J. Algebra Comb. Discrete Appl.
12(2) • 109–125

Received: 14 April 2024
Accepted: 29 August 2024

Journal of Algebra Combinatorics Discrete Structures and Applications

θ-generalized monomial codes

Research Article

Lhousain Mouatadid, Oussama Kabbouch, El Mahdi Mouloua, Mustapha Najmeddine

Abstract: In this paper, we generalize cyclic codes to another more large linear codes, namely θ-monomial codes. It is shown that for a θ-monomial code, its Euclidean and e-Galois dual are also θ-monomial codes. Furthermore, we present the equivalence between θ-monomial codes and generalized monomial codes. By considering the skew polynomial ring, we show that θ-monomial codes can relate to submodules under one condition and to ideals under another condition. This allows us to give a characterization of θ-monomial codes. More results on the e-Galois dual of θ-monomial codes are given with additional properties on self duality and self orthogonality. The generalized θ-monomial codes are discussed with their algebraic structure. The paper is closed by the investigation of the algebraic structure of θ-monomial codes over the ring $\mathbb{F}_q + v\mathbb{F}_q$ where $v^2 = v$.

2020 MSC: 94B15, 94B05, 11T71

Keywords: Cyclic codes, Monomial codes, Skew ring

1. introduction

Cyclic codes are considered as an extremely important class of codes. They were first introduced by E. Prange in [11, 12]. It is known that cyclic codes have a rich algebraic structure which allows the process of encoding and decoding more efficient. Let $\mathbb{F}_q$ be a finite field of q elements, where q is a prime power number, and let θ be an automorphism of $\mathbb{F}_q$. A linear code of length n over $\mathbb{F}_q$ is a subspace $\mathcal{C}$ of the vector space $\mathbb{F}_q^n$. The linear code $\mathcal{C}$ is called cyclic if, for each vector $\mathbf{c}=(c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$, the cyclic shift $\mathbf{sc}=(c_{n-1}, c_0, \ldots, c_{n-2})$ of $\mathbf{c}$ is still in $\mathcal{C}$. By considering the one to one correspondence between the vectors $\mathbf{c}=(c_0, c_1, \ldots, c_{n-1})$ in $\mathbb{F}_q^n$ and the polynomials $\mathbf{c(x)}=c_0 + c_1 x + \ldots, c_{n-1} x^{n-1}$ in $\mathbb{F}_q[x]$ of degree at most $n-1$, one can characterize cyclic codes as ideals of the residue class ring $\mathbb{F}_q[x]/(x^n - 1)$.

Lhousain Mouatadid, Oussama Kabbouch, El Mahdi Mouloua (Corresponding Author), Mustapha Najmeddine; Department of mathematics, ENSAM–Meknes, Moulay Ismail University, Morocco (email: lhousain.mouatadid@gmail.com, oussama.kabbouch@edu.umi.ac.ma, e.mouloua@edu.umi.ac.ma, m.najmeddine@umi.ac.ma).

In [1], authors generalize the notion of cyclic codes by considering the skew polynomial ring $\mathbb{F}_q[x,\theta]$, where θ is an automorphism of $\mathbb{F}_q$, to the so-called θ-cyclic codes. Indeed if $\mathcal{C}$ is a linear code, and for each $\mathbf{c}=(c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$ we have $(\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in \mathcal{C}$, then $\mathcal{C}$ is called a θ-cyclic code. It was shown that in the case where $\mathrm{ord}(\theta)|n$, θ-cyclic codes can be viewed as left ideals of the residue class ring $\mathbb{F}_q[x,\theta]/(x^n-1)$. The generalization of θ-cyclic codes are presented in [3] in the case of the residue class ring $\mathbb{F}_q[x,\theta]/(f)$, where f is a polynomial of degree n. The case where the order of θ is not necessarily a divisor of n is given in [3]. Such codes are modules over the ring $\mathbb{F}_q[x,\theta]$. In [13], authors studied the algebraic structure of θ-cyclic codes of an arbitrary length n. They investigated the equivalence between cyclic codes and θ-cyclic codes, and the equivalence between θ-cyclic codes and quasi-cyclic codes. The concept of θ-cyclic codes over finite fields was extended to the class of θ-cyclic codes over various types of finite rings. Indeed in [2], authors introduced the skew constacyclic codes over Galois rings. Dealing with Galois ring can be an extremely difficult issue in the context that the polynomial ring may not be left and right Euclidean, which is important to deal with ideals which are generated by monic polynomials. In [8], authors studied skew constacyclic codes over the finite chain ring $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, where $u^2 = 0$. Recently in [5], authors investigated the algebraic structure of skew constacyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$ with $v^2 = v$. Throughout this work, we present a new generalization of θ-cyclic codes under the name of θ-monomial codes which generalize also the class of monomial codes presented in [6], and recently in [9, 10]. We deal with the structure of the dual and the e-Galois dual of θ-monomial codes, with similar approach as in [13]. For the case where $\gcd(ord(\theta), n) = 1$, we present an equivalence between θ-monomial codes and monomial codes. The case where $\gcd(ord(\theta), n) = d$ such that $d > 1$, allows us to describe the equivalence between θ-monomial codes and generalized monomial codes. By considering the skew polynomial ring $\mathcal{R}_{\bar{a}} = \mathbb{F}_q[x,\theta]/\left\langle x^n - \prod_{i=0}^{n-1} a_i \right\rangle$, where $\bar{a} = (a_0, a_1, \ldots, a_{n-1}) \in \left(\mathbb{F}_q^{\theta}\right)^n$ such that $\prod_{i=0}^{m-1} a_i \neq 0$ and $\mathbb{F}_q^{\theta} = \{x \in \mathbb{F}_q : \theta(x) = x\}$ is the fixed field of θ. In the case where $ord(\theta)|n$, we present the characterization of θ-monomial codes as left ideals of $\mathcal{R}_{\bar{a}}$, and in the case where $\mathrm{ord}(\theta) \nmid n$ we characterize θ-monomial codes as left $\mathbb{F}_q[x,\theta]$-submodules of $\mathcal{R}_{\bar{a}}$. Some algebraic properties of the generator polynomial and the generator matrix of θ-monomial codes are given. By considering the e-Galois dual of θ-monomial codes, more results on the self duality and the self orthogonality are derived. To generalize such codes, we introduce the class of generalized θ-monomial codes under the name of $(\theta, \sigma, \bar{a})$-monomial codes with some properties on their duals and e-Galois duals. Finally, similar to the work of [5], we extend the class of θ-monomial code, over the finite non-chain ring $\mathbb{F}_q + v\mathbb{F}_q$, where $v^2 = v$. Indeed in Theorem 5.3, we give a characterization of θ-monomial codes. In this case, additional results on the e-Galois dual are given. The paper is organized as follows. Section 2 contains the preliminaries on skew polynomial rings. Section 3 defines the class of θ-monomial codes. Section 4 introduces the class of generalized θ-monomial codes. Section 5 is devoted to θ-monomial codes over the ring $\mathcal{R} = \mathbb{F}_q + v\mathbb{F}_q$ where $v^2 = v$.

2. Preliminaries

Let $q = p^h$ be a prime power and $\mathbb{F}_q$ be a finite field with q elements, e is an integer such that $0 \leq e \leq h-1$ and $\kappa = h - e$. A linear code $\mathcal{C}$ of length m over $\mathbb{F}_q$ is a linear subspace of $\mathbb{F}_q^m$. We define the Hamming weight of, $\mathbf{a} = (a_0, a_1, \ldots, a_{m-1}) \in \mathcal{C}$, $\mathrm{w}_H(\mathbf{a})$, as the number of non-zero components of $\mathbf{a}$ and the hamming distance of $\mathcal{C}$, $d_H(\mathcal{C}) = \min\{w_H(\mathbf{a}) \mid \mathbf{a} \in \mathcal{C}, \mathbf{a} \neq 0\}$. The code $\mathcal{C}$ is said to be an $[m, k, d]$ if it has the dimension k, and the Hamming distance d. The Euclidean inner product of two vectors of $\mathbb{F}_q^{m-1}$, $\mathbf{a} = (a_0, a_1, \ldots, a_{m-1})$ and

$\mathbf{b} = (b_0, b_1, \ldots, b_{m-1})$ is defined by $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=0}^{m-1} a_i b_i$ and the Euclidean dual code $\mathcal{C}^{\perp}$ of $\mathcal{C}$ is defined as $\mathcal{C}^{\perp} = \left\{\mathbf{a} \in \mathbb{F}_q^m \mid \langle \mathbf{b}, \mathbf{a} \rangle = 0, \text{ for all } \mathbf{b} \in \mathcal{C}\right\}$. The e-Galois inner product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined by $\langle \mathbf{a}, \mathbf{b} \rangle_e = \sum_{i=0}^{m} a_i b_i^{p^e}$ and the e-Galois dual code $\mathcal{C}^{\perp_e}$ of $\mathcal{C}$ is defined as $\mathcal{C}^{\perp_e} = \left\{\mathbf{a} \in \mathbb{F}_q^m \mid \langle \mathbf{b}, \mathbf{a} \rangle_e = 0, \text{ for all } \mathbf{b} \in \mathcal{C}\right\}$. We denote the Frobenius automorphism of $\mathbb{F}_q$ by Γ, which is given by $\Gamma(\alpha) = \alpha^p$ for $\alpha \in \mathbb{F}_q$.

Proposition 2.1 ([4]). *Let $\mathcal{C}$ be a linear code of length m over $\mathbb{F}_q$. Then we have*

1. *For all $\mathbf{a}, \mathbf{b} \in \mathbb{F}_q^m$, $\langle \mathbf{a}, \mathbf{b}\rangle_e = \Gamma^e(\langle \Gamma^\kappa(\mathbf{a}), \mathbf{b}\rangle) = \Gamma^e(\langle \mathbf{b}, \mathbf{a}\rangle_\kappa)$.*
2. *$\mathcal{C}^{\perp_e} = \Gamma^\kappa(\mathcal{C}^\perp) = \left(\mathcal{C}^{p^\kappa}\right)^\perp$, where $\mathcal{C}^{p^\kappa} = \{(x_1^{p^\kappa}, x_2^{p^\kappa}, \dots, x_n^{p^\kappa}) : (x_1, x_2, \dots, x_n) \in \mathcal{C}\}$.*
3. *If $\dim(\mathcal{C}) = k$, then $\dim\left(\mathcal{C}^\perp\right) = \dim\left(\mathcal{C}^{\perp_e}\right) = m - k$.*

Definition 2.2. *Let θ be an automorphism of $\mathbb{F}_q$. A linear code $\mathcal{C} \subseteq \mathbb{F}_q^n$ is said to be a θ-**monomial** code induced by the vector $\bar{a} = (a_0, a_1, \dots, a_{n-1})$ if and only if for each codeword $c = (c_0, c_1, \dots, c_{n-1}) \in \mathcal{C}$, the vector $(a_{n-1}\theta(c_{n-1}), a_0\theta(c_0), \dots, a_{n-2}\theta(c_{n-2}))$ is also a codeword of $\mathcal{C}$.*

Remark 2.3.

1. *If $a = (1, 1, \dots, 1)$ and $\theta = Id$, the code $\mathcal{C}$ is a cyclic code.*
2. *If $a = (1, 1, \dots, 1, \lambda)$ and $\theta = Id$, the code $\mathcal{C}$ is a λ-constacyclic code.*
3. *If $a = (1, 1, \dots, 1)$ and $\theta \neq Id$, the code $\mathcal{C}$ is a skew cyclic code.*
4. *If $a = (1, 1, \dots, 1, \lambda)$ and $\theta \neq Id$, the code $\mathcal{C}$ is a skew λ-constacyclic code.*
5. *$\mathcal{C}$ is a θ-monomial code induced by the vector $\bar{a} = (a_0, a_1, \dots, a_{n-1})$ if it is invariant under the following θ-monomial shift map*

$$\begin{array}{rccc} \varphi_{\bar{a},\theta}: & \mathbb{F}_q^n & \longrightarrow & \mathbb{F}_q^n \\ & (v_0, v_1 \dots, v_{n-1}) & \longmapsto & (a_{n-1}\theta(v_{n-1}), a_0\theta(v_0), \dots, a_{n-2}\theta(v_{n-2})). \end{array}$$

6. *If $\prod_{i=0}^{n-1} a_i \neq 0$, then $\varphi_{\bar{a},\theta}$ is an isomorphism and its inverse map is given by*

$$\begin{array}{rccc} \overset{-1}{\varphi}_{\bar{a},\theta} : & \mathbb{F}_q^n & \longrightarrow & \mathbb{F}_q^n \\ & (y_0, y_1, \dots, y_{n-1}) & \longmapsto & \left(\theta^{-1}(a_0^{-1}y_1), \theta^{-1}(a_1^{-1}y_2), \dots, \theta^{-1}(a_{n-2}^{-1}y_{n-1}), \theta^{-1}(a_{n-1}^{-1}y_0)\right). \end{array}$$

Lemma 2.4. *Let $\bar{a} = (a_0, a_1, ..., a_{n-1}) \in \left(\mathbb{F}_q^\theta\right)^n$ such that $\prod_{i=0}^{n-1} a_i \neq 0$. Let s be a positive integer and $k \in \{0, 1, \dots, n-1\}$. Then we have*

$$\forall v \in \mathbb{F}_q^n : \overset{sn+k}{\varphi}_{\bar{a},\theta}(v) = \left(\prod_{i=0}^{n-1} a_i\right)^s \cdot \overset{k}{\varphi}_{\bar{a},\theta} \circ \theta^{ns}(v).$$

where $\theta^{ns}(v_0, v_1 \dots, v_{n-1}) = (\theta^{ns}(v_0), \theta^{ns}(v_1) \dots, \theta^{ns}(v_{n-1}))$.

In the following theorem we give a characterization of the dual of θ-monomial codes induced by the vector $\bar{a} = (a_0, a_1, \dots, a_{n-1})$.

Theorem 2.5. *Let $\mathcal{C}$ be a θ-monomial code with associated vector $\bar{a} = (a_0, a_1, \dots, a_{n-1}) \in \left(\mathbb{F}_q^\theta\right)^n$ such that $\prod_{i=0}^{n-1} a_i \neq 0$. Then $\mathcal{C}^\perp$ is a θ-monomial code with associated vector $\bar{\alpha} = \left(a_0^{-1}, a_1^{-1}, \dots, a_{n-1}^{-1}\right)$.*

Proof. Let $x = (x_0, x_1, \ldots, x_{n-1}) \in \mathcal{C}^{\perp}$ and $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$. We have

$$\begin{aligned}
\langle \varphi_{\bar{\alpha},\theta}(x), c\rangle =& \left\langle \left(a_{n-1}^{-1}\theta(x_{n-1}), a_0^{-1}\theta(x_0), \ldots, a_{n-2}^{-1}\theta(x_{n-2})\right), (c_0, c_1, \ldots, c_{n-1})\right\rangle \\
=& a_{n-1}^{-1}\theta(x_{n-1})c_0 + \sum_{i=1}^{n-1} a_{i-1}^{-1}\theta(x_{i-1})c_i \\
=& \theta\left(\theta^{-1}(a_{n-1}^{-1}c_0)x_{n-1} + \sum_{i=1}^{n-1}\theta^{-1}(a_{i-1}^{-1}c_i)x_{i-1}\right) \\
=& \theta\left(\left\langle (x_0, x_1, \ldots, x_{n-1}), \left(\theta^{-1}(a_0^{-1}c_1), \theta^{-1}(a_1^{-1}c_2), \ldots, \theta^{-1}(a_{n-2}^{-1}c_{n-1}), \theta^{-1}(a_{n-1}^{-1}c_0)\right)\right\rangle\right) \\
=& \theta\left(\langle x, \varphi_{\bar{a},\theta}^{-1}(c)\rangle\right).
\end{aligned}$$

By Lemma 2.4, $\varphi_{\bar{a},\theta}^{m} = \left(\prod_{i=0}^{n-1} a_i\right)^s \cdot I_d$, where $m = \mathrm{lcm}(n, \mathrm{ord}(\theta))$ and $s = \dfrac{m}{n}$. Then we get $\varphi_{\bar{a},\theta}^{-1}(c) = \left(\prod_{i=0}^{n-1} a_i^{-1}\right)^s \cdot \varphi_{\bar{a},\theta}^{m-1}(c) \in \mathcal{C}$. Therefore

$$\langle \varphi_{\bar{\alpha},\theta}(x), c\rangle = 0.$$

Hence, $\mathcal{C}^{\perp}$ is a θ-monomial code with associated vector $\bar{\alpha} = \left(a_0^{-1}, a_1^{-1}, \ldots, a_{n-1}^{-1}\right)$. $\square$

Corollary 2.6. *Let h be a positive integer such that $q = p^h$, $0 \leq e < h$ and $\kappa = h - e$. If $\mathcal{C}$ is a θ-monomial code with associated vector $\bar{a} = (a_0, a_1, \ldots, a_{n-1}) \in \left(\mathbb{F}_q^{\theta}\right)^n$ such that $\prod_{i=0}^{n-1} a_i \neq 0$, then the e-Galois dual $\mathcal{C}^{\perp_e}$ of $\mathcal{C}$ is a θ-monomial code with associated vector $\left(a_0^{-p^{\kappa}}, a_1^{-p^{\kappa}}, \ldots, a_{n-1}^{-p^{\kappa}}\right)$.*

Proof. By Proposition 2.1, we have $\mathcal{C}^{\perp_e} = \left(\mathcal{C}^{p^{\kappa}}\right)^{\perp}$, then it is sufficient to show that $\mathcal{C}^{p^{\kappa}}$ is a θ-monomial code with associated vector $\left(a_0^{p^{\kappa}}, a_1^{p^{\kappa}}, \ldots, a_{n-1}^{p^{\kappa}}\right)$. Let $\left(c_0^{p^{\kappa}}, c_1^{p^{\kappa}}, \ldots, c_{n-1}^{p^{\kappa}}\right) \in \mathcal{C}^{p^{\kappa}}$, then $(c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$. Since $\mathcal{C}$ is a θ-monomial code with associated vector $\bar{a}$, then $(a_{n-1}\theta(c_{n-1}), a_0\theta(c_0), \ldots, a_{n-2}\theta(c_{n-2})) \in \mathcal{C}$. Therefor

$$\left(a_{n-1}^{p^{\kappa}}\theta(c_{n-1}^{p^{\kappa}}), a_0^{p^{\kappa}}\theta(c_0^{p^{\kappa}}), \ldots, a_{n-2}^{p^{\kappa}}\theta(c_{n-2}^{p^{\kappa}})\right) = \left(a_{n-1}^{p^{\kappa}}\theta(c_{n-1})^{p^{\kappa}}, a_0^{p^{\kappa}}\theta(c_0)^{p^{\kappa}}, \ldots, a_{n-2}^{p^{\kappa}}\theta(c_{n-2})^{p^{\kappa}}\right) \in \mathcal{C}^{p^{\kappa}}.$$

Hence $\mathcal{C}^{p^{\kappa}}$ is a θ-monomial code with associated vector $\left(a_0^{p^{\kappa}}, a_1^{p^{\kappa}}, \ldots, a_{n-1}^{p^{\kappa}}\right)$. Therefore $\mathcal{C}^{\perp_e}$ is a θ-monomial code with associated vector $\left(a_0^{-p^{\kappa}}, a_1^{-p^{\kappa}}, \ldots, a_{n-1}^{-p^{\kappa}}\right)$.

$\square$

The two following theorems give a relationship between θ-monomial codes and monomial codes and generalized monomial codes in the case where $\mathrm{ord}(\theta) \nmid n$.

Theorem 2.7. *Let $\bar{a} = (a_0, a_1, ..., a_{n-1}) \in \left(\mathbb{F}_q^{\theta}\right)^n$ such that $\prod_{i=0}^{n-1} a_i \neq 0$. If $\gcd(n, \mathrm{ord}(\theta)) = 1$ and $\mathcal{C}$ is a θ-monomial code with associated vector $\bar{a}$, then $\mathcal{C}$ is equivalent to a monomial code with the associated vector $\bar{a}$.*

Proof. Let us denote $\mathrm{ord}(\theta) = t$. Since $\gcd(n, t) = 1$, there exist two integers r and $s > 0$ such that $rt - sn = 1$. Let $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$, let us show that $(a_{n-1}c_{n-1}, a_0c_0, \ldots, a_{n-2}c_{n-2}) \in \mathcal{C}$. Using

Lemma 2.4, we get

$$\begin{aligned}
\varphi_{\bar{a},\theta}^{rt}(c) = \varphi_{\bar{a},\theta}^{sn+1}(c) &= \left(\prod_{i=0}^{n-1} a_i\right)^s \cdot \varphi_{\bar{a},\theta} \circ \theta^{ns}(c) \\
&= \left(\prod_{i=0}^{n-1} a_i\right)^s \cdot \varphi_{\bar{a},\theta} \circ \theta^{rt-1}(c) \\
&= \left(\prod_{i=0}^{n-1} a_i\right)^s \cdot \varphi_{\bar{a},\theta} \circ \theta^{-1}(c) \\
&= \left(\prod_{i=0}^{n-1} a_i\right)^s \cdot (a_{n-1}c_{n-1}, a_0c_0, \ldots, a_{n-2}c_{n-2}).
\end{aligned}$$

Then $(a_{n-1}c_{n-1}, a_0c_0, \ldots, a_{n-2}c_{n-2}) \in \mathcal{C}$, therefore $\mathcal{C}$ is a monomial code with associated vector $\bar{a}$. $\square$

Exercise 2.8. *Let θ be the Frobenius automorphism of $\mathbb{F}_9 = \mathbb{F}_3[a]$ and $\mathcal{C}$ the θ-monomial code with associated vector $\bar{\alpha} = (2,2,2,2,2,1,2,1,2,2,1,1,2)$ generated by the following matrix*

$$G = \begin{pmatrix} 0 & 2 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 0 & 2 & 0 & 1 & 1 \\ 2 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 2 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

We have $\mathcal{C}$ is a $[13,3,9]$ linear code and its 1-Galois dual is a $[13,10,3]$ θ-monomial code with associated vector $\bar{\alpha}^{-3} = \bar{\alpha}$.
Moreover, we have $\gcd(13, \mathrm{ord}(\theta)) = \gcd(13,2) = 1$. Then $\mathcal{C}$ is equivalent to a monomial code with the associated vector $\bar{\alpha}$.

Definition 2.9. *A linear code $\mathcal{C}$ of length n over $\mathbb{F}_q$ is said to be a generalized monomial code if and only if there exists a permutation σ and $\bar{a} = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{F}_q^n$ such that $\prod_{i=0}^{n-1} a_i \neq 0$ and for each codeword $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$, the vector*

$$c' = \left(a_{\sigma(0)}c_{\sigma(0)}, a_{\sigma(1)}c_{\sigma(1)}, \ldots, a_{\sigma(n-1)}c_{\sigma(n-1)}\right)$$

is also a codeword.
In this case $\mathcal{C}$ is said to be a $(\sigma, \bar{a})$-monomial code.

Theorem 2.10. *Let $\overline{a} = (a_0, a_1, ..., a_{n-1}) \in \left(\mathbb{F}_q^{\theta}\right)^n$ such that $\prod_{i=0}^{n-1} a_i \neq 0$. If $\gcd(n, \mathrm{ord}(\theta)) = d$, where $d > 1$, and $\mathcal{C}$ is a θ-monomial code with associated vector $\overline{a}$, then $\mathcal{C}$ is equivalent to a generalized monomial code.*

Proof. Let us denote $\mathrm{ord}(\theta) = t$. Since $\gcd(n,t) = d$, there exist two integers r and $s > 0$ such that $rt - sn = d$. By Lemma 2.4, we have

$$\varphi_{\bar{a},\theta}^{rt}(c) = \varphi_{\bar{a},\theta}^{sn+d}(c) = \left(\prod_{i=0}^{n-1} a_i\right)^s \cdot \varphi_{\bar{a},\theta}^{d} \circ \theta^{-d}(c).$$

On the other hand, if $x = (x_0, x_1, \ldots, x_{n-1}) \in \mathbb{F}_q^n$, we have

$$\varphi_{\bar{a},\theta}^{d}(x) = (b_{\tau(0)}\theta^{d}(x_{\tau(0)}),\, b_{\tau(1)}\theta^{d}(x_{\tau(1)}), \ldots, b_{\tau(n-1)}\theta^{d}(x_{\tau(n-1)})).$$

where

$$\begin{aligned}\tau : \{0, 1 \ldots, n-1\} &\longrightarrow \{0, 1 \ldots, n-1\}\\ k &\longmapsto n - d + k \bmod n\end{aligned}$$

and for all $j \in \{0, 1 \ldots, n-1\}$, $b_j = \prod_{k \in A_j} a_j$, where $A_j = \{k \bmod n : k \in \{j, \ldots, j+d-1\}\}$.
Hence, if $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$, then $(b_{\tau(0)}c_{\tau(0)}, b_{\tau(1)}c_{\tau(1)}, \ldots, b_{\tau(n-1)}c_{\tau(n-1)}) \in \mathcal{C}$. Therefore $\mathcal{C}$ is equivalent to a generalized monomial code. □

Exercise 2.11. *Over $\mathbb{F}_{16} = \mathbb{F}_2[a]$, let θ be the automorphism defined by $x \mapsto x^{2^2}$. Let $\mathcal{C}$ the θ-monomial code with associated vector $\bar{\alpha} = (a^5, a^{10}, 1, a^5)$ generated by the following matrix*

$$G = \begin{pmatrix} 0 & a^5 & a^5 & 1 \\ 0 & 0 & a^4 & a^7 \end{pmatrix}.$$

We have $\mathcal{C}$ is a $[4, 2, 2]$ linear code. Moreover, we have $\gcd(4, \operatorname{ord}(\theta)) = \gcd(4, 2) = 2$. Then $\mathcal{C}$ is equivalent to a $(\sigma, \bar{b})$-monomial code, where $\sigma = (0, 2) \circ (1, 3)$ and $\bar{b} = (1, a^{10}, a^5, a)$.

3. θ-monomial codes and skew polynomial ring

Before starting this section, let us recall some properties of the structure of the set $\mathcal{R}_{\bar{a}} = \mathbb{F}_q[x, \theta]/\left\langle x^n - \prod_{i=0}^{n-1} a_i \right\rangle$.

Lemma 3.1. *Let $\mathbb{F}_q$ be a finite field, θ an automorphism of $\mathbb{F}_q$ and n an integer divisible by the order of θ. The ring $\mathcal{R}_{\bar{a}}$ is a principal left ideal ring in which left ideals are generated by $\mu(G)$, where G is a right divisor of $x^n - \prod_{i=0}^{n-1} a_i$ in $\mathbb{F}_q[x, \theta]$ and $\mu : \mathbb{F}_q[x, \theta] \longrightarrow \mathcal{R}_{\bar{a}}$ is the canonical morphism that associate a polynomial with its remainder by the right division with $x^n - \prod_{i=0}^{n-1} a_i$ in $\mathbb{F}_q[x, \theta]$.*

Proof. The similar result is showed in [1] lemma 1, then one can deduce by substituting $x^n - 1$ by $x^n - \prod_{i=0}^{n-1} a_i$. □

Since $\bar{a} = (a_0, a_1, \ldots, a_{n-1}) \in \left(\mathbb{F}_q^{\theta}\right)^n$, two cases arise: the first where $\operatorname{ord}(\theta) \nmid n$ and the second where $\operatorname{ord}(\theta) \mid n$.

Proposition 3.2 (Lemma 1 [5]). ***1.*** *If $\operatorname{ord}(\theta) \nmid n$, then $\mathcal{R}_{\bar{a}}$ is a left $\mathbb{F}_q[x, \theta]$-module where the multiplications is defined for all $\left(f(x) + \left(x^n - \prod_{i=0}^{n-1} a_i\right)\right) \in \mathcal{R}_{\bar{a}}$ and $b(x) \in \mathbb{F}_q[x; \theta]$ by:*

$$b(x)\left(f(x) + \left(x^n - \prod_{i=0}^{n-1} a_i\right)\right) = b(x)f(x) + \left(x^n - \prod_{i=0}^{n-1} a_i\right).$$

2. *If $\operatorname{ord}(\theta) \mid n$, then $\mathcal{R}_{\bar{a}}$ is a non commutative ring, where the multiplications is defined by:*

$$\left(f(x) + \left(x^n - \prod_{i=0}^{n-1} a_i\right)\right)\left(g(x) + \left(x^n - \prod_{i=0}^{n-1} a_i\right)\right) = f(x)g(x) + \left(x^n - \prod_{i=0}^{n-1} a_i\right).$$

The following theorems show how θ-monomial codes induced by $\bar{a}$ relate to submodules of $\mathcal{R}_{\bar{a}}$ when $\operatorname{ord}(\theta) \nmid n$, and to ideals of $\mathcal{R}_{\bar{a}}$ when $\operatorname{ord}(\theta) \mid n$.

Theorem 3.3. *Let $\overline{a} = (a_0, a_1, ..., a_{n-1}) \in \left(\mathbb{F}_q^\theta\right)^n$ such that $\prod_{i=0}^{n-1} a_i \neq 0$, and consider the following map*

$$\begin{array}{rccc} \pi_{\bar{a},\theta}: & \mathbb{F}_q^n & \longrightarrow & \mathcal{R}_{\bar{a}} \\ & (v_0, v_1 \ldots, v_{n-1}) & \longmapsto & \sum_{i=0}^{n-2} \left(v_{i+1} \prod_{k=0}^{i} a_k^{-1}\right) x^i + \left(v_0 \prod_{i=0}^{n-1} a_i^{-1}\right) x^{n-1}. \end{array}$$

1. *If $\mathrm{ord}(\theta) \nmid n$, then $\mathcal{C}$ is a θ-monomial code induced by $\bar{a}$ if and only if $\pi_{\overline{a},\theta}(\mathcal{C})$ is a left $\mathbb{F}_q[x;\sigma]$-submodule of $\mathcal{R}_{\bar{a}}$.*
2. *If $\mathrm{ord}(\theta) \mid n$, then $\mathcal{C}$ is a θ-monomial code induced by $\bar{a}$ if and only if $\pi_{\overline{a},\theta}(\mathcal{C})$ is a left ideal of $\mathcal{R}_{\bar{a}}$.*

Proof. Let $v = (v_0, v_1 \ldots, v_{n-1}) \in \mathbb{F}_q^n$. We have

$$\begin{aligned} x\pi_{\bar{a},\theta}(v) &= x \sum_{i=0}^{n-2} \left(v_{i+1} \prod_{k=0}^{i} a_k^{-1}\right) x^i + x \left(v_0 \prod_{i=0}^{n-1} a_i^{-1}\right) x^{n-1} \\ &= \sum_{i=0}^{n-2} \left(\theta(v_{i+1}) \prod_{k=0}^{i} a_k^{-1}\right) x^{i+1} + \left(\theta(v_0) \prod_{i=0}^{n-1} a_i^{-1}\right) x^n \\ &= \sum_{i=0}^{n-2} \left(a_i\theta(v_i) \prod_{k=0}^{i} a_k^{-1}\right) x^i + \left(a_{n-1}\theta(v_{n-1}) \prod_{i=0}^{n-1} a_i^{-1}\right) x^{n-1} \\ &= \pi_{\bar{a},\theta}\left(a_{n-1}\theta(v_{n-1}), a_0\theta(v_0), \ldots, a_{n-2}\theta(v_{n-2})\right) \\ &= \pi_{\bar{a},\theta}\left(\varphi_{\bar{a},\theta}(v)\right). \end{aligned}$$

□

Remark 3.4. *The reciprocal map of $\pi_{\bar{a},\theta}$ is given by*

$$\begin{array}{rccc} \pi_{\bar{a},\theta}^{-1}: & \mathcal{R}_{\bar{a}} & \longrightarrow & \mathbb{F}_q^n \\ & \sum_{i=0}^{n-1} u_i x^i & \longmapsto & (v_0, v_1 \ldots, v_{n-1}), \end{array}$$

where $v_0 = \left(\prod_{j=0}^{n-1} a_j\right) u_{n-1}$ and for all $i \in \{1, 2, \ldots, n-1\}$, $v_i = \left(\prod_{j=0}^{i-1} a_j\right) u_{i-1}$.

From this point forward until the end of this section, we assume that $\mathrm{ord}(\theta) \mid n$.
As known, $\mathcal{R}_{\bar{a}}$ is a principal left ideal domain. Then we have the following characterization of monomial codes.

Proposition 3.5. *Let $\mathcal{C}$ be a θ-monomial code of length n over $\mathbb{F}_q$. Then*

1. *There is a unique monic polynomial of least degree $g(x) \in \mathbb{F}_q[x, \theta]$ such that $\pi_{\overline{a},\theta}(\mathcal{C}) = \langle g(x) \rangle$ and $g(x)$ is a right divisor of the polynomial $x^n - \prod_{i=0}^{n-1} a_i$.*
2. *The family $\{g(x), xg(x), \ldots, x^{k-1}g(x)\}$ forms a basis of $\pi_{\overline{a},\theta}(\mathcal{C})$, as an $\mathbb{F}_q$ vector space, where $k = n - \deg(g)$.*

3. A generator matrix G of $\mathcal{C}$ is given by:

$$G = \begin{pmatrix} \pi^{-1}_{\overline{a},\theta}(g(x)) \\ \\ \pi^{-1}_{\overline{a},\theta}(xg(x)) \\ \vdots \\ \vdots \\ \pi^{-1}_{\overline{a},\theta}\left(x^{k-1}g(x)\right) \end{pmatrix}. \tag{1}$$

Proof. **1.** Let $g(x) \in \pi_{\overline{a},\theta}(\mathcal{C})$ be a monic polynomial of minimal degree such that $g(x) \neq 0$. Suppose that there is a monic polynomial $f(x) \in \pi_{\overline{a},\theta}(\mathcal{C})$ of the same degree, then $g(x) - f(x) \in \pi_{\overline{a},\theta}(\mathcal{C})$ and $g(x) - f(x)$ is of degree less than the degree of $g(x)$, necessary $g(x) - f(x) = 0$, hence $g(x) = f(x)$. Let $c(x)$ be any element in $\pi_{\overline{a},\theta}(\mathcal{C})$. By the right division algorithm, there are two unique polynomials q and r such that

$$c(x) = q(x)g(x) + r(x) \text{ where } r(x) = 0 \text{ or } \deg(r(x)) < \deg(g(x)).$$

$\pi_{\overline{a},\theta}(\mathcal{C})$ is a left ideal, then $r(x) = c(x) - q(x)g(x) \in \pi_{\overline{a},\theta}(\mathcal{C})$. Since $g(x)$ is of minimal degree in $\pi_{\overline{a},\theta}(\mathcal{C})$, then $r(x) = 0$, hence $c(x) = q(x)g(x)$. Therefore $\pi_{\overline{a},\theta}(\mathcal{C}) = \langle g(x) \rangle$.

Now, let us show that $g(x)$ is a right divisor of $x^n - \prod_{i=0}^{n-1} a_i$. Again, by the right division algorithm, there are two unique polynomials $q(x)$ and $r(x)$ such that

$$x^n - \prod_{i=0}^{n-1} a_i = q(x)g(x) + r(x) \text{ where } \deg(r(x)) < \deg(g(x)).$$

Since $g(x)$ and $x^n - \prod_{i=0}^{n-1} a_i = 0$ are in $\pi_{\overline{a},\theta}(\mathcal{C})$, then $r(x) = 0$ and hence $g(x)$ is a right divisor of $x^n - \prod_{i=0}^{n-1} a_i$.

2. We have $g(x)$ is a right divisor of $x^n - \prod_{i=0}^{n-1} a_i$, then there exist a polynomial $h(x) \in \mathbb{F}[x;\theta]$ such that $h(x)g(x) = x^n - \prod_{i=0}^{n-1} a_i$. We will show that the family $\{g(x), xg(x), \ldots, x^{n-\deg(g)-1}g(x)\}$ forms a basis of $\pi_{\overline{a},\theta}(\mathcal{C})$. Let $c(x) \in \pi_{\overline{a},\theta}(\mathcal{C})$, then $c(x) = f(x)g(x)$ for a polynomial $f(x) \in \mathbb{F}[x;\theta]$. By the right division algorithm of $f(x)$ by $h(x)$, there are two polynomials $q(x)$ and $r(x)$ such that

$$f(x) = q(x)h(x) + r(x) \text{ with } r(x) = 0 \text{ or } \deg(r(x)) < n - \deg(g(x))$$

Multiplying by $g(x)$ on the right, we get

$$f(x)g(x) = q(x)h(x)g(x) + r(x)g(x) = r(x)g(x) \quad \text{ in } \mathcal{R}_{\bar{a}}.$$

Hence $c(x) = r(x)g(x)$, whit $\deg(r) \leq n - \deg(g) - 1$. Therefore, the set $\{g(x), xg(x), \ldots, x^{n-\deg(g)-1}g(x)\}$ is a spanning set of $\pi_{\overline{a},\theta}(\mathcal{C})$. To show that $\{g(x), xg(x), \ldots, x^{n-\deg(g)-1}g(x)\}$ is linearly independent, suppose that

$$c_0 g(x) + c_1 x g(x) + \ldots + c_{n-r-1} x^{n-\deg(g)-1} g(x) = 0.$$

Comparing coefficients yields the fact that $c_i = 0$ for all $i = 0, 1, \ldots, n - \deg(g) - 1$. Hence $\{g(x), xg(x), \ldots, x^{n-\deg(g)-1}g(x)\}$ is linearly independent and therefore it is a basis for $\pi_{\overline{a},\theta}(\mathcal{C})$ and $\dim \pi_{\overline{a},\theta}(\mathcal{C}) = n - \deg(g)$. $\square$

As in the case of skew cyclic and skew constacyclic codes, we give a parity check matrix of a θ-monomial code.

Definition 3.6. *Let e be an integer such that $q = p^h$, $0 \leq e < h$ and $\kappa = h - e$. Let $h = \sum_{i=0}^{\ell} h_i x^i$ be a polynomial of degree ℓ in $\mathcal{R}_{\bar{a}}$ such that $h_0 \neq 0$, the e-skew reciprocal polynomial of h is $h^{*_e} = \sum_{i=0}^{\ell} \theta^i \left(h_{\ell-i}^{p^\kappa}\right) x^i$, and the left monic e-skew reciprocal polynomial of h is $h^{\natural_e} = \left(1/\theta^\ell\left(h_0^{p^\kappa}\right)\right) h^{*_e}$.*
If a skew polynomial is equal to its left monic e-skew reciprocal polynomial, then it is called e-self-reciprocal.

Lemma 3.7. *Let $f \in \mathbb{F}_q[x;\theta]$ be a skew polynomial of degree m such that $f = hg$, where h and g are skew polynomials of degrees $m-k$ and k, respectively. Then*

1. $f^{*_e} = \theta^{m-k}\left(g^{*_e}\right) h^{*_e}$.
2. $\left(f^{*_e}\right)^{*_\kappa} = \theta^m(f)$.

Proof. **1.** Let us denote $g = \sum_{i=0}^{k} g_i x^i$ and $h = \sum_{i=0}^{m-k} h_i x^i$. We have $f^{*_e} = \sum_{s=0}^{m} \theta^s(f_{m-s}^{p^\kappa})x^s$, where

$$\theta^s(f_{m-s}^{p^\kappa}) = \sum_{\substack{i+j=m-s\\0\leq i\leq m-k\\0\leq j\leq k}} \theta^s(h_i^{p^\kappa})\theta^{i+s}\left(g_j^{p^\kappa}\right) = \sum_{\substack{m-k-t+k-r=m-s\\0\leq m-k-t\leq m-k\\0\leq k-r\leq k}} \theta^s(h_{m-k-t}^{p^\kappa})\theta^{m-k-t+s}\left(g_{k-r}^{p^\kappa}\right)$$
$$= \sum_{\substack{r+t=s\\0\leq t\leq m-k\\0\leq r\leq k}} \theta^s(h_{m-k-t}^{p^\kappa})\theta^{m-k+r}\left(g_{k-r}^{p^\kappa}\right).$$

On the other hand, we have $g^{*_e} = \sum_{r=0}^{k} \theta^i\left(g_{k-r}^{p^\kappa}\right)x^r$ and $h^{*_e} = \sum_{t=0}^{m-k} \theta^t\left(h_{m-k-t}^{p^\kappa}\right)x^t$. Then

$$\theta^{m-k}\left(g^{*_e}\right)h^{*_e} = \sum_{s=0}^{m} c_s x^s, \quad \text{where} \quad c_s = \sum_{\substack{r+t=s\\0\leq r\leq k\\0\leq t\leq m-k}} \theta^{m-k+r}\left(g_{k-r}^{p^\kappa}\right)\theta^{r+t}(h_{m-k-t}^{p^\kappa}).$$

Then, we get that $f^{*_e} = \theta^{m-k}\left(g^{*_e}\right)h^{*_e}$.
2. We have

$$\left(f^{*_e}\right)^{*_\kappa} = \left(\sum_{s=0}^{m} \theta^s(f_{m-s}^{p^\kappa})x^s\right)^{*_\kappa} = \sum_{s=0}^{m} \theta^s(\theta^{m-s}(f_s^{p^h}))x^s = \theta^m(f).$$

□

Theorem 3.8. *Let $\mathcal{C}$ be a θ-monomial code induced by the vector $\bar{a} = (a_0, a_1, \ldots, a_{n-1}) \in \left(\mathbb{F}_q^\theta\right)^n$ such that $\prod_{i=0}^{n-1} a_i \neq 0$, and generated by the polynomial $g = x^k + \sum_{i=0}^{k-1} g_i x^i \in \mathbb{F}_q[x;\theta]$, and $h \in \mathbb{F}_q[x;\theta]$ such that $x^n - \prod_{i=0}^{n-1} a_i = hg$. Then e-Galois dual $\mathcal{C}^{\perp_e}$ of $\mathcal{C}$ is generated by $h^{\natural_e}$.*

Proof. Since $\mathrm{ord}(\theta) \mid n$ and $\prod_{i=0}^{n-1} a_i \in \mathbb{F}_q^{\theta}$, then $x^n - \prod_{i=0}^{n-1} a_i = gh = hg$, hence

$$
\begin{aligned}
x^n - \prod_{i=0}^{n-1} a_i^{-p^\kappa} &= -\left(\prod_{i=0}^{n-1} a_i^{-p^\kappa}\right)\theta^{n-k}\left(g^{*_e}\right)h^{*_e} \\
&= h^{*_e}\left(-\prod_{i=0}^{n-1} a_i^{-p^\kappa}\right)\theta^{n-k}\left(g^{*_e}\right).
\end{aligned}
$$

Let $\overline{\mathcal{C}} = \overset{-1}{\pi}_{\bar{\alpha},\theta}(\langle h^{*_e}(x)\rangle)$, where $\bar{\alpha} = \bar{a}^{-p^\kappa}$. Then $\overline{\mathcal{C}}$ is a θ-monomial code induced by the vector $\bar{\alpha}$. Moreover, $\dim\left(\overline{\mathcal{C}}\right) = k$. Now, let us show that $\overline{\mathcal{C}} = \mathcal{C}^{\perp_e}$.

For $s \in \{0, 1, \ldots, k-1\}$ and $r \in \{0, 1, \ldots n-k-1\}$, let us denote $x^s g = \sum_{i=0}^{n-1} \tilde{g}_i x^i$ and $x^r h^{*_e} = \sum_{i=0}^{n-1} \tilde{h}_i x^i$.
We have

$$
\tilde{g}_i = \begin{cases} 0 & \text{if } 0 \le i \le s-1 \\ \theta^s(g_{i-s}) & \text{if } s \le i \le s+k \\ 0 & \text{if } s+k+1 \le i \end{cases} \quad \text{and} \quad \tilde{h}_i = \begin{cases} 0 & \text{if } 0 \le i \le r-1 \\ \theta^r(h_{n-k-i+r}^{p^\kappa}) & \text{if } r \le i \le r+n-k \\ 0 & \text{if } n-k+r+1 \le i \end{cases}
$$

Denote $\overset{-1}{\pi}_{\bar{a},\theta}(x^s g) = (u_0, u_1, \ldots, u_{n-1})$ and $\overset{-1}{\pi}_{\bar{\alpha},\theta}(x^r h^{*_e}) = (v_0, v_1, \ldots, v_{n-1})$. Then we get

$$
\begin{aligned}
\left\langle \overset{-1}{\pi}_{\bar{a},\theta}(x^s g), \overset{-1}{\pi}_{\bar{\alpha},\theta}(x^r h^{*_e}) \right\rangle_e &= \sum_{i=0}^{n-1} \tilde{g}_i \tilde{h}_i^{p^e} \\
&= \sum_{i=\max(r,s)}^{\min(n-k+r,k+s)} \tilde{g}_i \tilde{h}_i^{p^e} \\
&= \sum_{i=\max(0,r-s)}^{\min(n-k+r-s,k)} \tilde{g}_{i+s} \tilde{h}_{(i+s-r)+r}^{p^e} \\
&= \sum_{i=\max(0,r-s)}^{\min(n-k+r-s,k)} \theta^s(g_i)\theta^{s+i}(h_{(n-k+r-s)-i}).
\end{aligned}
$$

On the other hand, we have $x^n - \prod_{i=0}^{n-1} = gh = \sum_{i=0}^{n} \mu_i x^i$, where

$$
\mu_\ell = \sum_{\substack{i+j=\ell \\ 0 \le i \le k \\ 0 \le j \le n-k}} g_i \theta^i(h_j) = \sum_{i=\max(0,\ell-(n-k))}^{\min(\ell,k)} g_i \theta^i(h_{\ell-i}).
$$

Hence, for $\ell = n-k+r-s$ we get

$$
\left\langle \overset{-1}{\pi}_{\bar{a},\theta}(x^s g), \overset{-1}{\pi}_{\bar{\alpha},\theta}(x^r h^{*_e}) \right\rangle_e = \theta^s(\mu_\ell).
$$

Since $\ell \in \{1, 2, \ldots, n-1\}$, then $\mu_\ell = 0$, and $\left\langle \overset{-1}{\pi}_{\bar{a},\theta}(x^s g), \overset{-1}{\pi}_{\bar{\alpha},\theta}(x^r h^{*_e}) \right\rangle_e = \theta^s(\mu_\ell) = 0$. Then $\overline{\mathcal{C}} \subset \mathcal{C}^{\perp_e}$. Since they have the same dimension, then $\overline{\mathcal{C}} = \mathcal{C}^{\perp_e}$. Let us show that $\mathcal{C}^{\perp_e}$ is generated by $h^{\natural_e}$. Since $\mathrm{ord}(\theta)|n$, then we have

$$
\theta^{n-k}(h_0^{p^\kappa})\left(x^n - \prod_{i=0}^{n-1} a_i^{-p^\kappa}\right) = \left(x^n - \prod_{i=0}^{n-1} a_i^{-p^\kappa}\right)\theta^{n-k}(h_0^{p^\kappa}).
$$

Then

$$x^n - \prod_{i=0}^{n-1} a_i^{-p^{\kappa}} = h^{\natural_e}\left(-\prod_{i=0}^{n-1} a_i^{-p^{\kappa}}\right)\theta^{n-k}\left(g^{*_e}\right)\theta^{n-k}(h_0^{p^{\kappa}})$$

Hence $h^{\natural_e}$ is a right divisor of $x^n - \prod_{i=0}^{n-1} a_i^{-p^{\kappa}}$, and the code $\mathcal{C}' = \pi_{\bar{\alpha},\theta}^{-1}(\langle h^{*_e}(x)\rangle)$ is θ-monomial code induced by the vector $\bar{\alpha}$, and $\dim(\mathcal{C}') = k$. Let $v(x) \in \overline{\mathcal{C}}$, then $v(x) = t(x)h^{*_e}(x)$ for some polynomial $t(x)$, hence $v(x) = t(x)\theta^{n-k}(h_0^{p^{\kappa}})h^{\natural_e}(x)$, therefor $v(x) \in \mathcal{C}'$ and then $\mathcal{C}' = \overline{\mathcal{C}}$. $\square$

Corollary 3.9. *Let $\mathcal{C}$ be a θ-monomial code induced by the vector $\bar{a} = (a_0, a_1, \ldots, a_{n-1}) \in \left(\mathbb{F}_q^{\theta}\right)^n$, such that $\prod_{i=0}^{n-1} a_i \neq 0$, and generated by the polynomial $g = x^k + \sum_{i=0}^{k-1} g_i x^i \in \mathbb{F}_q[x;\theta]$, and $h \in \mathbb{F}_q[x;\theta]$ such that $x^n - \prod_{i=0}^{n-1} a_i = hg$. If the order of $\prod_{i=0}^{n-1} a_i$ in the multiplicative group $\mathbb{F}_q^{\times}$ divides $p^{\kappa}+1$, then the following holds*

1. *$\mathcal{C}$ is an e-Galois self-orthogonal if and only if $h^{\natural_e}$ is a right divisor of g.*
2. *$\mathcal{C}$ is an e-Galois self-dual if and only if $h^{\natural_e} = g$.*

4. Generalized θ-monomial codes

Before starting this section, let us recall some basics about permutations.

Definition 4.1. *Let ℓ be an integer such that $\ell \geq 2$, and $\tau \in \mathcal{S}_n$. We say that τ is a ℓ-cycle if there are integers $a_1, a_2, ..., a_\ell \in \{1, 2, \ldots, n\}$ such that $\tau(a_1) = a_2, \ldots, \tau(a_{\ell-1}) = a_\ell$, and $\tau(a_\ell) = a_1$ and τ fixes every other integers. In this case τ will be denoted by*

$$\tau = (a_1, a_2, \ldots, a_\ell) \quad \textit{and} \quad \mathrm{supp}(\tau) = \{i : \tau(i) \neq i\} = \{a_1, a_2, \ldots, a_\ell\}.$$

Based on the notion of ℓ-cycles, we give the following theorem, which gives the decomposition of a permutation into disjoint cycles.

Theorem 4.2. *Let σ be any element of $\mathcal{S}_n$. Then σ can be uniquely factored as a product of disjoint cycles, this factorization is unique.*

$$\sigma = \tau_1\tau_2\ldots\tau_r$$

Definition 4.3. *Let $\sigma = (a_1, \ldots, a_\ell)$ be an ℓ-cycle of $\mathcal{S}_n$. We call the index of σ the smallest integer a_i and we denote it by*

$$ind(\sigma) = \min\{a_i : i \in \{1, \ldots, \ell\}\}.$$

Now, we will give the definition of a generalized θ-monomial code.

Definition 4.4. *A linear code $\mathcal{C}$ of length n over $\mathbb{F}_q$ is said to be a generalized θ-monomial code if and only if there exists a permutation $\sigma \in \mathcal{S}_n$ and $\bar{a} = (a_1, a_2, \ldots, a_n) \in \left(\mathbb{F}_q^{\theta}\right)^n$ such that $\prod_1^n a_i \neq 0$ and for each codeword $c = (c_1, c_2, \ldots, c_n) \in \mathcal{C}$, the vector*

$$c' = \left(a_{\sigma(1)}\theta(c_{\sigma(1)}), a_{\sigma(2)}\theta(c_{\sigma(2)}), \ldots, a_{\sigma(n)}\theta(c_{\sigma(n)})\right).$$

is also a codeword.
In this case $\mathcal{C}$ is said to be a $(\theta, \sigma, \bar{a})$-monomial code.

Exercise 4.5. *1. A θ-cyclic code is a $(\theta,\sigma,\bar{a})$-monomial code, where $\sigma = (n, n-1, \ldots, 1)$ and $\bar{a} = (1, 1, \ldots, 1)$.*

2. *A θ-λ-constacyclic code is a $(\theta,\sigma,\bar{a})$-monomial code, where $\sigma = (n, n-1, \ldots, 1)$, $\bar{a} = (1, 1, \ldots, \lambda)$ and $\theta = Id$.*

3. *A θ-monomial code induced by the vector $\bar{a} = (a_1, a_2, \ldots, a_n)$ is a $(\theta,\sigma,\bar{a})$-monomial code, where $\sigma = (n, n-1, \ldots, 1)$ and $\bar{a} = (a_1, a_2, \ldots, a_n)$*

4. *A θ-quasi-cyclic codes of length $n = \ell m$ is a $(\theta,\sigma,\bar{a})$-monomial code, where σ is defined for all $i \in \{1, 2, \ldots, m\}$ by $\sigma(i) = (\ell-1)m + i$ and for all $i \in \{m+1, \ldots, \ell m\} : \sigma(i) = i - m$ and $\bar{a} = (1, 1, \ldots, 1)$.*

5. *Let $\mathcal{C}$ be a θ-generalized quasi-cyclic codes of length $n = n_1 + n_2 + \ldots + n_r$. Denote $n_0 = 0$, and for all $i \in \{0, 1, \ldots, r\} : s_i = \displaystyle\sum_{j=0}^{i} n_j$. Then $\mathcal{C}$ is a $(\theta,\sigma,\bar{a})$-monomial code, where $\sigma = \sigma_1\sigma_2\ldots\sigma_r$ such that $\sigma_i = (s_i, s_i - 1, \ldots, s_{i-1}+1)$ and $\bar{a} = (1, 1, \ldots, 1)$.*

6. *A θ-multi-twisted codes of length $n = n_1 + n_2 + \ldots + n_r$ and parameters $(\lambda_1, \ldots, \lambda_r)$ is a $(\theta,\sigma,\bar{a})$-monomial code, where $\sigma = \sigma_1\sigma_2\ldots\sigma_r$ such that $\sigma_i = (s_i, s_i - 1, \ldots, s_{i-1}+1)$, for all $i \in \{0, 1, \ldots, r\} : s_i = \displaystyle\sum_{j=0}^{i} n_j$ where $n_0 = 0$. And $\bar{a} = (a_1, a_2 \ldots, a_n)$ such that for $j \in \{s_i : i \in \{1, 2 \ldots, r\}\}$ $a_j = \lambda_j$ and $a_j = 1$ otherwise.*

Remark 4.6. *1. $\mathcal{C}$ is a θ-monomial code induced by the vector $\bar{a} = (a_0, a_1, \ldots, a_{n-1}) \in \left(\mathbb{F}_q^\theta\right)^n$ such that $\displaystyle\prod_{i=0}^{n-1} a_i \neq 0$, if it is invariant under the following θ-monomial shift map*

$$\begin{array}{rccl} \Phi_{\bar{a},\theta,\sigma}: & \mathbb{F}_q^n & \longrightarrow & \mathbb{F}_q^n \\ & (v_1, v_2 \ldots, v_n) & \longmapsto & \left(a_{\sigma(1)}\theta(v_{\sigma(1)}), a_{\sigma(2)}\theta(v_{\sigma(2)}), \ldots, a_{\sigma(n)}\theta(v_{\sigma(n)})\right). \end{array}$$

2. *The map $\Phi_{\bar{a},\theta,\sigma}$ is an isomorphism and its inverse map is given by*

$$\begin{array}{rccl} \Phi^{-1}_{\bar{a},\theta,\sigma} & : \mathbb{F}_q^n & \longrightarrow & \mathbb{F}_q^n \\ & (y_1, y_2, \ldots, y_n) & \longmapsto & \left(\theta^{-1}(a_1^{-1}y_{\tau(1)}), \theta^{-1}(a_2^{-1}y_{\tau(2)}), \ldots, \theta^{-1}(a_n^{-1}y_{\tau(n)})\right), \end{array}$$

where $\tau = \sigma^{-1}$

Lemma 4.7. *Using the same notation as in Remark 4.6, suppose that $\sigma = \tau_1\tau_2\ldots\tau_r$ the decomposition into disjoint cycles, where ℓ_i is the length of the cycle τ_i and $J = \{j \in \{1, 2, \ldots, n\} : \sigma(j) = j\}$. If we denote $\operatorname{ord}(\sigma) = m$*

and for all $i \in \{1, \ldots, r\}$ and we define $s_i = \dfrac{m}{\ell_i}$, then for $v = (v_1, v_2, \ldots, v_n) \in \mathbb{F}_q^n$, we have $\Phi^m_{\bar{a},\theta,\sigma}(v) = (\mu_1, \mu_2, \ldots, \mu_n)$, where

$$\forall k \in J: \ \mu_k = a_k^m \theta^m(v_k) \quad \text{ and } \quad \forall i \in \{1, \ldots, r\}: \mu_i = \left(\prod_{j \in \operatorname{supp}(\tau_i)} a_j\right)^{s_i} \theta^m(v_i).$$

Theorem 4.8. *Let $\mathcal{C}$ be a $(\theta, \sigma, \bar{a})$-monomial code, where $\bar{a} = (a_0, a_1, \ldots, a_{n-1}) \in \left(\mathbb{F}_q^\theta\right)^n$ such that $\displaystyle\prod_{i=0}^{n-1} a_i \neq 0$. Then $\mathcal{C}^\perp$ is a $(\theta, \sigma, \bar{\alpha})$-monomial code, where $\bar{\alpha} = \left(a_0^{-1}, a_1^{-1}, \ldots, a_{n-1}^{-1}\right)$.*

Proof. Let $x = (x_1, x_2, \dots, x_{n-1}) \in \mathcal{C}^\perp$ and $c = (c_1, c_2, \dots, c_n) \in \mathcal{C}$. We have

$$\begin{aligned}\langle \Phi_{\bar{\alpha},\theta,\sigma}(x), c\rangle &= \left\langle \left(a_{\sigma(1)}^{-1}\theta(x_{\sigma(1)}), a_{\sigma(2)}^{-1}\theta(x_{\sigma(2)}), \dots, a_{\sigma(n)}^{-1}\theta(x_{\sigma(n)})\right), (c_1, c_2, \dots, c_n)\right\rangle \\ &= \sum_{i=1}^{n} a_{\sigma(1)}^{-1}\theta(x_{\sigma(i)})c_i \\ &= \theta\left(\sum_{i=1}^{n} x_{\sigma(i)}\theta^{-1}\left(a_{\sigma(1)}^{-1}c_i\right)\right) \\ &= \theta\left(\sum_{i=1}^{n} x_{\sigma(i)}\theta^{-1}\left(a_{\sigma(i)}^{-1}c_{\sigma^{-1}\sigma(i)}\right)\right) \\ &= \theta\left(\sum_{i=1}^{n} x_i\theta^{-1}\left(a_i^{-1}c_{\sigma^{-1}(i)}\right)\right) \\ &= \theta\left(\left\langle (x_1, x_2, \dots, x_n), \left(\theta^{-1}(a_1^{-1}c_{\sigma^{-1}(1)}), \theta^{-1}(a_2^{-1}c_{\sigma^{-1}(2)}), \dots, \theta^{-1}(a_n^{-1}c_{\sigma^{-1}(n)})\right)\right\rangle\right) \\ &= \theta\left(\langle x, \Phi_{\bar{\alpha},\theta,\sigma}^{-1}(c)\rangle\right).\end{aligned}$$

Let $m = \mathrm{ord}(\sigma)\,\mathrm{lcm}\left(\mathrm{ord}(\theta), \mathrm{ord}(a_1), \dots, \mathrm{ord}(a_n)\right)$, where $\mathrm{ord}\,(a_i)$ denotes the multiplicative order of a_i in the multiplicative group $\mathbb{F}_q^\times$, for $1 \le i \le n$. Using Lemma 4.7, we get $\Phi_{\bar{\alpha},\theta,\sigma}^m = Id$, then $\Phi_{\bar{\alpha},\theta,\sigma}^{-1}(c) = \Phi_{\bar{\alpha},\theta,\sigma}^{m-1}(c) \in \mathcal{C}$. Hence

$$\langle \Phi_{\bar{\alpha},\theta,\sigma}(x), c\rangle = 0.$$

Therefore, $\mathcal{C}^\perp$ is a θ-monomial code with associated vector $\bar{\alpha} = \left(a_0^{-1}, a_1^{-1}, \dots, a_{n-1}^{-1}\right)$. □

Corollary 4.9. *Let h be a positive integer such that $q = p^h$, $0 \le e < h$ and $\kappa = h - e$. If $\mathcal{C}$ is a $(\theta, \sigma, \bar{a})$-monomial code with $a = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{F}_q^n$, then the e-Galois dual $\mathcal{C}^{\perp_e}$ of $\mathcal{C}$ is a $(\theta, \sigma, \bar{\alpha})$-monomial code code with $\bar{\alpha}$ $\left(a_0^{-p^\kappa}, a_1^{-p^\kappa}, \dots, a_{n-1}^{-p^\kappa}\right)$.*

Now, we will see θ-generalized monomial as left $\mathbb{F}_q[x,\theta]$-submodule of a left $\mathbb{F}_q[x,\theta]$-module. For this, let us consider the following maps defined for:

1- **A cycle in $\mathcal{S}_n$ and $\bar{a} = (a_1, a_2, \dots, a_n) \in \left(\mathbb{F}_q^*\right)^n$**

Let $\tau = (i_\ell, i_{\ell-1}, \dots, i_1)$ be a cycle of length ℓ with index i_1, we denote $\bar{a}_\tau = (a_{i_1}, a_{i_2}, \dots, a_{i_\ell})$ and $\Lambda_{(\bar{a},\theta,\tau)}$ the following map

$$\begin{array}{cccc}\Lambda_{(\bar{a},\theta,\tau)}: & \mathbb{F}_q^n & \longrightarrow & \mathcal{R}_{\bar{a}_\tau} \\ & (v_1, v_2, \dots, v_n) & \longmapsto & \pi_{\bar{a}_\tau,\theta}(v_{i_1}, v_{i_2}, \dots, v_{i_\ell})\end{array}$$

where $\pi_{\bar{a}_{\tau},\theta}$ is given as in Theorem 3.3.

2- **A permutation in $\mathcal{S}_n$ and $\bar{a} = (a_1, a_2, \dots, a_n) \in \left(\mathbb{F}_q^*\right)^n$**

Let $\sigma \in \mathcal{S}_n$ such that $\sigma = \tau_1\tau_2\dots\tau_r$ is the decomposition into disjoint cycles, where ℓ_i is the length of the cycle σ_i, and $J = \{j_1, \dots, j_s\} = \{j \in \{1, 2, \dots, n\} : \sigma(j) = j\}$ such that $j_1 < \dots < j_s$. We denote $\tilde{\varphi}_{(\bar{a},\theta,\sigma)}$ the following map

$$\begin{array}{cccc}\Pi_{(\bar{a},\theta,\sigma)}: & \mathbb{F}_q^n & \longrightarrow & \prod_{i=1}^{s}\mathcal{R}_{a_{i_j}} \times \prod_{i=1}^{r}\mathcal{R}_{\bar{a}_{\tau_i}} \\ & v = (v_1, v_2, \dots, v_n) & \longmapsto & \left(v_{j_1}, \dots, v_{j_s}, \Lambda_{(\bar{a},\theta,\tau_1)}(v), \dots, \Lambda_{(\bar{a},\theta,\tau_r)}(v)\right),\end{array}$$

where $\mathcal{R}_{a_{j_i}} = \mathbb{F}_q[x - a_{j_i}]/\langle x - a_i\rangle$ for all $i \in \{1, \dots, s\}$.

If $J = \emptyset$, then

$$\begin{array}{cccc} \Pi_{(\bar{a},\theta,\sigma)}: & \mathbb{F}_q^n & \longrightarrow & \prod_{i=1}^{r} \mathcal{R}_{\bar{a}_{\tau_i}} \\ & (v_1, v_2, \ldots, v_n) & \longmapsto & \left(\Lambda_{(\bar{a},\theta,\tau_1)}(v), \ldots, \Lambda_{(\bar{a},\theta,\tau_r)}(v)\right). \end{array}$$

Remark 4.10. *1.* $\prod_{i=1}^{s} \mathcal{R}_{a_i} \times \prod_{i=1}^{r} \mathcal{R}_{\bar{a}_{\tau_i}}$ *is a left* $\mathbb{F}_q[x,\theta]$*-module.*

2. The map $\Pi_{(\bar{a},\theta,\sigma)}$ *is an* $\mathbb{F}_q$*-isomorphism.*

Theorem 4.11. *Let* $\mathcal{C}$ *be a linear code of length* n *over* $\mathbb{F}_q$*. Then* $\mathcal{C}$ *is a* $(\theta,\, \sigma,\, \bar{a})$*-monomial if and only if* $\Pi_{\bar{a},\theta,\sigma}(\mathcal{C})$ *is a left* $\mathbb{F}_q[x;\sigma]$*-submodule of* $\prod_{i=1}^{s} \mathcal{R}_{a_i} \times \prod_{i=1}^{r} \mathcal{R}_{\bar{a}_{\tau_i}}$.

Proof. Let $v = (v_1, v_2, \ldots, v_n) \in \mathbb{F}_q^n$ and $v' = \left(a_{\sigma(1)}\theta(v_{\sigma(1)}), a_{\sigma(2)}\theta(v_{\sigma(2)}), \ldots, a_{\sigma(n)}\theta(v_{\sigma(n)})\right)$. Let us show that

$$\Pi_{(\bar{a},\theta,\sigma)}(v') = x\Pi_{(\bar{a},\theta,\sigma)}(v).$$

On one hand, we have

$$\Pi_{(\bar{a},\theta,\sigma)}(v') = \left(a_{j_1}\theta(v_{j_1}), \ldots, a_{j_s}\theta(v_{j_s}), \Lambda_{(\bar{a},\theta,\tau_1)}(v'), \ldots, \Lambda_{(\bar{a},\theta,\tau_r)}(v')\right).$$

On the other hand, let $i \in \{1, 2 \ldots, r\}$ and denote $\tau_i = (t_\ell, t_{\ell-1} \ldots, t_1)$, where $ind(\tau_i) = t_1$. We have

$$\begin{aligned} \Lambda_{(\bar{a},\theta,\tau_i)}(v') &= \pi_{\bar{a}_{\tau_i},\theta}\left(a_{\sigma(t_1)}\theta(v_{\sigma(t_1)}), a_{\sigma(t_2)}\theta(v_{\sigma(t_2)}), \ldots, a_{\sigma(t_\ell)}\theta(v_{\sigma(t_\ell)})\right) \\ &= \pi_{\bar{a}_{\tau_i},\theta}\left(a_{t_\ell}\theta(v_{t_\ell}), a_{t_1}\theta(v_{t_1}), \ldots, a_{t_{\ell-1}}\theta(v_{t_{\ell-1}})\right) \\ &= x\pi_{\bar{a}_{\tau_i},\theta}\left(v_{t_1}, v_{t_2}, \ldots, v_{t_\ell}\right) \\ &= x\Lambda_{(\bar{a},\theta,\tau_i)}(v). \end{aligned}$$

Hence

$$\Pi_{(\bar{a},\theta,\sigma)}(v') = \left(xv_{j_1}, \ldots, xv_{j_s}, x\Lambda_{(\bar{a},\theta,\tau_1)}(v), \ldots, x\Lambda_{(\bar{a},\theta,\tau_r)}(v)\right) = x\Pi_{(\bar{a},\theta,\sigma)}(v).$$

Therefor, $\mathcal{C}$ is a $(\theta,\, \sigma,\, \bar{a})$-monomial if and only if $\Pi_{\bar{a},\theta,\sigma}(\mathcal{C})$ is a left $\mathbb{F}_q[x;\sigma]$-submodule of $\prod_{i=1}^{s} \mathcal{R}_{a_i} \times \prod_{i=1}^{r} \mathcal{R}_{\bar{a}_{\tau_i}}$. □

5. θ-monomial codes over the ring $\mathcal{R} = \mathbb{F}_q + v\mathbb{F}_q$, where $v^2 = v$

Let us start by given some basic results on the finite non-chain ring $\mathcal{R} = \mathbb{F}_q + v\mathbb{F}_q = \mathbb{F}_q[v]/\left\langle v^2 - v\right\rangle$. Clearly, $\mathcal{R}$ is a semilocal ring with maximal ideals $\langle v\rangle$ and $\langle 1 - v\rangle$. Further, by the classical ring theory, we have that

$$\mathcal{R} = v\mathcal{R} \oplus (1 - v)\mathcal{R} = v\mathbb{F}_q \oplus (1 - v)\mathbb{F}_q.$$

Then, for any element $r \in \mathcal{R}$, there are unique $a, b \in \mathbb{F}_q$ such that $r = va + (1 - v)b$. We denote by ψ the Frobenius map over $\mathcal{R}$ defined by

$$\begin{gathered} \psi : \mathcal{R} \to \mathcal{R} \\ r = vx + (1 - v)y \mapsto \theta(r) = vx^p + (1 - v)y^p. \end{gathered}$$

For an automorphism θ of $\mathcal{R}$, we have $\theta = \psi^t$, where t is the order of θ. Now we give the definition of a linear code over $\mathcal{R}$.

Definition 5.1. *A nonempty subset $\mathcal{C}$ of $\mathcal{R}^n$ is said to be a linear code over $\mathcal{R}$ of length n if it is an $\mathcal{R}$-submodule of $\mathcal{R}^n$.*

Proposition 5.2 ([14])**.** *Let $\mathcal{C}$ be a linear code of length n over $\mathcal{R}$. Then $\mathcal{C}$ can be uniquely expressed as*

$$\mathcal{C} = v\mathcal{C}_1 \oplus (1-v)\mathcal{C}_2.$$

Where

$$\mathcal{C}_1 = \left\{\mathbf{x} \in \mathbb{F}_q^n \mid v\mathbf{x} + (1-v)\mathbf{y} \in \mathcal{C},\ \textit{for some } \mathbf{y} \in \mathbb{F}_q^n\right\}$$

and

$$\mathcal{C}_2 = \left\{\mathbf{y} \in \mathbb{F}_q^n \mid v\mathbf{x} + (1-v)\mathbf{y} \in \mathcal{C},\ \textit{for some } \mathbf{x} \in \mathbb{F}_q^n\right\}.$$

Based on the previous proposition, the following theorem gives a characterization of θ-monomial codes over $\mathcal{R}$.

Theorem 5.3. *Let $\bar{a} = (a_0, a_1, \ldots, a_{n-1})$ and $\bar{b} = (b_0, b_1, \ldots, b_{n-1})$ be in $\left(\mathbb{F}_q^\theta\right)^n$ such that $\prod_{i=0}^{n-1} a_i \neq 0$, $\prod_{i=0}^{n-1} b_i \neq 0$ and $\prod_{i=0}^{n-1}(a_i + b_i) \neq 0$. Let $\mathcal{C}$ be a linear code of length n over $\mathcal{R}$, and let $\mathcal{C} = v\mathcal{C}_1 \oplus (1-v)\mathcal{C}_2$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are linear codes of length n over $\mathbb{F}_q$. Then $\mathcal{C}$ is a θ-monomial code with respect to the vector $\bar{a} + v\bar{b} = (a_0 + vb_0, a_1 + vb_1, \ldots, a_{n-1} + vb_{n-1})$ if and only if $\mathcal{C}_1$ and $\mathcal{C}_2$ are θ-monomial codes over $\mathbb{F}_q$ with respect to the vectors $\bar{a} + \bar{b}$ and $\bar{a}$, respectively.*

Proof. Let $(x_0, y_1, \ldots, y_{n-1}) \in \mathcal{C}_1$ and $(y_0, y_1, \ldots, y_{n-1}) \in \mathcal{C}_2$. For all $i \in \{0, 1, \ldots, n-1\}$, put $c_i = vx_i + (1-v)y_i$. Then the vector $(c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$. Since $\mathcal{C}$ is a θ-monomial code with respect to $\bar{a} + v\bar{b}$ it follows that

$$c' = \left((a_{n-1} + b_{n-1}v)\theta\left(c_{n-1}\right), (a_0 + b_0 v)\theta\left(c_0\right), \ldots, (a_{n-2} + b_{n-2}v)\theta\left(c_{n-2}\right)\right) \in \mathcal{C}.$$

□

On the other hand, we have

$$\begin{aligned} c' = {} & v\left((a_{n-1} + b_{n-1})\theta\left(c_{n-1}\right), (a_0 + b_0)\theta\left(c_0\right), \ldots, (a_{n-2} + b_{n-2})\theta\left(c_{n-2}\right)\right) \\ & + (1-v)\left(a_{n-1}\theta\left(c_{n-1}\right), a_0\theta\left(c_0\right), \ldots, a_{n-2}\theta\left(c_{n-2}\right)\right) \end{aligned}.$$

Then $\left((a_{n-1} + b_{n-1})\theta\left(c_{n-1}\right), (a_0 + b_0)\theta\left(c_0\right), \ldots, (a_{n-2} + b_{n-2})\theta\left(c_{n-2}\right)\right) \in \mathcal{C}_1$ and $\left(a_{n-1}\theta\left(c_{n-1}\right), a_0\theta\left(c_0\right), \ldots, a_{n-2}\theta\left(c_{n-2}\right)\right) \in \mathcal{C}_2$.

Conversely, let $(c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$, then for all $i \in \{0, 1, \ldots n-1\}$, $c_i = vx_i + (1-v)y_i$, where $(x_0, x_1, \ldots, x_{n-1}) \in \mathcal{C}_1$ and $(y_0, y_1, \ldots, y_{n-1}) \in \mathcal{C}_2$. Let us show that $c' = \left((a_{n-1} + b_{n-1}v)\theta\left(c_{n-1}\right), (a_0 + b_0 v)\theta\left(c_0\right), \ldots, (a_{n-2} + b_{n-2}v)\theta\left(c_{n-2}\right)\right) \in \mathcal{C}$. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are θ-monomial codes over $\mathbb{F}_q$ with respect to $\bar{a} + \bar{b}$ and $\bar{a}$, respectively. Then we have

$$\begin{aligned} c' = {} & v\left((a_{n-1} + b_{n-1})\theta\left(c_{n-1}\right), (a_0 + b_0)\theta\left(c_0\right), \ldots, (a_{n-2} + b_{n-2})\theta\left(c_{n-2}\right)\right) \\ & + (1-v)\left(a_{n-1}\theta\left(c_{n-1}\right), a_0\theta\left(c_0\right), \ldots, a_{n-2}\theta\left(c_{n-2}\right)\right) \in v\mathcal{C}_1 \oplus (1-v)\mathcal{C}_2 = \mathcal{C}. \end{aligned}$$

Hence, $\mathcal{C}$ is a θ-monomial code with respect to the vector $\bar{a} + v\bar{b}$ over the ring $\mathcal{R}$.

Now, we will give some properties on the Galois dual codes.

Definition 5.4. *Let h be a positive integer such that $q = p^h$, $0 \leq e < h$ and $\kappa = h - e$. The e-Galois inner product of two elements $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ and $\mathbf{y} = (y_0, y_1, \ldots, y_{n-1})$ in $\mathcal{R}^n$ is defined by*

$$\langle \mathbf{x}, \mathbf{y} \rangle_e = \sum_{i=0}^{n-1} x_i \psi^e(y_i)$$

and the e-Galois dual code of a linear code $\mathcal{C}$ over $\mathcal{R}$ of length n is defined as

$$\mathcal{C}^{\perp_e} = \{\mathbf{x} \in \mathcal{R}^n \mid \langle \mathbf{c}, \mathbf{x} \rangle_e = 0 \text{ for any } \mathbf{c} \in \mathcal{C}\}.$$

If $\mathcal{C} \subseteq \mathcal{C}^{\perp_e}$, then $\mathcal{C}$ is called e-Galois self-orthogonal. And $\mathcal{C}$ is called e-Galois self-dual if $\mathcal{C} = \mathcal{C}^{\perp_e}$. Note that $\mathcal{C}^{\perp_0}$ is just the Euclidean dual code of $\mathcal{C}$, which we denote by $\mathcal{C}^{\perp}$.

Theorem 5.5 (Theorem 2 in [7]). *Let $\mathcal{C}$ be a linear code of length n over $\mathcal{R}$, and let $\mathcal{C} = v\mathcal{C}_1 \oplus (1-v)\mathcal{C}_2$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are linear codes of length n over $\mathbb{F}_q$. Then*

$$\mathcal{C}^{\perp} = v\mathcal{C}_1^{\perp} \oplus (1-v)\mathcal{C}_2^{\perp},$$

where $\mathcal{C}_1^{\perp}$ and $\mathcal{C}_2^{\perp}$ are dual codes of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Furthermore, $\mathcal{C}$ is self-dual if and only if both $\mathcal{C}_1$ and $\mathcal{C}_2$ are self-dual.

Corollary 5.6. *Let h be a positive integer such that $q = p^h$, $0 \leq e < h$ and $\kappa = h - e$. If $\mathcal{C} = v\mathcal{C}_1 \oplus (1-v)\mathcal{C}_2$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are linear codes of length n over $\mathbb{F}_q$ then*

$$\mathcal{C}^{\perp_e} = v\mathcal{C}_1^{\perp_e} \oplus (1-v)\mathcal{C}_2^{\perp_e},$$

where $\mathcal{C}_1^{\perp_e}$ and $\mathcal{C}_2^{\perp_e}$ are the e-Galois dual codes of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.

Proof. For a subset A of $\mathcal{R}^n$, let us denote $\psi(A) = \{(\psi(x_1), \ldots, \psi(x_n)) : (x_1, \ldots, x_n) \in A\}$. Then we can easily show that $\mathcal{C}^{\perp_e} = (\psi^{\kappa}(\mathcal{C}))^{\perp}$ and $\psi^{\kappa}(v\mathcal{C}_1 \oplus (1-v)\mathcal{C}_2) = v\Gamma^{\kappa}(\mathcal{C}_1) \oplus (1-v)\Gamma^{\kappa}(\mathcal{C}_2)$, where Γ is th Frobenius automorphism of $\mathbb{F}_q$. Then, by the previous Theorem we get

$$\begin{aligned}
\mathcal{C}^{\perp_e} &= (v\Gamma^{\kappa}(\mathcal{C}_1) \oplus (1-v)\Gamma^{\kappa}(\mathcal{C}_2))^{\perp} \\
&= v\,(\Gamma^{\kappa}(\mathcal{C}_1))^{\perp} \oplus (1-v)\,(\Gamma^{\kappa}(\mathcal{C}_2))^{\perp} \\
&= v\mathcal{C}_1^{\perp_e} \oplus (1-v)\mathcal{C}_2^{\perp_e}.
\end{aligned}$$

□

Proposition 5.7. *Let $\mathcal{C} = v\mathcal{C}_1 \oplus (1-v)\mathcal{C}_2$ be a θ-monomial code with respect to the vector $\bar{a} + v\bar{b}$ over the ring $\mathcal{R}$ such that $\prod_{i=0}^{n-1}(a_i + b_i) \neq 0$. Then the e-Galois dual code $\mathcal{C}^{\perp_e} = v\mathcal{C}_1^{\perp_e} \oplus (1-v)\mathcal{C}_2^{\perp_e}$ of $\mathcal{C}$ is a θ-monomial code, where $\mathcal{C}_1^{\perp_e}$ and $\mathcal{C}_2^{\perp_e}$ are θ-monomial codes over $\mathbb{F}_q$ with respect to the vectors $(\bar{a}+\bar{b})^{-p^{\kappa}}$ and $\bar{a}^{-p^{\kappa}}$, respectively.*

Proof. Since $\mathcal{C}_1$ and $\mathcal{C}_1$ are θ-monomial codes over $\mathbb{F}_q$ with respect to the vectors $\bar{a}+\bar{b}$ and $\bar{a}$, respectively. Then by Corollary 2.6, the codes $\mathcal{C}_1^{\perp_e}$ and $\mathcal{C}_2^{\perp_e}$ are θ-monomial codes over $\mathbb{F}_q$ with respect to the vectors $(\bar{a}+\bar{b})^{-p^{\kappa}}$ and $\bar{a}^{-p^{\kappa}}$, respectively . Then, by Theorem 5.3, we get that $\mathcal{C}^{\perp_e}$ is a θ-monomial code over $\mathcal{R}$, with respect to the vectors $\bar{a} + v\bar{b}$. □

References

[1] D. Boucher, W. Geiselmann, F. Ulmer, Skew-cyclic codes, Appl. Algebra Eng. Commun. Comput. 18 (2007) 379âĂŞ389.

[2] D. Boucher, P. SolÃľ, Ulmer, F. Ulmer, Skew constacyclic codes over Galois rings, Advances in mathematics of communications, 2(3) (2008), 273-292.

[3] D. Boucher, F. Ulmer, Coding with skew polynomial rings, J. Symb. Comput. 44 (2009) 1644âĂŞ1656.

[4] Y. Fan, L. Zhang, Galois self-dual constacyclic codes. Designs, Codes and Cryptography, 84 (2017) 473-492.

[5] J. Gao, F. Ma, F. Fu, Skew constacyclic codes over the ring $\mathbb{F}_q + v\mathbb{F}_q$, Appl. Comput. Math, 6(3) (2017) 286–295.

[6] M. I. Garcia-Planas, M. D. Magret, L. Emilie, Monomial codes seen as invariant subspaces, Open mathematics, 15(1) (2017) 1099–1107.

[7] F. Gursoy, I. Siap, B. Yildiz, Construction of skew cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$, Advances in Mathematics of Communications, 8(3) (2014) 313–322.

[8] S. Jitman, S. Ling, P. Udomkavanich, Skew constacyclic codes over finite chain rings, Advances in Mathematics of Communications, 6(1) (2012) 39–63.

[9] E. M. Mouloua, M. Najmeddine, M. I. Garcia-Planas, H. Ouazzou, Monomial codes under linear algebra point of view, ArXiv Preprint: arXiv:2304.00398(2023).

[10] H. Ouazzou, M. Najmeddine, L. Mouatadid, O. Kabbouch, On the structure of monomial codes and theirgeneralizations, Journal of Algebra Combinatorics Discrete Structures and Applications 10(3) (2022)175-âĂŞ192.

[11] E. Prange, Cyclic error-correcting codes in two symbols, Air Force Cambridge Research Center (1957).

[12] E. Prange, Some cyclic error-correcting codes with simple decoding algorithms, Air Force Cambridge Research Center Air Force Cambridge Research Center -TN (1985) 58-156.

[13] I. Siap, T. Abualrub, N. Aydin, P. Seneviratne, Skew cyclic codes of arbitrary length. International Journal of Information and Coding Theory, 2(1), 10-20(2011).

[14] S. Zhu, L. Wang, A class of constacyclic codes over $\mathbb{F}_p + v\mathbb{F}_p$ and its gray image, Discrete Math. 311 (23âĂŞ24) (2011) 2677âĂŞ2682.