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On a variant of k-plane trees

Research Article

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Abstract: In this paper, we introduce a class of plane trees whose vertices receive labels from the set $\{1,2,\ldots,k\}$ such that the sum of labels of adjacent vertices does not exceed $k+1$ and all vertices of label 1 are always on the left of all other vertices. Using generating functions, we enumerate these trees by number of vertices and label of the root, root degree, label of the eldest or youngest child of the root and forests.

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Keywords: k-plane tree, k_1-plane tree, Root degree, Eldest child, Youngest child, Forest

1. Introduction

One of the many structures counted by the famous Catalan numbers is the set of plane trees [22]. See sequence A000108 of Neil Sloane's celebrated online encyclopaedia [20] for a list of these structures. A *plane tree*, also called *ordered tree*, is a rooted tree drawn in the plane such that the positions of all children of internal vertices are taken into consideration. Given a plane tree, vertex u is a *child* of vertex v if u is adjacent to v but is on a lower level. Vertex v is the *parent* of u and all the children of v are called *siblings*. The child that appears on the far left is the *eldest child* and the *youngest child* is the one that is on the far right. So, ages of siblings decrease from left to right. The number of children of a vertex is its *degree* and a collection of trees is a *forest*. Plane trees have been enumerated by number of vertices, number of leaves, root degree, vertices of a given degree which reside on a certain level [2, 3], degree sequence and forests [18, 21].

Schröder numbers, both little and large, have been studied for decades [1, 16, 19]. They are known to enumerate Schröder paths [4], plane trees in which leaves come in two colours [21], block graphs [8, 13, 15], dissections of regular polygons [8, 23], categories of lattice paths [1] among many other structures listed in [20] as A001003 and A006318. In [7], Kariuki, Okoth and Nyamwala introduced and

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enumerated a class of plane trees in which vertices are labelled 1 or 2 such that there are no edges whose end points are labelled 2 and with a further condition that the labels of siblings are weakly increasing from left to right. They coined the name *non-decreasing 2-plane trees* for these trees. See Figure 1 for an example of a non-decreasing 2-plane tree. The number of non-decreasing 2-plane trees whose roots are

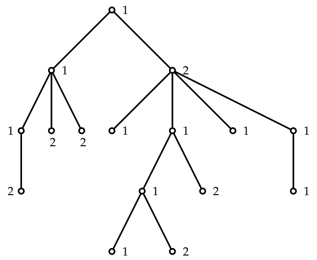

Figure 1: A non-decreasing 2-plane tree on 16 vertices with root label 1.

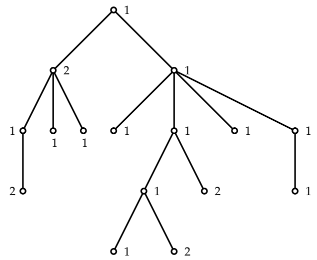

Figure 2: A 2-plane tree on 16 vertices with root label 1.

labelled 1 (resp. 2) is given by large (resp. little) Schröder numbers. The authors not only obtained the number of these trees by the number of vertices and label of the root but also by the degree of the root and label of the eldest child of the root. For non-decreasing 2-plane trees with roots labelled 2, the authors constructed bijections with little Schröder paths, plane trees whose leaves are labelled 1 or 2, lattice paths which allow unit vertical steps, unit horizontal steps and unit diagonal steps such that the paths lie weakly below the line $y = x$, and increasing tableaux. When the roots are labelled 1, bijections with large Schröder paths and row-increasing tableaux were obtained. Non-decreasing 2-plane trees are a special kind of 2-plane trees that were introduced and studied by Gu and Prodinger [5]. A *2-plane tree* is a plane tree in which vertices are given labels from the set $\{1, 2\}$ such that there are no edges with both end points labelled 2. Figure 2 is a 2-plane tree on 16 vertices with root labelled 1.

The number of 2-plane trees on n vertices is given by

$$\frac{3-j}{3n-j}\binom{3n-j}{n-1}$$

[5], where $j = 1, 2$ is the label of the root. The set of 2-plane trees has also been enumerated by degree of the root [12], label of the eldest child of the root [9] and number of vertices of each kind [12, 14]. In [12], these trees were related to noncrossing trees (trees drawn in the plane with vertices on the boundary of a circle such that edges do not cross inside the circle), ternary trees, certain Dyck paths and lattice paths which allow unit vertical steps and unit horizontal steps such that the paths never go above the line $y = 2x$. The set of 2-plane trees was generalized by Gu, Prodinger and Wagner [6] to the set of k-plane trees. These are plane trees in which vertices are labelled with integers in the set $\{1, 2, \ldots, k\}$ such that the sum of labels of any two adjacent vertices is no more than $k + 1$. The number of these trees on n vertices such that the root is labelled j was proved by Gu, Prodinger and Wagner [6] (using generating functions) to be

$$\frac{k-j+1}{(k+1)n-j}\binom{(k+1)n-j}{n-1}. \tag{1}$$

Setting $j = k$ in (1), we find that the number of k-plane trees on n vertices with root labelled k is given by the Fuss-Catalan number,

$$\frac{1}{k(n-1)+1}\binom{(k+1)(n-1)}{n-1}. \tag{2}$$

Formula (2) also counts $(k+1)$-ary trees with $n-1$ internal vertices or Dyck paths consisting of $n-1$ up-steps of size k and unit $k(n-1)$ down-steps that start at $(0, 0)$, stays above the line $y = 0$ and terminates at $(2k(n-1), 0)$. Bijections between the set of k-plane trees on n with roots labelled k and the sets of $(k+1)$-ary trees and Dyck paths described above were constructed in [6]. Bijections involving the set of k-plane trees have also been constructed by Okoth [11], and Nyariaro and Okoth [10]. In [14], Okoth and Wagner enumerated k-plane trees according to the number of vertices of each kind. In this paper, we are interested in a variant of k-plane trees which we shall call k_1-plane trees. We now formally define this class of combinatorial structures.

Definition 1.1. *A* k_1-plane tree *is a k-plane tree in which all children labelled 1 have to be to the left of all others.*

For an example of a k_1-plane tree, see Figure 3. Note that 2_1-plane trees are just non-decreasing 2-

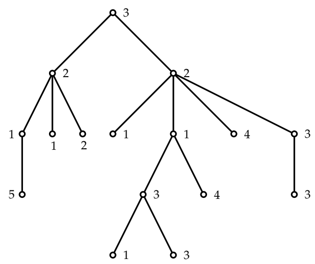

Figure 3: A 5_1-plane tree on 16 vertices with root label 3.

plane trees studied by Kariuki, Okoth and Nyamwala [8]. The results of this paper, therefore generalize the results obtained by Kariuki and her co-authors in [8]. The paper is organized as follows: We enumerate k_1-plane trees according to the label of the root and number of vertices in Section 2 and degree of the root in Section 3. Label of the eldest child and youngest child of the root are the parameters of enumeration in Section 4 and formulas for the number of forests with a given number of components are established in Section 5. This paper is concluded in Section 6 and therein we expose problems on how this research could be extended.

The tools employed in this paper are symbolic method which helps us get generating functions, using suitable substitutions (also used by Gu, Prodinger and Wagner [6]), application of binomial theorem, multinomial theorem, Vandermonde Convolution [17] , Lagrange Inversion Formula [24] and telescoping of binomial coefficients.

Theorem 1.2 (Lagrange Inversion Formula, [21, 24])**.** *Let $f(x)$ be a generating function that satisfies the functional equation $f(x) = x\phi(f(x)$, where $\phi(0) \neq 0$. Then, we have*

$$m[x^m]f(x)^k = k[g^{m-k}]\phi(g)^m.$$

Theorem 1.3 (Multinomial Theorem, [17])**.** *Let $x_1, x_2, \ldots, x_n$ and m be integers, then*

$$(x_1 + x_2 + \cdots + x_n)^m = \sum_{m_1+m_2+\cdots+m_n=m} \frac{m!}{m_1!m_2!\cdots m_n!} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}.$$

Setting $n = 2$ in Theorem 1.3, we get the well known binomial theorem. We also make use of the identity:

Identity 1 ([17])**.** *Let n and j be non-negative integers, then*

$$\binom{-n}{j}(-1)^j = \binom{n+j-1}{j}.$$

Identity 2 (Hockey Stick Identity, [17])**.** *Let n and j be positive integers, then*

$$\sum_{i=j}^{n} \binom{i}{j} = \binom{n+1}{j+1}.$$

Identity 3 (Vandermonde Convolution, [17])**.** *Let p, q and r be positive integers, then*

$$\sum_{i=0}^{r}\binom{p}{i}\binom{q}{r-i}=\binom{p+q}{r}.$$

2. Number of vertices and label of the root

Let $T_i(x) = T_i$ be the generating function for k_1-plane trees with root labelled by i such that x marks a node. These trees have as subtrees rooted at their children, a sequence of trees rooted at vertex labelled 1 followed by a sequence of subtrees rooted at vertices labelled by an integer $j \in \{2, 3, \ldots, k\}$ such that $i + j \leq k + 1$. This is shown in Figure 4.

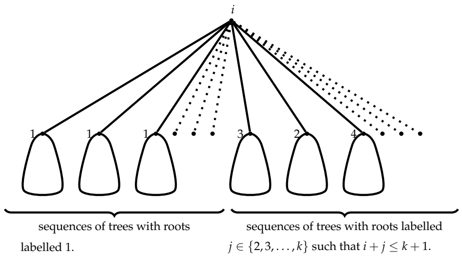

Figure 4: A k_1-plane tree with root label i

The generating function $T_i(x)$ is therefore given by

$$T_i(x) = x \cdot \frac{1}{1-T_1} \cdot \frac{1}{1-\sum_{j=2}^{k-i+1} T_j}. \tag{3}$$

We now strive to solve the system of functional equations (3). Let $T_i(x) = \dfrac{v}{(1+v)^{i-1}}$ and $x = \frac{v(1-v)}{(1+v)^{k-1}}$. Then

$$T_i(x) = x \cdot \frac{1}{1-v} \cdot \frac{1}{1-\sum_{j=2}^{k-i+1}\frac{v}{(1+v)^{j-1}}} = \frac{v(1-v)}{(1+v)^{k-1}} \cdot \frac{1}{1-v} \cdot \frac{1}{1-(1-(1+v)^{i-k})} = \frac{v}{(1+v)^{i-1}}.$$

Since the power series for $T_i(x)$ where $i = 1, 2, \ldots, k$ are uniquely determined by the functional equations, then $T_i(x) = \dfrac{v}{(1+v)^{i-1}}$ and $x = \frac{v(1-v)}{(1+v)^{k-1}}$ are the right substitutions to solve the system of functional equations (3).

Since $x = \frac{v(1-v)}{(1+v)^{k-1}}$ then $v = x(1-v)^{-1}(1+v)^{k-1}$. This is in a form we can apply Lagrange Inversion formula [21]. Note that $T_1 = v$. So,

$$\begin{aligned}
[x^n]T_1 = [x^n]v &= \frac{1}{n}[t^{n-1}](1-t)^{-n}(1+t)^{(k-1)n} \\
&= \frac{1}{n}[t^{n-1}]\sum_{a\geq 0}\binom{-n}{a}(-t)^a \sum_{b\geq 0}\binom{(k-1)n}{b}t^b \\
&= \frac{1}{n}[t^{n-1}]\sum_{a\geq 0}\binom{n+a-1}{a}\sum_{b\geq 0}\binom{(k-1)n}{b}t^{a+b} \\
&= \frac{1}{n}\sum_{a=0}^{n-1}\binom{n+a-1}{a}\binom{(k-1)n}{n-a-1}. \qquad (4)
\end{aligned}$$

Equation (4) is a generalization of large Schröder numbers and it counts k_1-plane trees on n vertices with root labelled 1. When we set $k = 2$, we get large Schröder numbers which count a number of combinatorial structures as given in [21]. When $k = 2$, we get non-decreasing 2-plane trees that were introduced by Kariuki, Okoth and Nyamwala in [8]. The aforementioned authors showed that these structures on n vertices such that the roots are labelled 1 (resp. 2) are counted by large (resp. little) Schröder numbers. Setting $k = 3$ in (4), we obtain a formula which also counts 3-Schröder numbers as recorded in the encyclopaedia [20] as sequence A064062.

Generally, let us compute $[x^n]T_i(x)$.

$$\begin{aligned}
[x^n]T_i = [x^n]v(1+v)^{1-i} = [x^n]v\sum_{s=0}^{1-i}\binom{1-i}{s}v^s &= \sum_{s=0}^{1-i}\binom{1-i}{s}[x^n]v^{s+1} \\
&= \sum_{s=0}^{1-i}\binom{1-i}{s}\frac{s+1}{n}[t^{n-s-1}](1-t)^{-n}(1+t)^{(k-1)n}.
\end{aligned}$$

Lagrange Inversion gives,

$$\begin{aligned}
[x^n]T_i &= \sum_{s=0}^{1-i}\binom{1-i}{s}\frac{s+1}{n}[t^{n-s-1}]\sum_{a\geq 0}\binom{-n}{a}(-t)^a\sum_{b\geq 0}\binom{(k-1)n}{b}t^b \\
&= \frac{1}{n}\sum_{s=0}^{1-i}\binom{1-i}{s}(s+1)\sum_{a=0}^{n-1}\binom{n+a-1}{a}\binom{(k-1)n}{n-a-s-1}.
\end{aligned}$$

Now, we have

$$\begin{aligned}
[x^n]T_i &= \frac{1}{n}\sum_{s=0}^{1-i}\left[(1-i)\binom{-i}{s-1}+\binom{1-i}{s}\right]\sum_{a=0}^{n-1}\binom{n+a-1}{a}\binom{(k-1)n}{n-a-s-1} \\
&= \frac{1}{n}\sum_{a=0}^{n-1}\left[(1-i)\binom{(k-1)n-i}{n-a-2}+\binom{(k-1)n-i+1}{n-a-1}\right]\binom{n+a-1}{a} \\
&= \frac{1}{n}\sum_{a=0}^{n-1}\frac{(k-i)n+(i-1)a}{(k-1)n-i+1}\binom{(k-1)n-i+1}{n-a-1}\binom{n+a-1}{a}.
\end{aligned}$$

The second last equality follows by Vandermonde Convolution. Let us formally advertise this result as a theorem.

Theorem 2.1. *The number of k_1-plane trees on n vertices with root labelled i is given by the sum*

$$\frac{1}{n}\sum_{a=0}^{n-1}\frac{(k-i)n+a(i-1)}{(k-1)n-i+1}\binom{(k-1)n-i+1}{n-a-1}\binom{n+a-1}{a}. \qquad (5)$$

Setting $i = k$ in (5), we find the following corollary:

Corollary 2.2. *There are*

$$\frac{1}{n(n-1)} \sum_{a=1}^{n-1} a \binom{(k-1)(n-1)}{n-a-1} \binom{n+a-1}{a} \tag{6}$$

k_1-plane trees on n vertices with root labelled k.

Substituting $k = 2$ in (6), we obtain little Schröder numbers. Bijections between the set of 2_1-plane trees (also called non-decreasing 2-plane trees) with roots labelled 2 and the set of other combinatorial structures were constructed in [8]. If $k = 1$ in (6) then $a = n - 1$, and thus we rediscover the Catalan number which enumerates 1-plane trees (plane trees).

Theorem 2.3. *There are*

$$\frac{1}{n} \sum_{a=0}^{n-1} \left[\binom{(k-1)n}{n-a-1} + (k-1) \binom{(k-1)n-k}{n-a-1} \right] \binom{n+a-1}{a} \tag{7}$$

k_1-plane trees on n vertices.

Proof. Let $T(x)$ be the generating function for k_1-plane trees. Then

$$T(x) = \sum_{i=1}^{k} \frac{v}{(1+v)^{i-1}} = v \frac{1 - \left(\frac{1}{1+v}\right)^k}{1 - \frac{1}{1+v}} = 1 + v - (1+v)^{-k+1}.$$

Now,

$$[x^n]T(x) = [x^n]v - [x^n](1+v)^{-k+1} = [x^n]v - \sum_{s \geq 0} \binom{-k+1}{s} [x^n]v^s.$$

By Lagrange Inversion, we get

$$\begin{aligned}
[x^n]T(x) &= \frac{1}{n}[t^{n-1}](1-t)^{-n}(1+t)^{(k-1)n} - \sum_{s \geq 0} \binom{-k+1}{s} \frac{s}{n} [t^{n-s}](1-t)^{-n}(1+t)^{(k-1)n} \\
&= \frac{1}{n} \sum_{a=0}^{n-1} \binom{n+a-1}{a} \binom{(k-1)n}{n-a-1} - \sum_{a \geq 0} \sum_{s \geq 0} \binom{-k+1}{s} \frac{s}{n} \binom{n+a-1}{a} \binom{(k-1)n}{n-s-a} \\
&= \frac{1}{n} \sum_{a=0}^{n-1} \binom{n+a-1}{a} \binom{(k-1)n}{n-a-1} + \frac{k-1}{n} \sum_{a \geq 0} \sum_{s \geq 0} \binom{-k}{s-1} \binom{n+a-1}{a} \binom{(k-1)n}{n-s-a}.
\end{aligned}$$

By Vandermonde Convolution, we have

$$[x^n]T(x) = \frac{1}{n} \sum_{a=0}^{n-1} \binom{n+a-1}{a} \binom{(k-1)n}{n-a-1} + \frac{k-1}{n} \sum_{a=0}^{n-1} \binom{n+a-1}{a} \binom{(k-1)n-k}{n-a-1}.$$

The result thus follows. □

Setting $k = 2$ in (7), we find that the sum of little and large Schröder numbers,

$$\frac{1}{n} \sum_{a=0}^{n-1} \frac{n(n-1) + a(a+1)}{(n-1)(a+1)} \binom{n-1}{a} \binom{n+a-1}{a},$$

counts 2_1-plane trees on n vertices. If $k = 1$ in (7) then $a = n - 1$ and thus we get the $(n-1)^{\text{th}}$ Catalan number,

$$\frac{1}{n} \binom{2n-2}{n-1},$$

as the number of plane trees with n vertices.

3. Root degree and label of eldest child

In this section, we obtain formulas for the number of k_1-plane trees based on the root degree and the label of the root. Moreover, we count these trees if the label of the eldest child of the root is also taken into consideration.

Theorem 3.1. *Let $\mathcal{T}_{i,j,d}$ be the set of k_1-plane trees on n vertices with root labelled j and of degree d such that all the children of the root are labelled i. Then,*

$$|\mathcal{T}_{i,j,d}| = \frac{d}{n-1}\sum_{a=0}^{n-d-1}\frac{(k-i)(n-1)+(i-1)a}{(k-1)(n-1)-di+d}\binom{(k-1)(n-1)-di+d}{n-a-d-1}\binom{n+a-2}{a}. \tag{8}$$

Proof. We extract the coefficient of x^n in xT_i^d. We have,

$$\begin{aligned}|\mathcal{T}_{i,j,d}| &= [x^n]xT_i^d = [x^{n-1}]v^d(1+v)^{d(1-i)}\\ &= [x^{n-1}]v^d\sum_{s=0}^{d(1-i)}\binom{d(1-i)}{s}v^s = \sum_{s=0}^{d(1-i)}\binom{d(1-i)}{s}[x^{n-1}]v^{s+d}\\ &= \sum_{s=0}^{d(1-i)}\binom{d(1-i)}{s}\frac{s+d}{n-1}[t^{n-s-d-1}](1-t)^{-(n-1)}(1+t)^{(k-1)(n-1)}.\end{aligned}$$

Now, we apply Lagrange Inversion to obtain

$$\begin{aligned}|\mathcal{T}_{i,j,d}| &= \sum_{s=0}^{d(1-i)}\binom{d(1-i)}{s}\frac{s+d}{n-1}[t^{n-s-d-1}]\sum_{a\geq 0}\binom{-(n-1)}{a}(-t)^a\sum_{b\geq 0}\binom{(k-1)(n-1)}{b}t^b\\ &= \sum_{s=0}^{d(1-i)}\binom{d(1-i)}{s}\frac{s+d}{n-1}[t^{n-s-d-1}]\sum_{a\geq 0}\binom{n+a-2}{a}\sum_{b\geq 0}\binom{(k-1)(n-1)}{b}t^{a+b}\\ &= \frac{1}{n-1}\sum_{s=0}^{d(1-i)}\binom{d(1-i)}{s}(s+d)\sum_{a\geq 0}\binom{n+a-2}{a}\binom{(k-1)(n-1)}{n-a-s-d-1}\\ &= \frac{1}{n-1}\left[d(1-i)\sum_{s=1}^{d(1-i)-1}\binom{-di+d-1}{s-1}+d\sum_{s=0}^{d(1-i)}\binom{d-di}{s}\right]\\ &\times\sum_{a\geq 0}\binom{n+a-2}{a}\binom{(k-1)(n-1)}{n-a-s-d-1}.\end{aligned}$$

By Vandermonde Convolution, we get

$$\begin{aligned}|\mathcal{T}_{i,j,d}| &= \frac{d}{n-1}\sum_{a\geq 0}\left[(1-i)\binom{(k-1)(n-1)-di+d-1}{n-a-d-2}+\binom{(k-1)(n-1)-di+d}{n-a-d-1}\right]\binom{n+a-2}{a}\\ &= \frac{d}{n-1}\sum_{a=0}^{n-d-1}\frac{(k-i)(n-1)+(i-1)a}{(k-1)(n-1)-di+d}\binom{(k-1)(n-1)-di+d}{n-a-d-1}\binom{n+a-2}{a}.\end{aligned}$$

□

Since formula (8) is independent of j, it follows that $|\mathcal{T}_{i,r,d}| = |\mathcal{T}_{i,s,d}|$ for all r and s satisfying the coherence condition $i+r \leq k+1$ and $i+s \leq k+1$. We obtain the following result upon setting $i=1$ in (8) and using the fact that if a vertex is labelled k then its adjacent vertex is labelled 1.

Corollary 3.2. *The number of k_1-plane trees on n vertices with roots labelled k and of degree d is given by*

$$\frac{d}{n-1}\sum_{a=0}^{n-d-1}\binom{(k-1)(n-1)}{n-a-d-1}\binom{n+a-2}{a}. \tag{9}$$

By setting $i = k$ in (8), we obtain:

Corollary 3.3. *The number of k_1-plane trees on n vertices with roots of degree d labelled 1 such that all children of the root are labelled k is given by*

$$\frac{d}{n-1}\sum_{a=1}^{n-d-1}\frac{a}{n-d-1}\binom{(k-1)(n-d-1)}{n-a-d-1}\binom{n+a-2}{a}. \tag{10}$$

Theorem 3.4. *There are*

$$\frac{1}{n-1}\sum_{a=0}^{n-d-1}\frac{(d(k-1)-e)(n-1)+ae}{(k-1)(n-1)-e}\binom{(k-1)(n-1)-e}{n-a-d-1}\binom{n+a-2}{a}\binom{d-d_1}{d_2,d_3,\dots,d_{k-i+1}} \tag{11}$$

k_1-plane trees on n vertices whose root is labelled i and the root has degree d such that among the children of the root, d_j are labelled j where $j = 1,2,3,\dots,k-i+1$ and $e := d_2+2d_3+\dots+(k-i)d_{k-i+1}$.

Proof. Let $T_j(x)$ be the generating function for k_1-plane trees rooted at a vertex labelled j, where x marks a vertex. Since there are d_j subtrees rooted at the children of the root for $j = 1,2,\dots,k$, there generating function for the desired k_1-plane trees in which the position of the subtrees is not taken into consideration is $xT_1(x)^{d_1}T_2(x)^{d_2}\cdots T_{k-i+1}(x)^{d_{k-i+1}}$. We now proceed as follows.

$$\begin{aligned}[x^n]xT_1{}^{d_1}T_2{}^{d_2}\cdots T_{k-i+1}{}^{d_{k-i+1}} &= [x^{n-1}]v^{d_1}\cdot\left(\frac{v}{1+v}\right)^{d_2}\cdots\left(\frac{v}{(1+v)^{k-i}}\right)^{d_{k-i+1}}\\ &= [x^{n-1}]v^{d_1+d_2+\dots+d_{k-i+1}}(1+v)^{-(d_2+2d_3+\dots+(k-i)d_{k-i+1})}\end{aligned}$$

where $v = x(1-v)^{-1}(1+v)^{k-1}$ as given in Section 2. Now, the total root degree is $d = d_1+d_2+\dots+d_{k-i+1}$. Define, $e := d_2+2d_3+\dots+(k-i)d_{k-i+1}$ so as to save us from writing the whole summation. Then,

$$[x^n]xT_1{}^{d_1}T_2{}^{d_2}\cdots T_{k-i+1}{}^{d_{k-i+1}} = [x^{n-1}]v^d(1+v)^{-e}.$$

By Binomial Theorem, we get

$$[x^n]xT_1{}^{d_1}T_2{}^{d_2}\cdots T_{k-i+1}{}^{d_{k-i+1}} = [x^{n-1}]v^d\sum_{s\geq 0}\binom{-e}{s}v^s = \sum_{s\geq 0}\binom{-e}{s}[x^{n-1}]v^{s+d}.$$

By Lagrange Inversion Formula, we have

$$\begin{aligned}[x^n]xT_1{}^{d_1}T_2{}^{d_2}\cdots T_{k-i+1}{}^{d_{k-i+1}} &= \sum_{s\geq 0}\binom{-e}{s}\frac{s+d}{n-1}[t^{n-s-d-1}](1-t)^{-(n-1)}(1+t)^{(k-1)(n-1)}\\ &= \sum_{s\geq 0}\binom{-e}{s}\frac{s+d}{n-1}[t^{n-s-d-1}]\sum_{a\geq 0}\binom{-(n-1)}{a}(-t)^a\sum_{b\geq 0}\binom{(k-1)(n-1)}{b}t^b\\ &= \frac{1}{n-1}\left[d\sum_{s\geq 0}\binom{-e}{s} - e\sum_{s\geq 1}\binom{-e-1}{s-1}\right]\sum_{a\geq 0}\binom{n+a-2}{a}\binom{(k-1)(n-1)}{n-a-s-d-1}.\end{aligned}$$

By Vandermonde Convolution, we get

$$\begin{aligned}&[x^n]x{T_1}^{d_1}{T_2}^{d_2}\cdots {T_{k-i+1}}^{d_{k-i+1}}\\&=\frac{1}{n-1}\sum_{a\geq 0}\left[d\binom{(k-1)(n-1)-e}{n-a-d-1}-e\binom{(k-1)(n-1)-e-1}{n-a-d-2}\right]\binom{n+a-2}{a}\\&=\frac{1}{n-1}\sum_{a=0}^{n-d-1}\frac{(d(k-1)-e)(n-1)+ae}{(k-1)(n-1)-e}\binom{(k-1)(n-1)-e}{n-a-d-1}\binom{n+a-2}{a}.\end{aligned}$$

Now, since all the all children labelled 1 for each internal vertex is on the left then there are

$$\binom{d-d_1}{d_2,d_3,\ldots,d_{k-i+1}}$$

ways of assigning labels to the children of the root so that there are d_j children labelled j for $j = 1,2,\ldots,k-i+1$. The proof follows by product rule of counting. □

We obtain Theorem 8, by setting $e = d(i-1)$ and $d_r = 0$ for all $r \neq i$ in (11). If $e = 0$ in Theorem 3.4 then $d_1 = d$, $d_2 = d_3 = \cdots = d_{k-i+1} = 0$ and thus, we find that there are

$$\frac{d}{n-1}\sum_{a=0}^{n-d-1}\binom{(k-1)(n-1)}{n-a-d-1}\binom{n+a-2}{a} \tag{12}$$

k_1-plane trees on n vertices such that the root is labelled k and is of degree d. Equation (12) was also obtained in Corollary 3.2. Setting $k = 1$ in (12), we find that there are

$$\frac{d}{2n-d-2}\binom{2n-d-2}{n-d-1}$$

plane trees on n vertices with root degree d. If $k = 2$ and $i = 1$ in (11) then $d_1 + d_2 = d$ and $d_2 = e$. This means that $d_2 = d - d_1$ and $e = d - d_1$. We thus obtain that there are

$$\frac{1}{n-1}\sum_{a=0}^{n-d-1}\frac{d_1(n-a-1)+ad}{n-d+d_1-1}\binom{n-d+d_1-1}{n-a-d-1}\binom{n+a-2}{a} \tag{13}$$

2_1-plane trees (called non-decreasing 2-plane trees in [8]) on n vertices with root labelled 1 and has d children of which d_1 are labelled 1. Summing over all d_1 and d in (13), we find the total number of 2_1-plane trees on n vertices with root labelled 1.

If $k = 2$ and $i = 2$ in (11) then $d_1 = d$ and $e = 0$. It follows that there

$$\frac{d}{n-1}\sum_{a=0}^{n-d-1}\binom{n-1}{n-a-d-1}\binom{n+a-2}{a}$$

2_1-plane trees on n vertices with root labelled 1 and has d children all labelled 2.

4. Eldest or youngest child of the root

In this section we enumerate k_1-plane trees by label of the root and label of the eldest or youngest child of the root. We prove the following result:

Theorem 4.1. *The number of k_1-plane trees on n vertices with roots labelled i such that the eldest child of the root is labelled 1 is*

$$\frac{1}{n}\sum_{a=0}^{n-2}\frac{(2k-i-1)n+(i-1)a}{(k-1)n-i+1}\binom{(k-1)n-i+1}{n-a-2}\binom{n+a-1}{a}. \tag{14}$$

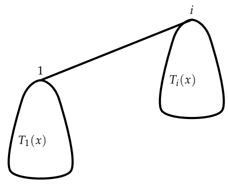

Figure 5: Decomposition of k_1-plane trees with root labelled i and eldest child of the root is labelled 1.

Proof. The decomposition is as shown in Figure 5.
By the decomposition, we find that the generating function is $T_1(x)T_i(x)$. We extract the coefficient of x^n.

$$[x^n]T_1(x)T_i(x) = [x^n]v^2(1+v)^{1-i} = \sum_{s=0}^{1-i}\binom{1-i}{s}[x^n]v^{s+2}.$$

Lagrange Inversion gives,

$$\begin{aligned}
[x^n]T_1T_i &= \sum_{s=0}^{1-i}\binom{1-i}{s}\frac{s+2}{n}[t^{n-s-2}](1-t)^{-n}(1+t)^{(k-1)n} \\
&= \sum_{s=0}^{1-i}\binom{1-i}{s}\frac{s+2}{n}[t^{n-s-2}]\sum_{a\geq 0}\binom{-n}{a}(-t)^a\sum_{b\geq 0}\binom{(k-1)n}{b}t^b \\
&= \frac{1}{n}\sum_{s=0}^{1-i}\binom{1-i}{s}(s+2)\sum_{a=0}^{n-1}\binom{n+a-1}{a}\binom{(k-1)n}{n-a-s-2}.
\end{aligned}$$

Now, we have

$$\begin{aligned}
[x^n]T_1T_i &= \frac{1}{n}\sum_{s=0}^{1-i}\left[(1-i)\binom{-i}{s-1}+2\binom{1-i}{s}\right]\sum_{a=0}^{n-1}\binom{n+a-1}{a}\binom{(k-1)n}{n-a-s-2} \\
&= \frac{1}{n}\sum_{a=0}^{n-2}\frac{(2k-i-1)n+(i-1)a}{(k-1)n-i+1}\binom{(k-1)n-i+1}{n-a-2}\binom{n+a-1}{a}.
\end{aligned}$$

□

Corollary 4.2. *There are*

$$\frac{1}{n}\sum_{a=0}^{n-2}\frac{(2k-i-1)n+(i-1)a}{(k-1)n-i+1}\binom{(k-1)n-i+1}{n-a-2}\binom{n+a-1}{a}$$

k_1-plane trees on n vertices whose root is labelled 1 *such that the eldest child of the root is also i.*

Proof. Detach the root and all the subtrees rooted at the children of the root except the eldest child and attach then to the eldest child of the root in a way that the initial root becomes the eldest child of the tree having the initial eldest child of the root as the root. This process is easily reversible. The formula thus follows from Theorem 4.1. □

Setting $i = 1$ in Theorem 4.1, we get the following corollary.

Corollary 4.3. *The number of k_1-plane trees on n vertices with roots labelled* 1 *such that the eldest child of the root is also labelled* 1 *is*

$$\frac{2}{n}\sum_{a=0}^{n-2}\binom{(k-1)n}{n-a-2}\binom{n+a-1}{a}. \tag{15}$$

We give an alternative proof.

Bijective proof of Corollary 4.3. Let T be a k_1-plane tree on n vertices with root labelled 1 such that the eldest child of the root is also labelled i. Let g be the edge connecting the root and its eldest child. We delete this edge to obtain an ordered pair of k_1-plane trees whose roots are all labelled by 1. The first k_1-plane tree, T_1, is the subtree of T whose root is the eldest child of the root of T. The second k_1-plane tree, T_2, is the subtree whose root is the root of T. Now, connect the roots of the two subtrees to a new vertex labelled k. The resultant tree is a k_1-plane tree on $n+1$ vertices such that the root is labelled k and of degree 2. Moreover, the children of the root are all labelled 1. The procedure is easily reversible. The bijection is illustrated in Figure 6. The formula therefore results by setting $n = n+1$, $j = 1$ and $d = 2$ in

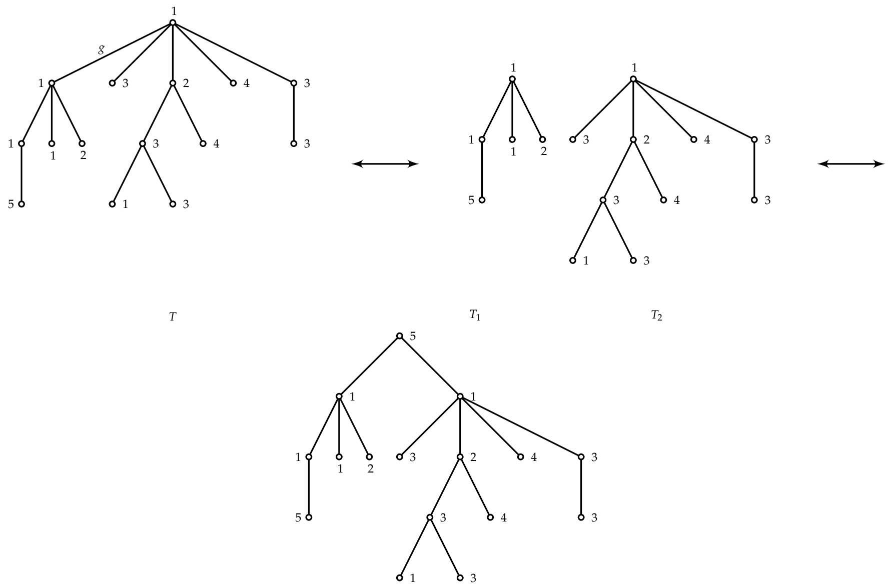

Figure 6: Bijection between a 5_1-plane tree with root labelled 1 such that the eldest child is labelled 1 and 5_1-plane tree with root labelled 5 of degree 2 and all children of the root are labelled 1.

(8). □

We now enumerate k_1-plane trees in which the youngest child of the root is of a given label. If the youngest child of the root is labelled 1 then all its older siblings must be labelled 1. This has already been obtained in Corollary 2.2. So, we concentrate to determine counting formulas for k_1-plane trees in which the youngest child of the root has a specified label, different from 1. We achieve this in the following theorem:

Theorem 4.4. *The number of k_1-plane trees on n vertices with roots labelled i such that the youngest child of the root is also labelled $j \neq 1$ is*

$$\frac{1}{n}\sum_{a=0}^{n-1}\frac{(2k-i-j)n+(i+j-2)a}{(k-1)n-i-j+2}\binom{(k-1)n-i-j+2}{n-a-2}\binom{n+a-1}{a}. \tag{16}$$

Proof. The decomposition is as shown in Figure 7.

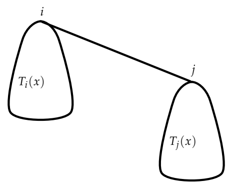

Figure 7: Decomposition of k_1-plane trees with root labelled i and youngest child of the root is labelled $j \neq 1$.

By the decomposition, the desired generating function is $T_i(x)T_j(x)$. We proceed to extract the coefficient of x^n in the generating function.

$$[x^n]T_i(x)T_j(x) = [x^n]v^2(1+v)^{2-i-j} = \sum_{s=0}^{2-i-j} \binom{2-i-j}{s}[x^n]v^{s+2}.$$

Lagrange Inversion gives,

$$\begin{aligned}[x^n]T_iT_j &= \sum_{s=0}^{2-i-j} \binom{2-i-j}{s} \frac{s+2}{n}[t^{n-s-2}] \sum_{a\geq 0}\binom{-n}{a}(-t)^a \sum_{b\geq 0}\binom{(k-1)n}{b}t^b \\ &= \frac{1}{n}\sum_{s=0}^{2-i-j}\binom{2-i-j}{s}(s+2)\sum_{a=0}^{n-1}\binom{n+a-1}{a}\binom{(k-1)n}{n-a-s-2}.\end{aligned}$$

By Vandermonde Convolution, we have

$$\begin{aligned}[x^n]T_iT_j &= \frac{1}{n}\sum_{a\geq 0}\left[(2-i-j)\binom{(k-1)n-i-j+1}{n-a-3} + 2\binom{(k-1)n-i-j+2}{n-a-2}\right]\binom{n+a-1}{a} \\ &= \frac{1}{n}\sum_{a=0}^{n-1}\frac{(2k-i-j)n+(i+j-2)a}{(k-1)n-i-j+2}\binom{(k-1)n-i-j+2}{n-a-2}\binom{n+a-1}{a}.\end{aligned}$$

□

Corollary 4.5. *There are*

$$\frac{2}{n}\sum_{a=0}^{n-1}\frac{(k-i)n+(i-1)a}{(k-1)n-2i+2}\binom{(k-1)n-2i+2}{n-a-2}\binom{n+a-1}{a}$$

k_1-plane trees on n vertices such that both the root and the youngest child of the root are labelled i.

Proof. If $i \neq 1$ then the result follows by setting $j = i$ in (16). If $i = 1$ then the problem is equivalent to determining the number of k-plane trees on $n+1$ vertices with root labelled k, of degree 2 and all the children of the root are labelled 1. This is based on modification of the bijective proof of Corollary 4.3, such that the deleted edge is the one joining the root and the youngest child of the root. So, we set $n = n+1$, and $d = 2$ in (9). □

5. Forests of k_1-plane trees

In this section, we enumerate forests of k_1-plane trees. The forests on n vertices considered are given further labels $1, 2, \ldots, n$. These forests are called *labelled forests*. We enumerate labelled forests so as to avoid redundancies.

Theorem 5.1. *There are*

$$(n-1)!\sum_{a=0}^{n-r}\frac{(r(k-1)-q)n+aq}{(k-1)n-q}\binom{(k-1)n-q}{n-a-r}\binom{n+a-1}{a}\binom{r}{r_1,r_2,\ldots,r_{k-i+1}} \tag{17}$$

labelled forests of k_1-plane trees on n vertices with r components such that there are r_j trees whose roots are labelled j where $j=1,2,3,\ldots,k-i+1$ and $q:=r_2+2r_3+\cdots+(k-i)r_{k-i+1}$.

Proof. Let $T_i(x)$ be the generating function for k_1-plane trees rooted at a vertex labelled i. Here x marks a vertex. The generating function for the number of unlabelled forests of k_1-plane trees with r components is $T_1^{r_1}T_2^{r_2}\cdots T_{k-i+1}^{r_{k-i+1}}$. So, we have

$$\begin{aligned}[x^n]{T_1}^{r_1}{T_2}^{r_2}\cdots {T_{k-i+1}}^{r_{k-i+1}} &= [x^n]v^{r_1}\cdot\left(\frac{v}{1+v}\right)^{r_2}\cdots\left(\frac{v}{(1+v)^{k-i}}\right)^{r_{k-i+1}}\\ &= [x^n]v^{r_1+r_2+\cdots+r_{k-i+1}}(1+v)^{-(r_2+2r_3+\cdots+(k-i)r_{k-i+1})}\end{aligned}$$

where $v=x(1-v)^{-1}(1+v)^{k-1}$. Since the number of components is r then $r=r_1+r_2+\cdots+r_{k-i+1}$. We let $q:=r_2+2r_3+\cdots+(k-i)r_{k-i+1}$. Then,

$$[x^n]T_1^{r_1}T_2^{r_2}\cdots T_{k-i+1}^{r_{k-i+1}}=[x^n]v^r(1+v)^{-q}.$$

By Binomial Theorem, we obtain

$$[x^n]{T_1}^{r_1}{T_2}^{r_2}\cdots {T_{k-i+1}}^{r_{k-i+1}}=[x^n]v^r\sum_{s\geq 0}\binom{-q}{s}v^s=\sum_{s\geq 0}\binom{-q}{s}[x^n]v^{s+d}.$$

Application of Lagrange Inversion Formula gives

$$\begin{aligned}[x^n]{T_1}^{r_1}{T_2}^{r_2}\cdots {T_{k-i+1}}^{r_{k-i+1}} &= \sum_{s\geq 0}\binom{-q}{s}\frac{s+r}{n}[t^{n-s-r}](1-t)^{-n}(1+t)^{(k-1)n}\\ &= \sum_{s\geq 0}\binom{-q}{s}\frac{s+r}{n}[t^{n-s-r}]\sum_{a\geq 0}\binom{-n}{a}(-t)^a\sum_{b\geq 0}\binom{(k-1)n}{b}t^b\\ &= \sum_{s\geq 0}\binom{-q}{s}\frac{s+r}{n}[t^{n-s-r}]\sum_{a\geq 0}\binom{n+a-1}{a}\sum_{b\geq 0}\binom{(k-1)n}{b}t^{a+b}\\ &= \frac{1}{n}\left[r\sum_{s\geq 0}\binom{-q}{s}-q\sum_{s\geq 1}\binom{-q-1}{s-1}\right]\sum_{a\geq 0}\binom{n+a-1}{a}\binom{(k-1)n}{n-a-s-r}.\end{aligned}$$

By Vandermonde Convolution, we get

$$\begin{aligned}[x^n]{T_1}^{r_1}{T_2}^{r_2}\cdots {T_{k-i+1}}^{r_{k-i+1}} &= \frac{1}{n-1}\sum_{a\geq 0}\left[r\binom{(k-1)n-q}{n-a-r}-q\binom{(k-1)n-q-1}{n-a-r-1}\right]\binom{n+a-1}{a}\\ &= \frac{1}{n}\sum_{a=0}^{n-r}\frac{(r(k-1)-q)n+aq}{(k-1)n-q}\binom{(k-1)n-q}{n-a-r}\binom{n+a-1}{a}.\end{aligned}$$

There are

$$\binom{r}{r_1,r_2,\ldots,r_{k-i+1}}$$

ways of assigning positions for the trees to form a forest and $n!$ ways to label the vertices. By product rule of counting the proof follows. □

If $r_i = r$ in (17) then $q = r(i-1)$ and $r_j = 0$ for all $j \neq i$ so that

$$r(n-1)! \sum_{a=0}^{n-r} \frac{((k-i))n + a(i-1)}{(k-1)n - r(i-1)} \binom{(k-1)n - r(i-1)}{n-a-r} \binom{n+a-1}{a} \tag{18}$$

is the number of labelled forests of k_1-trees with n vertices and r components such that the roots of all the trees are labelled i. If $i = 1$ in (18), we get that there are

$$r(n-1)! \sum_{a=0}^{n-r} \binom{(k-1)n}{n-a-r} \binom{n+a-1}{a} \tag{19}$$

labelled forests of k_1-trees on n vertices and r components such that the roots of all the trees are labelled 1. Further, on setting $k = 1$ in (19), we obtain the number of labelled forests of 1_1-plane trees (plane trees) on n vertices as

$$r(n-1)! \sum_{a=0}^{n-r} \binom{n+a-1}{a} = r(n-1)! \sum_{j=n-1}^{2n-r-1} \binom{j}{n-1} = r(n-1)! \binom{2n-r}{n}.$$

The last equality follows by Hockey Stick Identity.

6. Conclusion and Future work

In this paper, we have introduced and enumerated k_1-plane trees according to number of vertices, label of the root, root degree, label of the eldest child of the root, label of the youngest child of the root and number of forests of these trees. In [20, A064062], combinatorist David Callan recorded that the sequence $1, 3, 13, 67, 381, \ldots$ gives the number of Dyck paths with n unit up-steps such that each unit up-step not at ground level come in two colours. The number of these paths with n unit up-steps is given as

$$\frac{1}{n} \sum_{a=0}^{n-1} \binom{n+a-1}{a} \binom{2n}{n-a-1}.$$

This formula also counts 3_1-plane trees on n vertices with root labelled 1. It would be interesting to obtain a bijection between the sets of these structures. Further, bijections between the set of k_1-plane trees and the set of structures listed in sequences A003645 and A151374 of [20] can also be sought. The generalised forms of structures which are in bijection with non-decreasing 2-plane trees obtain by Kariuki, Okoth and Nyamwala in [8] can also be investigated if they are in bijections with k_1-plane trees. The following other variants of k-plane trees can also be considered and enumerate using various parameters.

(i) All children of internal vertices are such that the ones labelled r where $r = 1, 2, \ldots, k$, are on the left of all others. These are k_r-plane trees.

(ii) All children of internal vertices are such that the ones labelled 1 are followed by those labelled 2, then labelled 3, until the ones labelled k to be on the far right.

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