is an arbitrary ordinal.

For a proof of this assertion see [10]. In accordance with Lemma 3.3, since S is separable in M , and if P is any countably generated submodule of M . Then there exists a countably generated submodule Q of M containing P such that $Q \parallel S$. Similarly, there exists a countably generated submodule R of M containing Q that belongs to the collection $\mathcal{N}$. On continuing the same process to these properties, we construct a sequence

$$P \subseteq Q_0 \subseteq R_0 \subseteq Q_1 \subseteq R_1 \subseteq \dots$$

of countably generated submodules of M such that $Q_i \parallel S$ and $R_i \in \mathcal{N}$.

Next, with this in hand, we ascertain the same argument that there exists a countably generated submodule K of M such that $K \parallel S$ and $K \in \mathcal{N}$. Since $g(M) \leq \aleph_1$, it is plainly to see that there exists an ascending chain of countably generated submodules

$$0 = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_\beta \subseteq \dots$$

that leads up to $M = \cup K_\beta$ with the following conditions:

- (i⁺) $K_\gamma = \cup_{\beta < \gamma} K_\beta$ where γ is a limit.
- (ii⁺) $K_\beta \in \mathcal{N}$ for each β .
- (iii⁺) $K_\beta \parallel S$ for each β .

If we let $N_\beta = K_\beta \cap S$, it remains only to show that N_β is nice in S . Our future aim, which we pursue, is to give us the desired chain of submodules N_β of S . To that goal, we suppose for any $x \in S$ that

$$\alpha = \sup\{H_S(x + a_\beta) : a_\beta \in N_\beta\}.$$

This follows immediately that

$$\alpha \leq \sup\{H_S(x + b_\beta) : b_\beta \in K_\beta\}.$$

Since K_β is nice in M , there exists an element $u_\beta \in K_\beta$ such that $H_M(x + u_\beta) \geq \alpha$. Moreover, since $x \in S$ and $K_\beta \parallel S$, there exists an element $v_\beta \in N_\beta = K_\beta \cap S$ such that $H_M(x + v_\beta) \geq \alpha$. Finally, S being isotype in M implies that $H_S(x + v_\beta) \geq \alpha$. Hence, it consequently follows that N_β is nice in S . We complete the proof. $\square$

As a direct consequence, we have the following corollary.

Corollary 3.5. *Let M be a QTAG-module with an isotype submodule S of M such that M is a direct sum of uniserial modules with all its Ulm invariants not exceeding $\aleph_1$. Then S is separable in M if and only if S is totally projective.*

We are now ready to give our example.

Example 3.6. *For each countable ordinal β , let P_β be a countably generated module of length $\beta + 1$ such that $H_\beta(P_\beta) = \langle u_\beta R \rangle$ is a uniserial module of exponent one. Now we want to construct an ascending chain of countably generated modules Q_α defined as follows. Set $Q_0 = P_0$, and suppose that Q_γ is a countably generated module such that $H_\gamma(Q_\gamma) = \langle v_\gamma R \rangle$ is a uniserial module of exponent one. Then we have two cases to consider:*

Case (i). α is a limit. In this case, let us consider $Q_\alpha = \cup_{\gamma < \alpha} Q_\gamma$.

Case (ii). $\alpha - 1$ exists. Let $\gamma = \alpha - 1$ and let Q_α be the pushout associated with the diagram

$$\begin{array}{ccc} H_{\omega_1}(M) & \longrightarrow & P_\alpha \\ \downarrow & & \downarrow \\ Q_\gamma & \longrightarrow & Q_\alpha \end{array}$$

where $H_{\omega_1}(M)$ is mapped onto $H_\gamma(Q_\gamma) = \langle v_\gamma R \rangle$, and similarly onto $H_\alpha(P_\alpha) = \langle u_\alpha R \rangle$. Notice that $Q_\gamma \subseteq Q_\alpha = (Q_\gamma \oplus P_\alpha) / \langle (v_\gamma - u_\alpha)R \rangle$ with the property that $H_\alpha(Q_\alpha) = \langle v_\gamma R \rangle = \langle u_\alpha R \rangle$ is a uniserial module of exponent one.

Now, if x denotes v_α then $H_{Q_\alpha}(xR) = \alpha$ and $Q_{\alpha+1}/\langle xR \rangle = Q_\alpha/\langle xR \rangle \oplus P_{\alpha+1}/\langle xR \rangle$. Therefore, if $Q = \cup_{\alpha < \omega_1} Q_\alpha$ then $H_{\omega_1}(Q) = \langle xR \rangle$ and $Q/\langle xR \rangle$ is a direct sum of uniserial modules. Consider the exact sequence

$$S \hookrightarrow \Sigma \oplus Q_\alpha \twoheadrightarrow Q$$

where $\Sigma \oplus Q_\alpha \rightarrow Q$ is the natural map associated with the inclusion maps $Q_\alpha \hookrightarrow Q$. In order to show that S is isotype in $\Sigma \oplus Q_\alpha$, it remains only to show that $H_\beta(M) \cap S = H_\beta(S)$ for all β . To that goal, let us assume that $M = \Sigma \oplus Q_\alpha$ where the summation is over the countable ordinals α . The proof is by induction on β in conjunction with $H_\beta(M) \cap S = H_\beta(S)$ for $\beta \leq \sigma$ and let $y \in H_{\sigma+1}(M) \cap S$.

Choose $z \in H_\sigma(M)$ such that $z' = y$ where $d(zR/z'R) = 1$ and let $z = \Sigma \oplus u_\alpha$ where $u_\alpha \in Q_\alpha$. Since $z \in H_\sigma(M)$, $u_\alpha \in H_\sigma(Q_\alpha)$, we have $a = \Sigma v_\alpha \in H_\sigma(Q)$ for each α . Therefore to finish the induction, we choose $z' \in S$ such that $d(zR/z'R) = 1$ and $a' = 0$ where $d(aR/a'R) = 1$. Moreover, $a \in H_\sigma(Q)$ implies that $a \in H_\sigma(Q_\alpha)$ for some α . Then we have two cases to consider. Firstly, if a is a multiple of x , then $a \in H_\sigma(Q_\sigma)$. Secondly, if a is not a multiple of x , then $H_{Q_\alpha}(aR) = H_{Q_{\alpha+1}}(aR)$. Thus, we have $a \in H_\sigma(Q_\alpha)$ and $a \supseteq Q_\alpha$. Next, choose μ such that $a \in H_\sigma(Q_\mu)$ and let $\tilde{a} \in M$ have only one non-zero element. Observe that $H_1((z - \tilde{a})R) = y$ and $z - \tilde{a} \in H_\sigma(M) \cap S$. Hence the induction hypothesis, $H_\sigma(M) \cap S = H_\sigma(S)$, implies that $y \in H_{\sigma+1}(S)$, and S is isotype in M . Since S is not separable in M and

$$\sup\{H_M(x + a) : a \in S\} = \omega_1,$$

so referring to Theorem 3.2, we can conclude that S can not be a direct sum of countably generated modules.

Motivated by nice system [19] and weak nice system [12], we make the following definitions.

Definition 3.7. Let N be a submodule of a QTAG-module M . A family $\mathcal{S}$ of separable submodules $S \supseteq N$ of M is called a separable system in M if

- (i) $N \in \mathcal{S}$;
- (ii) if $\{S_i\}_{i \in I}$ is any subset of $\mathcal{S}$, then $\Sigma_{i \in I} S_i \in \mathcal{S}$;
- (iii) given $N \subseteq T \subseteq M$ and T/N is countably generated, there exists $S \in \mathcal{S}$ such that $S \supseteq T$ and S/N is countably generated.

Definition 3.8. Let α be an arbitrary ordinal and M a QTAG-module of countable length. A family $\mathcal{S}$ of separable submodules S of M containing a submodule N of M is called a weak separable system in M if there exists a chain of separable submodules

$$N = S_0 \subseteq S_1 \subseteq \dots S_\beta \subseteq \dots, \quad \beta < \gamma,$$

satisfying the following conditions:

- (i) $S_{\beta+1}/S_\beta$ is countably generated.
- (ii) $S_\alpha = \cup_{\beta < \gamma} S_\beta$ where α is a limit.
- (iii) $M = \cup_{\beta < \gamma} S_\beta$.

The next assertion gives a comparison between two of the defined above classes.

Theorem 3.9. Suppose M is a QTAG-module. In QTAG-modules, separable system and weak separable system are equivalent.

Proof. The first part is straightforward that separable system implies weak separable system.

As for the second part, we shall demonstrate that there exists a family $\mathcal{S}$ of separable submodules S of M containing a submodule N of M satisfying all the conditions of separable system when conditions of weak separable system are satisfied.

Let Γ denote the initial segment of ordinals less than γ ; note that Γ is the index set for the chain of separable submodules S_β associated with weak separable system. For each $\beta \in \Gamma$, let $\{x_{\beta,k}\}$ be a set of representatives for the non-zero cosets of S_β in $S_{\beta+1}$. Without loss of generality, we may assume that $S_\beta \neq S_{\beta+1}$. Then there exist positive integers k 's depending on β such that $S_{\beta+1}/S_\beta$ is countably generated.

In order to show that M satisfies separable system, let us suppose that

$$\mathcal{S} = \{S_\Omega \subseteq M : S_\Omega = \langle N, x_{\beta,k} \rangle\},$$

where Ω ranges over the closed subset of Γ . It is fairly to see that the empty set ϕ is closed. Thus $\mathcal{S}$ satisfies all the conditions of separable system. What remains to show is that S_Ω is separable in M whenever Ω is a closed subset of Γ . In order to do this, among all uniform elements in M , choose x such that $H_M(x + S_\Omega) = \sigma$. Our future aim, which we pursue, is to check $\text{cof}(\sigma) = \omega_0$. To that goal, we assume that $\text{cof}(\sigma) \neq \omega_0$. By hypothesis on separability on N , we have $H_M(x + N) < \sigma$. Letting x have the standard representation

$$x = x_{\mu_{(1)},k_{(1)}} + x_{\mu_{(2)},k_{(2)}} + \dots + x_{\mu_{(r)},k_{(r)}} + y,$$

we prove by induction on $\mu_{(1)}$ in conjunction with $H_M(x + z) = \sigma$ for some $z \in S_\Omega$. Without loss of generality, we assume that $\mu_{(i)} \notin \Omega$ for each i . We construct now a coset $x + S_\Omega$ of M , deleting the terms $x_{\mu_{(i)},k_{(i)}}$ from x as mentioned in the above standard representation. Therefore to finish the induction, we choose $a \in S_\Omega$ such that $H_M(x + a) > \sigma'$ with $\sigma' < \sigma$. Let

$$a = x_{\lambda_{(1)},t_{(1)}} + x_{\lambda_{(2)},t_{(2)}} + \dots + x_{\lambda_{(j)},t_{(j)}} + b$$

be the standard representation of a . Since $a \in S_\Omega$, $\lambda_{(i)} \in \Omega$ for each i . Bearing in mind this construction, it is apparent that $\mu_{(1)} \notin \Omega$ and $\lambda_{(1)} \neq \mu_{(1)}$. Finally, we let $\lambda_{(1)} > \mu_{(1)}$, and a routine computations reveals that

$$H_M(x_{\lambda_{(1)}, t_{(1)}} + S_{\lambda_{(1)}}) \geq H_M(x + a) > \sigma'.$$

Then, there exists $c \in S_\Omega \cap S_{\lambda_{(1)}}$ such that $H_M(x_{\lambda_{(1)}, t_{(1)}} + c) > \sigma'$. Certainly, if we set $u = a - x_{\lambda_{(1)}, t_{(1)}} - c$, then $u \in S_\Omega \cap S_{\lambda_{(1)}}$ and $H_M(x + u) > \sigma'$. We choose $a \in S_\Omega$ such that $\lambda_{(1)}$ is minimal. Observe that $\mu_{(1)} > \lambda_{(1)}$. Therefore $H_M(x + S_{\mu_{(1)}}) \geq \sigma$ and consequently $H_M(x - v) \geq \sigma$ for some $v \in S_{\mu_{(1)}}$, where $S_{\mu_{(1)}}$ is separable. Since $v \in S_{\mu_{(1)}}$, the initial term of its standard representation has a smaller index than $\mu_{(1)}$. Hence by induction hypothesis there exists z in S_Ω such that $H_M(v + z) \geq \sigma$. This means that $H_M(x + z) \geq \sigma$, as required. The proof is completed. $\square$

We are now endowed with enough information to proceed by proving the following criterion for a totally projective module.

Theorem 3.10. *Let M be a totally projective QTAG-module and S be an isotype submodule of M . Then S is totally projective if and only if M satisfies the weak separable system.*

Proof. Foremost, assume that S is totally projective. Let $\mathcal{N}_M$ and $\mathcal{N}_S$, respectively, be families of nice submodules of M and S satisfying nice system. Since S is totally projective, it now follows from Theorem 3.2 that S is separable in M . Suppose that

$$0 = P_0 \subseteq P_1 \subseteq \dots \subseteq P_\beta, \quad \beta < \lambda$$

is an ascending chain of submodules of M satisfying the following conditions:

- (i) $P_\beta \in \mathcal{N}_M$.
- (ii) $P_\beta \cap S \in \mathcal{N}_S$.
- (iii) $P_\beta \parallel S$.
- (iv) $g(P_{\beta+1}/P_\beta) \leq \aleph_0$, $\beta + 1 < \lambda$
- (v) $P_\gamma = \cup_{\beta < \gamma} P_\beta$, where $\gamma < \lambda$ is a limit,

We consider two possibilities. Firstly, if λ is a limit ordinal, we define $P_\lambda = \cup_{\beta < \lambda} P_\beta$ and see that conditions (i) – (v) are satisfied for the chain of submodules P_β . Secondly, if $\lambda - 1$ exists. Then there exists a countably generated extension P_λ of $P_{\lambda-1}$ that satisfies the desired properties.

We construct now a submodule P_λ such that $P_{\lambda-1} = P$. Since P satisfies conditions (i) and (ii), we have $\langle S, P \rangle / P \cong S / S \cap P$ is totally projective, and similarly for M / P . Therefore, $\langle S, P \rangle / P$ is isotype in M / P . Let x be an uniform element of M such that $x + P \in H_\gamma(M / P) = \langle H_\gamma(M), P \rangle / P$. Then $x + P = y + P$ where $y \in H_\gamma(M)$, and $x + z = y \in H_\gamma(M)$ where $z \in P$. Consequently, we can easily continue along condition (iii) that $x + a \in H_\gamma(M)$ for some $a \in S \cap P$. Therefore, since S is isotype in M , we get that $x + a \in H_\gamma(S)$. This, in turn, implies that $x + P \in H_\gamma(\langle S, P \rangle / P)$, and so $\langle S, P \rangle / P$ is isotype in M / P . It follows by hypothesis that S is an isotype submodule of M with S and M totally projective.

Now, we set

$$\mathcal{N}_{M/P} = \{Q/P : Q \in \mathcal{N}_M \text{ and } Q \supseteq P\}$$

is a family of nice submodules of M / P that satisfies the nice system. It is also easily verified that

$$\mathcal{N}_{\langle S, P \rangle / P} = \{U/P : U \in \mathcal{N}_S \text{ and } U \supseteq S \cap P\}$$

is a family of nice submodules of $\langle S, P \rangle / P$ that satisfies nice system. Now, using the condition (iii), together with the fact that S is isotype in M , it is straight forward to compute that P is nice in $\langle S, P \rangle$

and $\langle S, P \rangle / P \cong S / S \cap P$. Therefore, we plainly obtain the family of nice submodules of $\langle S, P \rangle / P$ from those in $\mathcal{N}_S$ containing $S \cap P$.

If V is any countably generated submodule of M , then there exists a countably generated submodule Q of M containing V such that $\langle P, Q \rangle / P \parallel \langle S, P \rangle / P$ from the utilization of Lemma 3.3. Then, there exists a submodule Q of M such that the following hold:

- (i⁺) $Q \in \mathcal{N}_M$.
- (ii⁺) $\langle P, Q \rangle / P \cap \langle S, P \rangle / P \in \mathcal{N}_{\langle S, P \rangle / P}$.
- (iii⁺) $\langle P, Q \rangle / P \parallel \langle S, P \rangle / P$.

Since P and Q belong to $\mathcal{N}_M$, it follows that $\langle P, Q \rangle \in \mathcal{N}_M$. Thus if we set $P_\lambda = \langle P, Q \rangle$, then P_λ satisfies conditions (i) and (ii). Moreover, if $P = P_{\lambda-1}$ with Q is countably generated, we observe that $P_\lambda / P_{\lambda-1}$ is countably generated. Thus, we see that condition (iv) from definition of P_β is satisfied. By the choice of λ , condition (v) holds for all $\gamma < \lambda$, and in view of condition (iii), it remains only to show that $P_\lambda \parallel S$. Suppose that $H_M(P_\lambda + x) = \gamma$ where $p_\lambda \in P_\lambda = \langle P, Q \rangle$ and $x \in S$. At the same time, by using (iii⁺), we get $H_{M/P}(p_\lambda + b + P) \geq \gamma$ for some $b \in \langle P, Q \rangle \cap \langle S, P \rangle$. It follows that $\langle P, Q \rangle \cap \langle S, P \rangle = \langle \langle P, Q \rangle \cap S, P \rangle$, where $b \in \langle P, Q \rangle \cap S$. Note that if P is nice in M , it immediately follows that $H_M(p_\lambda + b + z) \geq \gamma$ for some $z \in P$. This, in turn, implies that $H_M(b - x + z) \geq \gamma$. Now, since $b \in S$ with $P \parallel S$, there exists $c \in P \cap S$ such that $H_M(c + z) \geq \gamma$. Then we obtain $H_M(p_\lambda + b - c) \geq \gamma$ and hence $b - c \in \langle P, Q \rangle \cap S = p_\lambda \cap S$. Thus $p_\lambda \parallel S$ and see that conditions (i) – (v) are satisfied for the chain of submodules of P_β .

Turning, therefore, to the claim, we continue to extend the chain of submodules of P_β such that $M = \cup_{\beta < \lambda} T_\beta$. To that goal, let us define $T_\beta = \langle S, P_\beta \rangle$ and consider the chain

$$S = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_\beta \subseteq \cdots, \quad \beta < \lambda$$

Since condition (iv) for the chain of submodules of P_β implies that $T_{\beta+1}/T_\beta$ is countably generated and condition (v) yields $T_\gamma = \cup_{\beta < \gamma} T_\beta$, where γ is a limit. In order to prove the remaining necessity part, we have to show that $T_\beta = \langle S, P_\beta \rangle$ is separable in M . Suppose $H_M(y + T_\beta) = \sigma$ where $\text{cof}(\sigma) \neq \omega_0$. Letting $T = T_\beta$, we note that

$$H_{M/P}(y + \langle S, P \rangle / P) \geq H_M(y + \langle S, P \rangle) = \sigma.$$

Since $\langle S, P \rangle / P$ is isotype in M/P , and $\langle S, P \rangle / P$ is totally projective. Therefore, according to Theorem 3.2, there exists $x \in S$ such that $H_{M/P}(x + y + P) \geq \sigma$. Since P is nice in M , $H_M(x + y + z) = \sigma$ for some $z \in P$. This means that $T_\beta = \langle S, P \rangle$ is separable in M . One seeing readily in view of Theorem 3.9 that M satisfies the weak separable system.

Conversely, suppose that S is an isotype submodule of the totally projective module M and suppose that M satisfies the weak separable system. Let $\mathcal{N}$ be a family of nice submodules of M satisfying nice system and let $\mathcal{S}$ be a family of separable submodules of M satisfying separable system. Then, clearly $S \in \mathcal{S}$ is a separable submodule of M . Consequently, as early checked, P is a submodule of M satisfying the conditions:

- (1) $P \in \mathcal{N}$,
- (2) $\langle S, P \rangle \in \mathcal{S}$,
- (3) $P \parallel S$.

Since $\langle S, P \rangle$ is separable, there exists a countably generated submodule Q of M with $Q \not\subseteq \langle S, P \rangle$ and $M = \langle S, P \rangle$ such that

- (1⁺) $Q \in \mathcal{N}$,
- (2⁺) $\langle S, Q \rangle \in \mathcal{S}$,
- (3⁺) $Q \parallel \langle S, P \rangle$.

Of course we claim that

- (1*) $\langle P, Q \rangle \in \mathcal{N}$,
- (2*) $\langle S, P, Q \rangle \in \mathcal{S}$,
- (3*) $\langle P, Q \rangle \| S$.

The first two conditions are immediate, and the third is a direct cosequence of $Q \| \langle S, P \rangle$ and $P \| S$. It follows that there exists an ascending chain

$$0 = P_0 \subseteq P_1 \subseteq P_2 \subseteq \dots P_\beta \subseteq \dots$$

of submodules of M satisfying conditions (i) – (iii) such that $P_{\beta+1}/P_\beta$ is countably generated, $P_\gamma = \cup_{\beta < \gamma} P_\beta$ where γ is a limit, and $M = \cup P_\beta$. Setting $K_\beta = S \cap P_\beta$, and consider the corresponding chain

$$0 = K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots \subseteq K_\beta \subseteq \dots$$

of submodules of S such that $K_{\beta+1}/K_\beta$ is countably generated, $K_\gamma = \cup_{\beta < \gamma} K_\beta$ where γ is a limit, and $S = \cup K_\beta$. To complete the proof of the theorem, it remains only to show that S is totally projective. It suffices to show that K_β is nice in S . In the remaining part where $u \in S$ is a uniform element, we assume that $H_S(u + K_\beta) = \gamma$. Note that $H_M(u + P_\beta) \geq \gamma$ and $H_M(u + z) \geq \gamma$ for some $z \in P_\beta$ since $P_\beta \in \mathcal{N}$ is nice in M . Finally, since $P_\beta \| S$ provided that $H_M(u + b) \geq \gamma$ for some $b \in K_\beta$. Thus, we arrive at $H_S(u + b) \geq \gamma$ where S is isotype in M . Therefore, we conclude that K_β is nice in S . The proof of the theorem is now completed. $\square$

4. Left-open problems

In closing we pose the following still unsettled questions. We start with

Problem 4.1. Describe the structure of the class of those QTAG-modules M for which there exist $S \subseteq \text{Soc}(M)$ which are separable in M and countably generated nice submodules $N \subseteq M$ such that $M/(S + N)$ are totally projective?

Problem 4.2. Does it follow that the Lemma 3.3 remains true without the restriction of compatibility of submodules?

Problem 4.3. Suppose $\mathcal{S}_P$ and $\mathcal{S}_Q$ are the families of separable submodules of the QTAG-modules M satisfying separable system. What are the conditions under which $f_{(P \oplus Q)}(t) = f_{(P)}(t) + f_{(Q)}(t)$?

Problem 4.4. Suppose M_1 is a QTAG-module of length α such that whenever M_2 is a QTAG-module such that $M_1 \cong M_2/H_\alpha(M_2)$ and $H_\alpha(M_2)$ are isotype in M_2 . Can we conclude that M_1 is totally projective?

Acknowledgment: The authors express their appreciation to the referees and to the editor, for his/her valuable editorial work.

References

- [1] K. Benabdallah, S. Singh, On torsion Abelian groups like modules, In: Göbel, R., Lady, L., Mader, A. (Eds), Abelian group theory, Lect. Notes Math., Springer, Berlin, Heidelberg, 1006 (1983) 639–653.
- [2] H. Cartan, S. Eilenberg, Homological algebra, Princeton University Press, New Jersey (1956).
- [3] P. V. Danchev, Intersections of isotype subgroups in abelian group rings, Compt. rend. Acad. bulg. Sci. 53(2) (2000) 9–12.

- [4] P. V. Danchev, On the isomorphic group algebras of isotype subgroups of Warfield abelian groups, *Ukrain. Math. Bull.* 3(3) (2006) 305–314.
- [5] N. O. Ertaş, On almost projective modules, *Axioms*, 10(1) (2021) 21.
- [6] L. Fuchs, *Infinite Abelian Groups, Volume I*, Pure Appl. Math. 36, Academic Press, New York (1970).
- [7] L. Fuchs, *Infinite Abelian Groups, Volume II*, Pure Appl. Math. 36, Academic Press, New York (1973).
- [8] Y. Guo, S. Yang, Projective class rings of a kind of category of Yetter-Drinfeld modules, *AIMS Math.* 8(5) (2023) 10997–11014.
- [9] A. Hasan, Characterizations of generalized submodules of QTAG-modules, *Miskolc Math. Notes*, 24(3) (2023) 1351–1360.
- [10] A. Hasan, Countably generated extensions of QTAG-modules, *Eurasian Math. J.* 14(3) (2023) 26–34.
- [11] A. Hasan, J. C. Mba, Σ -uniserial modules and their properties, *Math. Stat.* 11(6) (2023) 917–922.
- [12] A. Hasan, Rafiquddin, On totally projective QTAG-modules characterized by its submodules, *Bol. Soc. Paran. Mat.* 36(4) (2018) 77–86.
- [13] A. Hasan, Rafiquddin, M. Hanzla, Isotypity and direct sum of countably generated modules, *Georgian Math. J.* (2025).
- [14] P. Hill, Isotype subgroups of totally projective groups, In: Göbel, R., Walker, E. (Eds) *Abelian Group Theory*, Lect. Notes Math. Springer, Berlin, Heidelberg, 874 (1981) 305–321.
- [15] M. Z. Khan, G. Varshney, On h-purifiable submodule of QTAG-module, *Tamkang J. Math.* 45(3) (2014) 251–258.
- [16] M. Z. Khan, A. Zubair, On quasi h-pure submodules of QTAG-modules, *Int. J. Math. Math. Sci.* 24(7) (2000) 493–499.
- [17] C. J. Leininger, D. B. McReynolds, Separable subgroups of mapping class groups, *Topology Appl.* 154(1) (2007) 1–10.
- [18] A. Mehdi, M. Y. Abbasi, F. Mehdi, Nice decomposition series and rich modules, *South East Asian J. Math. Math. Sci.* 4(1) (2005) 1–6.
- [19] A. Mehdi, M. Y. Abbasi, F. Mehdi, On $(\omega + n)$ -projective modules, *Ganita Sandesh*, 20(1) (2006) 27–32.
- [20] H. A. Mehran, S. Singh, On σ -pure submodules of QTAG-modules, *Arch. Math.* 46(6) (1986) 501–510.
- [21] F. Sikander, F. Begam, T. Fatima, On submodule transitivity of QTAG-modules, *AIMS Math.* 8(4) (2023) 9303–9313.
- [22] F. Sikander, T. Fatima, On totally projective QTAG-modules, *J. Taibah Univ. Sci.* 13(1) (2019) 892–896.
- [23] S. Singh, Some decomposition theorems in abelian groups and their generalizations, *Ring Theory: Proceedings of Ohio University Conference*, Marcel Dekker, New York, 25 (1976) 183–189.
- [24] S. Singh, Abelian groups like modules, *Act. Math. Hung.* 50 (1987) 85–95.
- [25] S. Singh, M. Z. Khan, TAG-modules with complement submodules h-pure, *Int. J. Math. Math. Sci.* 21(4) (1998) 801–814.
- [26] E. A. Walker, Ulm’s theorem for totally projective groups, *Proc. Amer. Math. Soc.* 37(2) (1973) 387–392.