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Additive constacyclic codes and the MacWilliams identities over mixed alphabets

Research Article

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Abstract: Let $\mathbb{Z}_p$ be the ring of integers modulo a prime integer p, where $p-1$ is a quadratic residue modulo p. This paper presents the study of constacyclic codes over chain rings $\mathcal{R} = \frac{\mathbb{Z}_p[u]}{\langle u^2 \rangle}$ and $\mathcal{S} = \frac{\mathbb{Z}_p[u]}{\langle u^3 \rangle}$. We also study additive constacyclic codes over $\mathcal{RS}$ and $\mathbb{Z}_p\mathcal{RS}$ using the generator polynomials over the rings $\mathcal{R}$ and $\mathcal{S}$, respectively. Further, by defining Gray maps on $\mathcal{R}$, $\mathcal{S}$ and $\mathbb{Z}_p\mathcal{RS}$, we obtain some results on the Gray images of additive codes. Then we provide the weight enumeration and MacWilliams identities corresponding to the additive codes over $\mathbb{Z}_p\mathcal{RS}$.

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1. Introduction

Initially, in algebraic coding theory, codes were studied over finite fields. From 1970 onward, the study of codes over rings has been started [7]. However, this study over rings found momentum and created a lot of interest among the researchers after the extraordinary work by Hammons et al. [14] in 1993. Recently, noncommutative rings have been considered in many works to study and obtain better codes. Still, the study has been done mostly on commutative rings for ease of computation. In 1997,

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Rifà and Pujol [22] first encountered codes over mixed alphabets. Later, Borges et al. [9] studied $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and defined the duality of such codes. Abualrub et al. [1] studied algebraic structures of $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes and determined a set of generator polynomials of those codes. They showed that the duals of $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes are also cyclic and further obtained some optimal codes from the $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. As a natural generalization of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, Aydogdu and Siap [3] in 2013 investigated the algebraic structure of $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes and presented the standard form of the generator and parity-check matrices. Further, Aydogdu et al. determined the algebraic structure of $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes [4] and $\mathbb{Z}_2\mathbb{Z}_2[u^3]$-linear and cyclic codes [5]. Later, Islam et al. [17] presented the $\mathbb{Z}_4\mathbb{Z}_4[u]$-additive cyclic and constacyclic codes. On the other hand, Prakash et al. [21] considered the ring $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ to study additive cyclic and constacyclic codes. First, they obtained the generator polynomials along with the minimal generating set of additive cyclic codes and then extended the results to determine the structure of additive constacyclic codes. In 2022, Borges et al. thoroughly discussed the $\mathbb{Z}_2\mathbb{Z}_4$-linear codes in their book entitled $\mathbb{Z}_2\mathbb{Z}_4$**-Linear Codes** [8]. The authors investigated various properties such as dual structure, rank and kernel, and encoding and decoding of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes to show their importance.

One of the most important and frequently studied linear codes is the class of constacyclic codes. This family of linear codes has a wide range of applications in information technology. Due to its rich algebraic structure, constacyclic codes are easy to implement, and shift registers can encode them. Over the years, researchers have considered constacyclic codes to study different aspects of coding theory extensively, see [2, 10, 16, 19, 20, 26].

Weight distribution in coding theory is another important aspect. The weight enumerator of a linear code of length n indicates the number of codewords of each possible weight $0, 1, \ldots, n$. In 1963, a remarkable work of MacWilliams [18] proposed a formula that relates the weight enumerator of a code with that of its dual. Yildiz and Karadeniz [28] considered linear codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ and proved the MacWilliams identities for complete, symmetrized, and Lee weight enumerators. Aydogdu et al. [4] introduced a new class of additive codes, $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes and they proved a MacWilliams-type identity on the weight enumerator of such codes. Later, Tang et al. [25] studied the existence of the MacWilliams-type identities for the Lee and Euclidean weight enumerators and provided necessary and sufficient conditions for the existence of those identities over $\mathbb{Z}_l$. In 2021, Bhaintwal and Biswas [6] studied the algebraic structure of $\mathbb{Z}_p\mathbb{Z}_p[u]/\langle u^k\rangle$-cyclic codes and established the MacWilliams identities for complete weight enumerators of $\mathbb{Z}_p\mathbb{Z}_p[u]/\langle u^k\rangle$-linear codes. Recently, in 2024, Sagar et al. [23] presented the form of the generators of constacyclic codes over the ring $\mathbb{Z}_2[u]/\langle u^2\rangle \times \mathbb{Z}_2[u]/\langle u^3\rangle$. They also derived the MacWilliams identities corresponding to several weight enumerators.

The above research works motivate us to study $\mathbb{Z}_p\mathcal{RS}$-additive codes where $\mathcal{R} = \frac{\mathbb{Z}_p[u]}{\langle u^2\rangle}$, $\mathcal{S} = \frac{\mathbb{Z}_p[u]}{\langle u^3\rangle}$ and $\mathbb{Z}_p$ is the ring of integers modulo a prime p where $p-1$ is a quadratic residue modulo p. The ring $\mathbb{Z}_p\mathcal{RS}$ is a generalization of the rings such as $\mathbb{Z}_2\mathbb{Z}_2[u]$, $\mathbb{Z}_2\mathbb{Z}_2[u^3]$ and $\mathbb{Z}_p\mathbb{Z}_p[u]/\langle u^k\rangle$, on which additive codes have already been studied. But the study of additive constacyclic codes over the ring $\mathbb{Z}_p\mathcal{RS}$ is not yet available in the literature. Specifically, we study the $\mathbb{Z}_p\mathcal{RS}$-additive constacyclic codes of block length (q, r, s) and derive the form of generators of these codes. First, we derive the form of generators of additive constacyclic codes over $\mathcal{R}$ and $\mathcal{S}$ each. Then using these generators, we find the generator of $\mathbb{Z}_p\mathcal{RS}$-additive constacyclic codes. We define a suitable inner product on both $\mathcal{R}^r\mathcal{S}^s$ and $\mathbb{Z}_p^q\mathcal{R}^r\mathcal{S}^s$, and use them to find a relation between an additive constacyclic code and its dual. Next, we define Gray maps on $\mathcal{R}^r$, $\mathcal{S}^s$ and $\mathbb{Z}_p^q\mathcal{R}^r\mathcal{S}^s$, respectively, and study the Gray images of additive cyclic codes and additive constacyclic codes over $\mathcal{R}$, $\mathcal{S}$ and $\mathbb{Z}_p\mathcal{RS}$. Then we obtain the MacWilliams identities of $\mathbb{Z}_p\mathcal{RS}$-additive codes corresponding to the complete, Hamming, Symmetrized, and Lee weight enumerators.

This paper is arranged as follows: In Section 2, we recall some basic definitions and results, which will be required later. Section 3 deals mainly with the generators of additive constacyclic codes over $\mathcal{R}$, $\mathcal{S}$ and $\mathbb{Z}_p\mathcal{RS}$ each. Section 4 revolves around the Gray maps defined over $\mathcal{R}^r$, $\mathcal{S}^s$ and $\mathbb{Z}_p^q\mathcal{R}^r\mathcal{S}^s$, and also their images. In Section 5, we obtain the MacWilliams identities of $\mathbb{Z}_p\mathcal{RS}$-additive codes over several weight enumerators. Section 6 concludes our work.

2. Preliminaries

In this section, we state some basic definitions and results. Let $\mathcal{R} = \frac{\mathbb{Z}_p[u]}{\langle u^2 \rangle}$ and $\mathcal{S} = \frac{\mathbb{Z}_p[u]}{\langle u^3 \rangle}$, where $\mathbb{Z}_p$ is a ring of integers modulo a prime p which is chosen in such a way that $p-1$ becomes a quadratic residue in $\mathbb{Z}_p$. Note that $\mathcal{R}$ and $\mathcal{S}$ are chain rings of order p^2 and p^3, respectively.

For any two positive integers r and s, $\mathcal{R}^r \times \mathcal{S}^s$ is an additive group and $\mathcal{R}^r \times \mathcal{S}^s$ forms an $\mathcal{S}$-module with the scalar multiplication

$$\begin{aligned} d \cdot (x, y) &= d \cdot (x_0, x_1, \ldots, x_{r-1}; y_0, y_1, \ldots, y_{s-1}) \\ &= (d''x_0, d''x_1, \ldots, d''x_{r-1}; dy_0, dy_1, \ldots, dy_{s-1}), \end{aligned}$$

where $d \in \mathcal{S},\ x \in \mathcal{R}^r,\ y \in \mathcal{S}^s,\ d'' = d \pmod{u^2}$.

Similarly, for any three positive integers q, r and s, $\mathbb{Z}_p^q \times \mathcal{R}^r \times \mathcal{S}^s$ also forms a module over $\mathcal{S}$ with the scalar multiplication

$$\begin{aligned} d \cdot (w, x, y) &= d \cdot (w_0, w_1, \ldots, w_{q-1}; x_0, x_1, \ldots, x_{r-1}; y_0, y_1, \ldots, y_{s-1}) \\ &= (d'w_0, d'w_1, \ldots, d'w_{q-1}; d''x_0, d''x_1, \ldots, d''x_{r-1}; dy_0, dy_1, \ldots, dy_{s-1}), \end{aligned}$$

where $d \in \mathcal{S},\ w \in \mathbb{Z}_p^q,\ x \in \mathcal{R}^r,\ y \in \mathcal{S}^s,\ d' = d \pmod{u}$ and $d'' = d \pmod{u^2}$.

Throughout the paper, we denote the direct product of $\mathcal{R}$ and $\mathcal{S}$ by $\mathcal{RS}$ and the direct product of $\mathbb{Z}_p$, $\mathcal{R}$ and $\mathcal{S}$ by $\mathbb{Z}_p\mathcal{RS}$. Now, we define an $\mathcal{RS}$-additive code and a $\mathbb{Z}_p\mathcal{RS}$-additive code.

Definition 2.1. *An $\mathcal{RS}$-additive code of block length (r, s) is an $\mathcal{S}$-submodule of $\mathcal{R}^r\mathcal{S}^s$ and similarly, a $\mathbb{Z}_p\mathcal{RS}$-additive code of block length (q, r, s) is an $\mathcal{S}$-submodule of $\mathbb{Z}_p^q\mathcal{R}^r\mathcal{S}^s$.*

Example 2.2. *Consider the code $\mathfrak{C}_1 = \langle\{(1, 0; 0, u; 1 + u^2, 0), (0, 1; 1 + u, 0; 0, 1 + u)\}\rangle$ of block length $(2, 2, 2)$ over $\mathbb{Z}_2\mathcal{RS}$. Then*

$$\begin{aligned} \mathfrak{C}_1 = \langle\{&(1, 0; 0, u; 1 + u^2, 0), (0, 0; 0, 0; u, 0), (0, 0; 0, 0; u^2, 0), (0, 1; 1 + u, 0; 0, 1 + u), \\ &(0, 0; u, 0; 0, u + u^2), (0, 0; 0, 0; 0, u^2)\}\rangle \end{aligned}$$

over $\mathbb{Z}_2$, i.e., $\mathfrak{C}_1$ is a vector space over $\mathbb{Z}_2$ of dimension 6.

Next, we recall the definition of a Frobenius ring.

Definition 2.3. *[13] Let R be a ring with unity. Then R is called a Frobenius ring if R is Artinian and $R/Rad(R) \cong Soc(R)$ (both left as well as right R-modules). $Rad(R)$ denotes the Jacobson radical of R and $Soc(R)$ denotes the socle of R as an R-module.*

Lemma 2.4. *The rings $\mathcal{R}$, $\mathcal{S}$ and $\mathbb{Z}_p\mathcal{RS}$ are Frobenius.*

Definition 2.5. *[11] A character χ of a ring R is a group homomorphism from R to $\mathbb{C}^*$ where $\mathbb{C}^*$ is the group of all non-zero complex numbers.*

Definition 2.6. *[27] Let R be a finite ring and let $\widehat{R}$ be the set of all characters of R. If there exists an R-module isomorphism $f : R \to \widehat{R}$, then $\chi = f(1)$ is said to be a generating character of R.*

Lemma 2.7. *[11] Let χ be a character of a finite ring R. Then χ is a generating character if and only if $ker(\chi)$ contains no non-zero ideals of R.*

Now, let us recall the definitions of a few special classes of codes.

Definition 2.8. *Let R be a ring, n be a positive integer, and λ be a unit in R. We denote the λ-constacyclic shift operator by σ_λ and it is defined on R^n by*

$$\sigma_\lambda(x_0, x_1, \ldots, x_{n-1}) = (\lambda x_{n-1}, x_0, x_1, \ldots, x_{n-2}),$$

where $x_i \in R$ for $i = 0, 1, \ldots, n-1$. An R-additive code $\mathcal{C}$ of length n is said to be a λ-constacyclic code if $\mathcal{C}$ is invariant under the map σ_λ. In particular, when $\lambda = 1$, we denote the operator σ_1 simply as σ, known as the cyclic shift operator. An R-additive code invariant under σ is called a cyclic code.

Definition 2.9. *[12] Let R be a ring and $n = lm$ where l, m are positive integers. We denote the l-quasi-cyclic shift operator by θ_l and it is defined on R^n by*

$$\theta_l(x_0|x_1|\dots|x_{l-1}) = (\sigma(x_0)|\sigma(x_1)|\dots|\sigma(x_{l-1})),$$

where $x_i \in R^m$ for $i = 0, 1, \dots, l-1$ and σ is the cyclic shift operator on R^m. An R-additive code $\mathcal{C}$ of length n is said to be an l-quasi-cyclic code if $\mathcal{C}$ is invariant under the map θ_l.

Definition 2.10. *[15] Let R be a ring and $n = lm$, where l, m are positive integers. We denote the (λ, l)-quasi-twisted shift operator by $\theta_{\lambda,l}$ and it is defined on R^n by*

$$\theta_{\lambda,l}(x_0|x_1|\dots|x_{l-1}) = (\sigma_\lambda(x_0)|\sigma_\lambda(x_1)|\dots|\sigma_\lambda(x_{l-1})),$$

where $\lambda \in R$ is a unit, $x_i \in R^m$ for $i = 0, 1, \dots, l-1$ and σ_λ is the λ-constacyclic shift operator on R^m. An R-additive code $\mathcal{C}$ of length n is said to be a (λ, l)-quasi-twisted code if $\mathcal{C}$ is invariant under the map $\theta_{\lambda,l}$.

Definition 2.11. *Let R be a ring and $R_i = \frac{R[x]}{\langle x^{m_i} - \lambda_i \rangle}$, $i = 1, 2, \dots, l$, where $m_1, m_2, \dots, m_l$ are positive integers and $\lambda_1, \lambda_2, \dots, \lambda_l$ are units in R. Then any $R[x]$-submodule of $R_1 \times R_2 \times \cdots \times R_l$ is called a generalized $(\lambda_1, \lambda_2, \dots, \lambda_l)$-quasi-twisted code of block length $(m_1, m_2, \dots, m_l)$.*

We can observe that a generalized $(\lambda_1, \lambda_2, \dots, \lambda_l)$-quasi-twisted code of block length $(m_1, m_2, \dots, m_l)$, where $\lambda_1 = \lambda_2 = \cdots = \lambda_l = \lambda$, $m_1 = m_2 = \cdots = m_l = m$, is a (λ, l)-quasi-twisted code of length lm.

Now, by using the standard inner product (Euclidean inner product), we define the dual of a linear code, self-orthogonal code, self-dual code, and dual-containing code, respectively.

Definition 2.12. *Let R be a ring and $\mathcal{C}$ be a linear code of length n over R. Then the dual code $\mathcal{C}^\perp$ of code $\mathcal{C}$ is defined as*

$$\mathcal{C}^\perp = \{v \in R^n \mid v \cdot c = 0 \text{ for all } c \in \mathcal{C}\}.$$

A code $\mathcal{C}$ is called self-orthogonal if $\mathcal{C} \subseteq \mathcal{C}^\perp$, dual-containing if $\mathcal{C}^\perp \subseteq \mathcal{C}$, and self-dual if $\mathcal{C} = \mathcal{C}^\perp$.

3. Constacyclic codes over $\mathcal{R}$, $\mathcal{S}$, $\mathcal{RS}$ and $\mathbb{Z}_p\mathcal{RS}$

In this section, first, we study additive constacyclic codes over $\mathcal{R}$ and $\mathcal{S}$ and then generalize the results over $\mathcal{RS}$ and $\mathbb{Z}_p\mathcal{RS}$. Here, our main objective is to find the generators of additive constacyclic codes. We define suitable inner products, and under those inner products, we also find the generators of the dual of additive constacyclic codes.

3.1. Constacyclic codes over $\mathcal{R}$ and $\mathcal{S}$

Here, we first study all the units of $\mathcal{R}$ and $\mathcal{S}$, and then we find the generators of constacyclic codes over $\mathcal{R}$ and $\mathcal{S}$ respectively.

Let us denote the group of units of $\mathcal{R}$ and $\mathcal{S}$ by $U(\mathcal{R})$ and $U(\mathcal{S})$, respectively. Then $U(\mathcal{R}) = \{a+ub \mid a, b \in \mathbb{Z}_p \text{ and } a \neq 0\}$ and $U(\mathcal{S}) = \{a + ub + u^2d \mid a, b, d \in \mathbb{Z}_p \text{ and } a \neq 0\}$. We denote the set of all non-zero elements of $\mathbb{Z}_p$ by $\mathbb{Z}_p^*$.

Now, we define a few maps.

(a) Define $\eta_0 : \mathcal{R} \to \mathbb{Z}_p$ by $\eta_0(a + bu) = a$.

(b) Define $\eta_1 : \mathcal{S} \to \mathbb{Z}_p$ by $\eta_1(a + bu + du^2) = a$.

(c) Define $\eta_2 : \mathcal{S} \to \mathcal{R}$ by $\eta_2(a + bu + du^2) = a + bu$.

The following lemma shows the interrelation between the units of $\mathbb{Z}_p, \mathcal{R}$ and $\mathcal{S}$.

Lemma 3.1. *Let $\mu_1 \in \mathcal{R}$ and $\mu_2 \in \mathcal{S}$. Then, we have the following.*

1. *$\mu_1 \in U(\mathcal{R})$ if and only if $\eta_0(\mu_1) \in \mathbb{Z}_p^*$;*
2. *$\mu_2 \in U(\mathcal{S})$ if and only if $\eta_1(\mu_2) \in \mathbb{Z}_p^*$;*
3. *$\mu_2 \in U(\mathcal{S})$ if and only if $\eta_2(\mu_2) \in U(\mathcal{R})$.*

Let $\mu_0 \in \mathbb{Z}_p^*$, $\mu_1 \in U(\mathcal{R})$, $\mu_2 \in U(\mathcal{S})$, and q, r, s are three positive integers. Then there is a one-to-one correspondence between $\mathbb{Z}_p^q \times \mathcal{R}^r \times \mathcal{S}^s$ and $\frac{\mathbb{Z}_p[x]}{\langle x^q - \mu_0 \rangle} \times \frac{\mathcal{R}[x]}{\langle x^r - \mu_1 \rangle} \times \frac{\mathcal{S}[x]}{\langle x^s - \mu_2 \rangle}$ with respect to the identification

$$(a_0, a_1, \ldots, a_{q-1}; b_0, b_1, \ldots, b_{r-1}; c_0, c_1, \ldots, c_{s-1}) \to (a(x), b(x), c(x)),$$

where $a(x) = a_0 + a_1 x + \cdots + a_{q-1}x^{q-1}$, $b(x) = b_0 + b_1 x + \cdots + b_{r-1}x^{r-1}$, and $c(x) = c_0 + c_1 x + \cdots + c_{s-1}x^{s-1}$. Now, we recall one result from [24].

Theorem 3.2. *Let $\mathfrak{C}$ be a cyclic code of length n over $R = \frac{\mathbb{Z}_p[u]}{\langle u^k \rangle}$. If n is relatively prime to p then $\mathfrak{C} = \langle f_0(x) + uf_1(x) + \cdots + u^{k-1}f_{k-1}(x) \rangle$, where $f_0(x), f_1(x), \ldots, f_{k-1}(x) \in \mathbb{Z}_p[x]$ and $f_{k-1}(x) \mid f_{k-2}(x) \mid \cdots \mid f_0(x) \mid (x^n - 1)$ mod p.*

Let R be a ring. We denote the multiplicative order of an element $\alpha \in R$ by $\mathrm{ord}(\alpha)$. The following theorem gives the form of the generator of a constacyclic code over $\mathcal{R}$.

Theorem 3.3. *Let $\mathcal{R} = \frac{\mathbb{Z}_p[u]}{\langle u^2 \rangle}$ and $\mu_1 \in U(\mathcal{R})$. Then for any positive integer r satisfying $\gcd(p, r) = 1$ and $r \equiv 1 \pmod{\mathrm{ord}(\mu_1)}$, every μ_1-constacyclic code $\mathfrak{C}$ of length r over $\mathcal{R}$ is given by*

$$\mathfrak{C} = \langle f_0(x) + uf_1(x) \rangle,$$

where $f_1(x) \mid f_0(x) \mid (x^r - \mu_1)$ mod p.

Proof. We define a map $\rho_{\mathcal{R}} : \frac{\mathcal{R}[x]}{\langle x^r - 1 \rangle} \to \frac{\mathcal{R}[x]}{\langle x^r - \mu_1 \rangle}$ by $\rho_{\mathcal{R}}(f(x)) = f(\mu_1^{-1}x)$. Let $a(x), b(x)$ and $h(x)$ be three polynomials in $\mathcal{R}[x]$ such that $a(x) - b(x) = (x^r - 1)h(x)$. Now, $a(x) - b(x) = (x^r - 1)h(x) \Leftrightarrow a(\mu_1^{-1}x) - b(\mu_1^{-1}x) = ((\mu_1^{-1}x)^r - 1)h(\mu_1^{-1}x) \Leftrightarrow a(\mu_1^{-1}x) - b(\mu_1^{-1}x) = \mu_1^{-1}(x^r - \mu_1)h(\mu_1^{-1}x)$. This shows that $\rho_{\mathcal{R}}$ is an isomorphism of rings and if I is an ideal of $\frac{\mathcal{R}[x]}{\langle x^r - 1 \rangle}$ then $\rho_{\mathcal{R}}(I)$ is an ideal of $\frac{\mathcal{R}[x]}{\langle x^r - \mu_1 \rangle}$. Thus if $\mathfrak{C}$ is a μ_1-constacyclic code over $\mathcal{R}$, then $\rho_{\mathcal{R}}^{-1}(\mathfrak{C})$ is a cyclic code over $\mathcal{R}$ and by Theorem 3.2, $\rho_{\mathcal{R}}^{-1}(\mathfrak{C}) = \langle a_0(x) + ua_1(x) \rangle$, where $a_1(x) \mid a_0(x)$ and $a_0(x) \mid (x^r - 1)$ mod p. Hence, $\mathfrak{C} = \langle a_0(\mu_1^{-1}x) + ua_1(\mu_1^{-1}x) \rangle$. Consider $f_0(x) = a_0(\mu_1^{-1}x)$ and $f_1(x) = a_1(\mu_1^{-1}x)$. Then one can easily verify that $f_1(x) \mid f_0(x) \mid (x^r - \mu_1)$ mod p. □

Next, we find the generator of a constacyclic code over $\mathcal{S}$.

Theorem 3.4. *Let $\mathcal{S} = \frac{\mathbb{Z}_p[u]}{\langle u^3 \rangle}$ and $\mu_2 \in U(\mathcal{S})$. Then for any positive integer s satisfying $\gcd(p, s) = 1$ and $s \equiv 1 \pmod{\mathrm{ord}(\mu_2)}$, every μ_2-constacyclic code $\mathfrak{C}$ of length s over $\mathcal{S}$ is given by*

$$\mathfrak{C} = \langle f_0(x) + uf_1(x) + u^2 f_2(x) \rangle,$$

where $f_2(x) \mid f_1(x) \mid f_0(x) \mid (x^s - \mu_2)$ mod p.

Proof. We define a map $\rho_{\mathcal{S}} : \frac{\mathcal{S}[x]}{\langle x^s-1\rangle} \to \frac{\mathcal{S}[x]}{\langle x^s-\mu_2\rangle}$ by $\rho_{\mathcal{S}}(f(x)) = f(\mu_2^{-1}x)$. Let $a(x), b(x)$ and $h(x)$ be three polynomials in $\mathcal{S}[x]$ such that $a(x) - b(x) = (x^s - 1)h(x)$. Now, $a(x) - b(x) = (x^s - 1)h(x) \Leftrightarrow a(\mu_2^{-1}x) - b(\mu_2^{-1}x) = ((\mu_2^{-1}x)^s - 1)h(\mu_2^{-1}x) \Leftrightarrow a(\mu_2^{-1}x) - b(\mu_2^{-1}x) = \mu_2^{-1}(x^s - \mu_2)h(\mu_2^{-1}x)$. This shows that $\rho_{\mathcal{S}}$ is an isomorphism of rings and if I is an ideal of $\frac{\mathcal{S}[x]}{\langle x^s-1\rangle}$ then $\rho_{\mathcal{S}}(I)$ is an ideal of $\frac{\mathcal{S}[x]}{\langle x^s-\mu_2\rangle}$. Thus if $\mathfrak{C}$ is a μ_2-constacyclic code over $\mathcal{S}$ then $\rho_{\mathcal{S}}^{-1}(\mathfrak{C})$ is a cyclic code over $\mathcal{S}$ and by Theorem 3.2, $\rho_{\mathcal{S}}^{-1}(\mathfrak{C}) = \langle a_0(x) + ua_1(x) + u^2a_2(x)\rangle$, where $a_2(x) \mid a_1(x) \mid a_0(x)$ and $a_0(x) \mid (x^s - 1) \bmod p$. Hence $\mathfrak{C} = \langle a_0(\mu_2^{-1}x) + ua_1(\mu_2^{-1}x) + u^2a_2(\mu_2^{-1}x)\rangle$. Consider $f_0(x) = a_0(\mu_2^{-1}x), f_1(x) = a_1(\mu_2^{-1}x)$ and $f_2(x) = a_2(\mu_2^{-1}x)$. Then one can easily verify that $f_2(x) \mid f_1(x) \mid f_0(x) \mid (x^s - \mu_2) \bmod p$. □

3.2. Constacyclic codes over $\mathbb{Z}_p\mathcal{RS}$

Let $\mu_0 \in \mathbb{Z}_p^*$, $\mu_1 \in U(\mathcal{R})$, $\mu_2 \in U(\mathcal{S})$ and $q \equiv 1 \pmod{\text{ord}(\mu_0)}$, $r \equiv 1 \pmod{\text{ord}(\mu_1)}$, $s \equiv 1 \pmod{\text{ord}(\mu_2)}$, $\gcd(p, q) = 1$, $\gcd(p, r) = 1$ and $\gcd(p, s) = 1$. From here onwards, we will continue with these conditions. One can check that the three maps η_0, η_1 and η_2, defined in Section 3.1, are ring epimorphisms and using these maps, we can define the module structures of $\frac{\mathbb{Z}_p[x]}{\langle x^q-\mu_0\rangle} \times \frac{\mathcal{R}[x]}{\langle x^r-\mu_1\rangle}$, $\frac{\mathcal{R}[x]}{\langle x^r-\mu_1\rangle} \times \frac{\mathcal{S}[x]}{\langle x^s-\mu_2\rangle}$ and $\frac{\mathbb{Z}_p[x]}{\langle x^q-\mu_0\rangle} \times \frac{\mathcal{R}[x]}{\langle x^r-\mu_1\rangle} \times \frac{\mathcal{S}[x]}{\langle x^s-\mu_2\rangle}$. Let us denote $\mathcal{M}_{0,1}^{q,r} = \frac{\mathbb{Z}_p[x]}{\langle x^q-\mu_0\rangle} \times \frac{\mathcal{R}[x]}{\langle x^r-\mu_1\rangle}$, $\mathcal{M}_{1,2}^{r,s} = \frac{\mathcal{R}[x]}{\langle x^r-\mu_1\rangle} \times \frac{\mathcal{S}[x]}{\langle x^s-\mu_2\rangle}$ and $\mathcal{M}_{0,1,2}^{q,r,s} = \frac{\mathbb{Z}_p[x]}{\langle x^q-\mu_0\rangle} \times \frac{\mathcal{R}[x]}{\langle x^r-\mu_1\rangle} \times \frac{\mathcal{S}[x]}{\langle x^s-\mu_2\rangle}$. Then $\mathcal{M}_{0,1}^{q,r}$ has an $\mathcal{R}[x]$-module structure with the scalar multiplication defined as

$$g(x) \cdot (a(x), b(x)) = (\eta_0(g(x))a(x), g(x)b(x)).$$

Similarly, $\mathcal{M}_{1,2}^{r,s}$ has an $\mathcal{S}[x]$-module structure with the scalar multiplication defined as

$$h(x) \cdot (b(x), d(x)) = (\eta_2(h(x))b(x), h(x)d(x))$$

and $\mathcal{M}_{0,1,2}^{q,r,s}$ has an $\mathcal{S}[x]$-module structure with the scalar multiplication defined as

$$h(x) \cdot (a(x), b(x), d(x)) = (\eta_1(h(x))a(x), \eta_2(h(x))b(x), h(x)d(x)),$$

where $g(x) \in \mathcal{R}[x]$, $h(x) \in \mathcal{S}[x]$, $a(x) \in \frac{\mathbb{Z}_p[x]}{\langle x^q-\mu_0\rangle}$, $b(x) \in \frac{\mathcal{R}[x]}{\langle x^r-\mu_1\rangle}$, $d(x) \in \frac{\mathcal{S}[x]}{\langle x^s-\mu_2\rangle}$. Note that if $g(x) = \sum_{j=0}^{l} g_j x^j$ then $\eta_0(g(x)) = \sum_{j=0}^{l} \eta_0(g_j)x^j$ and if $h(x) = \sum_{j=0}^{l} h_j x^j$, then $\eta_i(h(x)) = \sum_{j=0}^{l} \eta_i(h_j)x^j$ for $i = 1, 2$.

Definition 3.5. *The operator $T_{\mu_0,\mu_1} : \mathbb{Z}_p^q\mathcal{R}^r \to \mathbb{Z}_p^q\mathcal{R}^r$ defined by $T_{\mu_0,\mu_1}(a_0, a_1, \ldots, a_{q-1}|b_0, b_1, \ldots, b_{r-1}) = (\mu_0 a_{q-1}, a_0, a_1, \ldots, a_{q-2}|\mu_1 b_{r-1}, b_0, b_1, \ldots, b_{r-2})$ is called the (μ_0, μ_1)-constacyclic shift operator. A $\mathbb{Z}_p\mathcal{R}$-additive code $\mathfrak{C}$ having block length (q, r) is said to be a (μ_0, μ_1)-constacyclic code if $T_{\mu_0,\mu_1}(\mathfrak{C}) \subset \mathfrak{C}$. Similarly, one can define a (μ_1, μ_2)-constacyclic code of block length (r, s) over $\mathcal{RS}$ with the (μ_1, μ_2)-constacyclic shift operator T_{μ_1,μ_2} defined over $\mathcal{R}^r\mathcal{S}^s$ and a (μ_0, μ_1, μ_2)-constacyclic code of block length (q, r, s) over $\mathbb{Z}_p\mathcal{RS}$ with the (μ_0, μ_1, μ_2)-constacyclic shift operator T_{μ_0,μ_1,μ_2} defined over $\mathbb{Z}_p^q\mathcal{R}^r\mathcal{S}^s$, respectively. For $q = 0$, it is said to be a (μ_1, μ_2)-constacyclic code over $\mathcal{RS}$ and for $q = 0, r = 0$, it is said to be a μ_2-constacyclic code over $\mathcal{S}$, respectively.*

Let $\zeta_0 : \mathbb{Z}_p^q\mathcal{R}^r \to \mathcal{M}_{0,1}^{q,r}$ be defined by $\zeta_0(c) = c(x)$, where $c = (a_0, a_1, \ldots, a_{q-1}|b_0, b_1, \ldots, b_{r-1})$ and $c(x) = (a(x), b(x))$ with $a(x) = \sum_{i=0}^{q-1} a_i x^i$, $b(x) = \sum_{i=0}^{r-1} b_i x^i$. It is easy to observe that ζ_0 is an isomorphism. Now, we have the following lemma.

Lemma 3.6. *The isomorphism ζ_0 maps the constacyclic shift T_{μ_0,μ_1} of an element c of $\mathbb{Z}_p\mathcal{R}$ to the multiplication of its image by x, i.e.,*

$$\zeta_0(T_{\mu_0,\mu_1}(c)) = x \cdot \zeta_0(c).$$

Using the above lemma, we have two equivalent statements.

Proposition 3.7. *The following two statements are equivalent:*

1. *The $\mathbb{Z}_p\mathcal{R}$-additive code $\mathfrak{C}$ having block length (q, r) is a (μ_0, μ_1)-constacyclic code.*
2. *$\zeta_0(\mathfrak{C})$ is a submodule of $\mathcal{M}_{0,1}^{q,r}$ over $\mathcal{R}[x]$.*

Proof. Let $\mathfrak{C}$ be an $\mathcal{R}$-submodule of $\mathbb{Z}_p^q\mathcal{R}^r$. Then by Lemma 3.6, $T_{\mu_0,\mu_1}(\mathfrak{C}) \subset \mathfrak{C} \Leftrightarrow x \cdot \zeta_0(\mathfrak{C}) \subset \zeta_0(\mathfrak{C}) \Leftrightarrow f(x) \cdot \zeta_0(\mathfrak{C}) \subset \zeta_0(\mathfrak{C})$ for all $f(x) \in \mathcal{R}[x]$. □

Note that in a similar way as above, a (μ_1, μ_2)-constacyclic code over $\mathcal{RS}$ and a (μ_0, μ_1, μ_2)-constacyclic code over $\mathbb{Z}_p\mathcal{RS}$ can be considered as an $\mathcal{S}[x]$-submodule of $\mathcal{M}_{1,2}^{r,s}$ and an $\mathcal{S}[x]$-submodule of $\mathcal{M}_{0,1,2}^{q,r,s}$, respectively.

In the following theorem, we find the generator of a (μ_1, μ_2)-constacyclic code over $\mathcal{RS}$.

Theorem 3.8. *Any (μ_1, μ_2)-constacyclic code $\mathfrak{C}$ of block length (r, s) over $\mathcal{RS}$ can be given as*

$$\mathfrak{C} = \langle (g_0(x) + ug_1(x), 0), (l(x), h_0(x) + uh_1(x) + u^2h_2(x)) \rangle,$$

where $g_0(x), g_1(x) \in \mathcal{R}[x]$, $h_0(x), h_1(x), h_2(x) \in \mathcal{S}[x]$ are polynomials satisfying $g_1(x) \mid g_0(x) \mid (x^r - \mu_1)$ mod p, $h_2(x) \mid h_1(x) \mid h_0(x) \mid (x^s - \mu_2)$ mod p, and $l(x) \in \mathcal{R}[x]$ is such that $(l(x), h_0(x) + uh_1(x) + u^2h_2(x)) \in \mathfrak{C}$.

Proof. Consider the projection map $p_2 : \mathcal{M}_{1,2}^{r,s} \to \frac{\mathcal{S}[x]}{\langle x^s - \mu_2 \rangle}$ defined by $p_2(a(x), b(x)) = b(x)$. It is verified that p_2 is an $\mathcal{S}[x]$-linear map. Denote $\hat{p}_2 = p_2|_{\mathfrak{C}}$, the restriction map of p_2 on $\mathfrak{C}$. Since $\mathfrak{C}$ is an $\mathcal{S}[x]$-submodule of $\mathcal{M}_{1,2}^{r,s}$, $\hat{p}_2(\mathfrak{C})$ is also an $\mathcal{S}[x]$-submodule of $\frac{\mathcal{S}[x]}{\langle x^s - \mu_2 \rangle}$. Hence, $\hat{p}_2(\mathfrak{C})$ is a μ_2-constacyclic code of length s over $\mathcal{S}$ and hence

$$\hat{p}_2(\mathfrak{C}) = \langle h_0(x) + uh_1(x) + u^2h_2(x) \rangle,$$

where $h_2(x) \mid h_1(x) \mid h_0(x) \mid (x^s - \mu_2) \bmod p$. Also, $\ker(\hat{p}_2) = \{(a(x), 0) \in \mathfrak{C} \mid a(x) \in \frac{\mathcal{R}[x]}{\langle x^r - \mu_1 \rangle}\}$ is an $\mathcal{S}[x]$-submodule of $\mathfrak{C}$. Let $I = \{a(x) \in \frac{\mathcal{R}[x]}{\langle x^r - \mu_1 \rangle} \mid (a(x), 0) \in \ker(\hat{p}_2)\}$. Then I is an $\mathcal{R}[x]$-submodule of $\frac{\mathcal{R}[x]}{\langle x^r - \mu_1 \rangle}$ and so I is a μ_1-constacyclic code of length r over $\mathcal{R}$. From Theorem 3.3, we have $I = \langle g_0(x) + ug_1(x) \rangle$ where $g_1(x) \mid g_0(x) \mid (x^r - \mu_1) \bmod p$. Therefore,

$$\ker(\hat{p}_2) = \langle (g_0(x) + ug_1(x), 0) \rangle.$$

Let $l(x) \in \mathcal{R}[x]$ be such that $(l(x), h_0(x) + uh_1(x) + u^2h_2(x)) \in \mathfrak{C}$. Denote $c_1(x) = (l(x), h_0(x) + uh_1(x) + u^2h_2(x))$ and $c_2(x) = (g_0(x) + ug_1(x), 0)$. Take $c(x) = (a_1(x), a_2(x)) \in \mathfrak{C}$. Then $a_2(x) \in \hat{p}_2(\mathfrak{C})$ and thus there exists a polynomial $q_1(x) \in \mathcal{S}[x]$ such that $a_2(x) = q_1(x)(h_0(x) + uh_1(x) + u^2h_2(x))$. Now, $c(x) - q_1(x)c_1(x) = (a_1(x) - \eta_2(q_1(x))l(x), 0) \in \ker(\hat{p}_2)$ and hence $a_1(x) - \eta_2(q_1(x))l(x) = q_2(x)(g_0(x) + ug_1(x))$ for some $q_2(x) \in \mathcal{R}[x]$. Thus, $c(x) = q_1(x)c_1(x) + q_2(x)c_2(x)$. □

In the following example, we find a (μ_0, μ_1, μ_2)-constacyclic code over $\mathcal{RS}$ using Theorem 3.8.

Example 3.9. *Consider $p = 5$, $\mu_1 = 1$, $\mu_2 = 4$, $r = 2 = s$. Over $\mathcal{R}$, we have $x^2 - \mu_1 = (x + 1)(x + 4)$. Over $\mathcal{S}$, the polynomial $x^2 - \mu_2$ has can be factorized as $x^2 - \mu_2 = (x + 2)(x + 3)$. Also, consider $g_0(x) = 4 + x^2$, $g_1(x) = 4 + x$, $h_0(x) = 1 + x^2$, $h_1(x) = 3 + x$, $h_2(x) = 3 + x$, $l(x) = 0$. Then from Theorem 3.8, we have*

$$\begin{aligned}\mathfrak{C} &= \langle (4 + x^2 + u(4 + x), 0),\ (0, 1 + x^2 + u(3 + x) + u^2(3 + x)) \rangle \\ &= \langle 4u, u; 0, 0),\ (0, 0; 3u + 3u^2, u + u^2) \rangle.\end{aligned}$$

Thus, $\mathfrak{C}$ is a $(1, 4)$-constacyclic code of block length $(2, 2)$ over $\mathbb{Z}_5\mathcal{RS}$. Moreover, $\mathfrak{C}$ is a vector space over $\mathbb{Z}_5$ of dimension 3.

Now, we define an inner product on $\mathcal{R}^r\mathcal{S}^s$.

Definition 3.10. *Let* $v = (y_0, y_1, \ldots, y_{r-1}|z_0, z_1, \ldots, z_{s-1}), w = (y'_0, y'_1, \ldots, y'_{r-1}|z'_0, z'_1, \ldots, z'_{s-1}) \in \mathcal{R}^r\mathcal{S}^s$. *Define the inner product of* v *and* w *by*

$$< v, w > \; = u\sum_{i=0}^{r-1} y_i y'_i + \sum_{i=0}^{s-1} z_i z'_i.$$

From hereon, for an $\mathcal{RS}$*-additive code* $\mathfrak{C}$*, its dual will be defined with respect to this inner product, and the dual will be denoted by* $\mathfrak{C}^\perp$*. Note that the block length of* $\mathfrak{C}^\perp$ *is the same as that of* $\mathfrak{C}$*.*

Remark 3.11. *In Definition 3.10, we have defined the inner product on* $\mathcal{R}^r\mathcal{S}^s$ *in a similar way as defined by Aydogdu et al. in [5, Section 2.2] on* $\mathbb{Z}_2^r \times \mathcal{R}_3^s$*, where* $\mathcal{R}_3 = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$ *with* $u^3 = 0$*. In the proof of Theorem 2.5 in [5], the authors have shown that if* $\mathcal{C} \subseteq \mathbb{Z}_2^r \times \mathcal{R}_3^s$ *is a linear code then* $|\mathcal{C}||\mathcal{C}^\perp| = 2^{r+3s}$*. Thus, in our case, for an* $\mathcal{RS}$*-additive code* $\mathfrak{C} \subseteq \mathcal{R}^r\mathcal{S}^s$*, we have* $|\mathfrak{C}||\mathfrak{C}^\perp| = p^{2r+3s}$*. Replacing* $\mathfrak{C}$ *by* $\mathfrak{C}^\perp$ *in* $|\mathfrak{C}||\mathfrak{C}^\perp| = p^{2r+3s}$*, we obtain* $|\mathfrak{C}^\perp||(\mathfrak{C}^\perp)^\perp| = p^{2r+3s}$*. Thus,* $|\mathfrak{C}| = |(\mathfrak{C}^\perp)^\perp|$*.*

Theorem 3.12. *Let* $\mu_1 \in U(\mathcal{R})$*,* $\mu_2 \in U(\mathcal{S})$*. Then the code* $\mathfrak{C}$ *is an* $\mathcal{RS}$*-additive* (μ_1, μ_2)*-constacyclic code if and only if* $\mathfrak{C}^\perp$ *is an* $\mathcal{RS}$*-additive* (μ_1^{-1}, μ_2^{-1})*-constacyclic code.*

Proof. Let $\mathfrak{C}$ be a (μ_1, μ_2)-constacyclic code of block length (r, s). Let $c = (y_0, y_1, \ldots, y_{r-1}|z_0, z_1, \ldots, z_{s-1})$, $c' = (y'_0, y'_1, \ldots, y'_{r-1}|z'_0, z'_1, \ldots, z'_{s-1}) \in \mathfrak{C}^\perp$. Let $\mathrm{ord}(\mu_i) = l_i$ for $i = 1, 2$, and consider the integer $m = l_1 l_2 rs$. Then $T^m_{\mu_1,\mu_2}(c) = c$ and hence

$$T^{m-1}_{\mu_1,\mu_2}(c) = (y_1, \ldots, y_{r-2}, y_{r-1}, \mu_1^{-1}y_0|z_1, \ldots, z_{s-2}, z_{s-1}, \mu_2^{-1}z_0).$$

Since $c \in \mathfrak{C}$, then $\mathfrak{C}$ being a (μ_1, μ_2)-constacyclic code we get $T^{m-1}_{\mu_1,\mu_2}(c) \in \mathfrak{C}$.
Now, we have

$$\begin{aligned}
& < T_{\mu_1^{-1},\mu_2^{-1}}(c'), c > \\
= \; & < (\mu_1^{-1}y'_{r-1}, y'_0, \ldots, y'_{r-2}|\mu_2^{-1}z'_{s-1}, z'_0, \ldots, z'_{s-2}), (y_0, y_1, \ldots, y_{r-1}|z_0, z_1, \ldots, z_{s-1}) > \\
= \; & u\{\mu_1^{-1}y'_{r-1}y_0 + y'_0y_1 + \cdots + y'_{r-2}y_{r-1}\} + \{\mu_2^{-1}z'_{s-1}z_0 + z'_0z_1 + \cdots + z'_{s-2}z_{s-1}\} \\
= \; & u\{y'_0y_1 + y'_1y_2 + \cdots + \mu_1^{-1}y'_{r-1}y_0\} + \{z'_0z_1 + z'_1z_2 + \cdots + \mu_2^{-1}z'_{s-1}z_0\} \\
= \; & < (y'_0, y'_1, \ldots, y'_{r-1}|z'_0, z'_1, \ldots, z'_{s-1}), (y_1, \ldots, y_{r-1}, \mu_1^{-1}y_0|z_1, \ldots, z_{s-1}, \mu_2^{-1}z_0) > \\
= \; & < c', T^{m-1}_{\mu_1,\mu_2}(c) > \\
= \; & 0.
\end{aligned}$$

This implies that $T_{\mu_1^{-1},\mu_2^{-1}}(c') \in \mathfrak{C}^\perp$, and hence $\mathfrak{C}^\perp$ is a (μ_1^{-1}, μ_2^{-1})-constacylic code.
We observe that $\mathfrak{C} \subseteq (\mathfrak{C}^\perp)^\perp$. Also, from Remark 3.11, we have $|\mathfrak{C}| = |(\mathfrak{C}^\perp)^\perp|$. Thus, $(\mathfrak{C}^\perp)^\perp = \mathfrak{C}$. Now, by interchanging the role of $\mathfrak{C}$ and $\mathfrak{C}^\perp$, we get the converse of the theorem. □

Corollary 3.13. *Let* $\mathfrak{C}$ *be a* (μ_1, μ_2)*-constacyclic code of block length* (r, s) *over* $\mathcal{RS}$*. Then* $\mathfrak{C}^\perp$ *is a* (μ_1^{-1}, μ_2^{-1})*-constacyclic code of block length* (r, s) *over* $\mathcal{RS}$ *and*

$$\mathfrak{C}^\perp = \langle (g'_0(x) + ug'_1(x), 0), (l'(x), h'_0(x) + uh'_1(x) + u^2h'_2(x)) \rangle,$$

where $g'_0(x), g'_1(x) \in \mathcal{R}[x]$*,* $h'_0(x), h'_1(x), h'_2(x) \in \mathcal{S}[x]$ *are polynomials satisfying* $g'_1(x) \mid g'_0(x) \mid (x^r - \mu_1)$ *mod* p*,* $h'_2(x) \mid h'_1(x) \mid h'_0(x) \mid (x^s - \mu_2)$ *mod* p*, and* $l'(x) \in \mathcal{R}[x]$ *is such that* $(l'(x), h'_0(x) + uh'_1(x) + u^2h'_2(x)) \in \mathfrak{C}^\perp$*.*

Proof. It follows directly from Theorem 3.8 and Theorem 3.12. □

In the following result, we find the generator of a (μ_0, μ_1, μ_2)-constacyclic code over $\mathbb{Z}_p\mathcal{RS}$.

Theorem 3.14. *Any (μ_0, μ_1, μ_2)-constacyclic code $\mathfrak{C}$ of block length (q, r, s) over $\mathbb{Z}_p\mathcal{RS}$ can be given as*

$$\mathfrak{C} = \langle (f_0(x), 0, 0), (l_1(x), g_0(x) + ug_1(x), 0), (l_2(x), l_3(x), h_0(x) + uh_1(x) + u^2h_2(x)) \rangle,$$

where $f_0(x) \in \mathbb{Z}_p[x]$, $g_0(x), g_1(x) \in \mathcal{R}[x]$, $h_0(x), h_1(x), h_2(x) \in \mathcal{S}[x]$ are polynomials satisfying $f_0(x) \mid (x^q - \mu_0) \bmod p$, $g_1(x) \mid g_0(x) \mid (x^r - \mu_1) \bmod p$, $h_2(x) \mid h_1(x) \mid h_0(x) \mid (x^s - \mu_2) \bmod p$, and the polynomials $l_1(x), l_2(x) \in \mathbb{Z}_p[x]$, $l_3(x) \in \mathcal{R}[x]$ are such that $(l_1(x), g_0(x) + ug_1(x), 0), (l_2(x), l_3(x), h_0(x) + uh_1(x) + u^2h_2(x)) \in \mathfrak{C}$.

Proof. Consider the projection map $p_3 : \mathcal{M}_{0,1,2}^{q,r,s} \to \frac{\mathcal{S}[x]}{\langle x^s - \mu_2 \rangle}$ defined as $p_3(a(x), b(x), d(x)) = d(x)$. It can be verified that p_3 is an $\mathcal{S}[x]$-linear map. Denote $\hat{p}_3 = p_3|_{\mathfrak{C}}$, the restriction map of p_3 on $\mathfrak{C}$. Since $\mathfrak{C}$ is an $\mathcal{S}[x]$-submodule of $\mathcal{M}_{0,1,2}^{q,r,s}$, $\hat{p}_3(\mathfrak{C})$ is also an $\mathcal{S}[x]$-submodule of $\frac{\mathcal{S}[x]}{\langle x^s - \mu_2 \rangle}$. Therefore, $\hat{p}_3(\mathfrak{C})$ is a μ_2-constacyclic code of length s over $\mathcal{S}$ and hence

$$\hat{p}_3(\mathfrak{C}) = \langle h_0(x) + uh_1(x) + u^2h_2(x) \rangle,$$

where $h_2(x) \mid h_1(x) \mid h_0(x) \mid (x^s - \mu_2) \bmod p$. Also, $\ker(\hat{p}_3) = \{(a(x), b(x), 0) \in \mathfrak{C} \mid a(x) \in \frac{\mathbb{Z}_p[x]}{\langle x^q - \mu_0 \rangle}, b(x) \in \frac{\mathcal{R}[x]}{\langle x^r - \mu_1 \rangle}\}$ is an $\mathcal{S}[x]$-submodule of $\mathfrak{C}$. Let $I = \{(a(x), b(x)) \in \mathcal{M}_{0,1}^{q,r} \mid (a(x), b(x), 0) \in \ker(\hat{p}_3)\}$. Then I is an $\mathcal{R}[x]$-submodule of $\mathcal{M}_{0,1}^{q,r}$ and so I is a (μ_0, μ_1)-constacyclic code of block length (q, r) over $\mathbb{Z}_p\mathcal{R}$. Using similar arguments as in Theorem 3.8, we have $I = \langle (f_0(x), 0), (l_1(x), g_0(x) + ug_1(x)) \rangle$, where $f_0(x), l_1(x)$ are polynomials over $\mathbb{Z}_p$ with $f_0(x) \mid (x^q - \mu_0) \bmod p$ and $g_1(x) \mid g_0(x) \mid (x^r - \mu_1) \bmod p$. Hence,

$$\ker(\hat{p}_3) = \langle (f_0(x), 0, 0), (l_1(x), g_0(x) + ug_1(x), 0) \rangle.$$

Let $l_2(x) \in \mathbb{Z}_p[x]$, $l_3(x) \in \mathcal{R}[x]$ are such that $(l_2(x), l_3(x), h_0(x) + uh_1(x) + u^2h_2(x)) \in \mathfrak{C}$. Denote $c_1(x) = (l_2(x), l_3(x), h_0(x) + uh_1(x) + u^2h_2(x))$, $c_2(x) = (f_0(x), 0, 0)$ and $c_3(x) = (l_1(x), g_0(x) + ug_1(x), 0)$. Note that $c_2(x) = (f_0(x), 0, 0), c_3(x) = (l_1(x), g_0(x) + ug_1(x), 0) \in \ker(\hat{p}_3) \subset \mathfrak{C}$. Take $c(x) = (a_1(x), a_2(x), a_3(x)) \in \mathfrak{C}$. Then $a_3(x) \in \hat{p}_3(\mathfrak{C})$ and thus there exists a polynomial $q_1(x) \in \mathcal{S}[x]$ such that $a_3(x) = q_1(x)(h_0(x) + uh_1(x) + u^2h_2(x))$. Now $c(x) - q_1(x)c_1(x) = (a_1(x) - \eta_1(q_1(x))l_2(x), a_2(x) - \eta_2(q_1(x))l_3(x), 0) \in \ker(\hat{p}_3)$. Hence, $a_1(x) - \eta_1(q_1(x))l_2(x) = q_2(x)(f_0(x) + q_3(x)l_1(x))$ for some $q_2(x) \in \mathbb{Z}_p[x]$ and $a_2(x) - \eta_2(q_1(x))l_3(x) = q_3(x)(g_0(x) + ug_1(x))$ for some $q_3(x) \in \mathcal{R}[x]$. Thus, $c(x) = q_1(x)c_1(x) + q_2(x)c_2(x) + q_3(x)c_3(x)$. □

In the following example, we find a (μ_0, μ_1, μ_2)-constacyclic code over $\mathbb{Z}_p\mathcal{RS}$ using Theorem 3.14.

Example 3.15. *Consider $p = 2$, $\mu_0 = 1 = \mu_1$, $\mu_2 = 1 + u^2$, $q = 2 = r = s$. Over $\mathbb{Z}_p$ and $\mathcal{R}$, we have $x^2 - \mu_i = (x + 1)^2$ for $i = 0, 1$. Now, over $\mathcal{S}$, the polynomial $x^2 - \mu_2$ can be factorized as $x^2 - \mu_2 = (x + 1 + u)^2$. Also, consider $f_0(x) = 1 + x$, $g_0(x) = 1 + x^2$, $g_1(x) = 1 + x$, $h_0(x) = 1 + u^2 + x^2$, $h_1(x) = 1 + u + x$, $h_2(x) = 1 + u + x$, $l_0(x) = 0 = l_1(x) = l_2(x)$. Then, from Theorem 3.14, we have*

$$\begin{aligned}\mathfrak{C} &= \langle (1 + x, 0, 0),\ (0, 1 + x^2 + u(1 + x), 0),\ (0, 0, 1 + u^2 + x^2 + u(1 + u + x) + u^2(1 + u + x)) \rangle \\ &= \langle (1, 1; 0, 0; 0, 0),\ (0, 0; u, u; 0, 0),\ (0, 0; 0, 0; u, u + u^2) \rangle.\end{aligned}$$

Thus, $\mathfrak{C}$ is a $(1, 1, 1 + u^2)$-constacyclic code of block length $(2, 2, 2)$ over $\mathbb{Z}_2\mathcal{RS}$. Moreover, $\mathfrak{C}$ is a vector space over $\mathbb{Z}_2$ of dimension 4.

Similarly, as in Definition 3.10, we can define an inner product on $\mathbb{Z}_p^q\mathcal{R}^r\mathcal{S}^s$ as follows.

Definition 3.16. *Let $v = (x_0, x_1, \ldots, x_{q-1} | y_0, y_1, \ldots, y_{r-1} | z_0, z_1, \ldots, z_{s-1})$ and $w = (x'_0, x'_1, \ldots, x'_{q-1} | y'_0, y'_1, \ldots, y'_{r-1} | z'_0, z'_1, \ldots, z'_{s-1})$ are two members of $\mathbb{Z}_p^q\mathcal{R}^r\mathcal{S}^s$. Define the inner product of v and w by*

$$< v, w > \ = u^2 \sum_{i=0}^{q-1} x_i x'_i + u \sum_{i=0}^{r-1} y_i y'_i + \sum_{i=0}^{s-1} z_i z'_i.$$

From here onwards, the dual of a $\mathbb{Z}_p\mathcal{RS}$-additive code will be defined with respect to this inner product.

Theorem 3.17. *The code $\mathfrak{C}$ is a $\mathbb{Z}_p\mathcal{RS}$-additive (μ_0, μ_1, μ_2)-constacyclic code if and only if $\mathfrak{C}^\perp$ is a $\mathbb{Z}_p\mathcal{RS}$-additive $(\mu_0^{-1}, \mu_1^{-1}, \mu_2^{-1})$-constacyclic code.*

Proof. By using Definition 3.16, the proof follows similarly as in Theorem 3.12. □

The following corollary is an obvious consequence of Theorem 3.14 and Theorem 3.17.

Corollary 3.18. *If $\mathfrak{C}$ is any (μ_0, μ_1, μ_2)-constacyclic code of block length (q, r, s) over $\mathbb{Z}_p\mathcal{RS}$, then $\mathfrak{C}^\perp$ is a $(\mu_0^{-1}, \mu_1^{-1}, \mu_2^{-1})$-constacyclic code of block length (q, r, s) over $\mathbb{Z}_p\mathcal{RS}$ and*

$$\mathfrak{C}^\perp = \langle (f_0'(x), 0, 0), (l_1'(x), g_0'(x) + ug_1'(x), 0), (l_2'(x), l_3'(x), h_0'(x) + uh_1'(x) + u^2h_2'(x)) \rangle,$$

where $f_0'(x) \in \mathbb{Z}_p[x]$, $g_0'(x), g_1'(x) \in \mathcal{R}[x]$, $h_0'(x), h_1'(x), h_2'(x) \in \mathcal{S}[x]$ are polynomials satisfying $f_0'(x) \mid (x^q - \mu_0) \bmod p$, $g_1'(x) \mid g_0'(x) \mid (x^r - \mu_1) \bmod p$, $h_2'(x) \mid h_1'(x) \mid h_0'(x) \mid (x^s - \mu_2) \bmod p$, and the polynomials $l_1'(x), l_2'(x) \in \mathbb{Z}_p[x]$, $l_3'(x) \in \mathcal{R}[x]$ are such that $(l_1'(x), g_0'(x) + ug_1'(x), 0), (l_2'(x), l_3'(x), h_0'(x) + uh_1'(x) + u^2h_2'(x)) \in \mathfrak{C}^\perp$.

Definition 3.19. *Let $\mathfrak{C}$ be a $\mathbb{Z}_p\mathcal{RS}$-additive code of block length (q, r, s). Let $\mathfrak{C}_q$ be the code obtained by removing all the coordinates from $\mathcal{R}$ and $\mathcal{S}$, $\mathfrak{C}_r$ be the code obtained by removing all the coordinates from $\mathbb{Z}_p$ and $\mathcal{S}$, $\mathfrak{C}_s$ be the code obtained by removing all the coordinates from $\mathbb{Z}_p$ and $\mathcal{R}$. Then $\mathfrak{C}$ is called separable if $\mathfrak{C} = \mathfrak{C}_q \times \mathfrak{C}_r \times \mathfrak{C}_s$. If $\mathfrak{C}$ is separable then $\mathfrak{C}^\perp = \mathfrak{C}_q^\perp \times \mathfrak{C}_r^\perp \times \mathfrak{C}_s^\perp$, i.e., $\mathfrak{C}^\perp$ is also separable.*

It is easy to observe that $\mathfrak{C}_q$ is an additive code over $\mathbb{Z}_p$ of length q, $\mathfrak{C}_r$ is an additive code over $\mathcal{R}$ of length r and $\mathfrak{C}_s$ is an additive code over $\mathcal{S}$ of length s. Now, we have the following two results.

Proposition 3.20. *Let $\mathfrak{C}$ be a $\mathbb{Z}_p\mathcal{RS}$-additive code of block length (q, r, s) and suppose $\mathfrak{C}$ is separable. Then $\mathfrak{C}$ is a (μ_0, μ_1, μ_2)-constacyclic code if and only if $\mathfrak{C}_q$ is a μ_0-constacyclic code, $\mathfrak{C}_r$ is a μ_1-constacyclic code and $\mathfrak{C}_s$ is a μ_2-constacyclic code.*

Proof. Since $\mathfrak{C}$ is separable, we have $\mathfrak{C} = \mathfrak{C}_q \times \mathfrak{C}_r \times \mathfrak{C}_s$. First, suppose that $\mathfrak{C}$ is a (μ_0, μ_1, μ_2)-constacyclic code. Take $(a_0, a_1, \ldots, a_{q-1}) \in \mathfrak{C}_q$, $(b_0, b_1, \ldots, b_{r-1}) \in \mathfrak{C}_r$, $(d_0, d_1, \ldots, d_{s-1}) \in \mathfrak{C}_s$ such that $(a_0, a_1, \ldots, a_{q-1} | b_0, b_1, \ldots, b_{r-1} | d_0, d_1, \ldots, d_{s-1}) \in \mathfrak{C}$. Then $(\mu_0 a_{q-1}, a_0, \ldots, a_{q-2} | \mu_1 b_{r-1}, b_0, \ldots, b_{r-2} | \mu_2 d_{s-1}, d_0, \ldots, d_{s-2}) \in \mathfrak{C}$ and hence $(\mu_0 a_{q-1}, a_0, \ldots, a_{q-2}) \in \mathfrak{C}_q$, $(\mu_1 b_{r-1}, b_0, \ldots, b_{r-2}) \in \mathfrak{C}_r$ and $(\mu_2 d_{s-1}, d_0, \ldots, d_{s-2}) \in \mathfrak{C}_s$. Thus $\mathfrak{C}_q$ is a μ_0-constacyclic code, $\mathfrak{C}_r$ is a μ_1-constacyclic code and $\mathfrak{C}_s$ is a μ_2-constacyclic code.

The converse can be proved by reversing the above arguments. □

Proposition 3.21. *Let $\mathfrak{C}$ be a $\mathbb{Z}_p\mathcal{RS}$-additive code of block length (q, r, s) and suppose $\mathfrak{C}$ is separable. Then $\mathfrak{C}$ is a (μ_0, μ_1, μ_2)-constacyclic code if and only if $\mathfrak{C}^\perp$ is separable and it is a $(\mu_0^{-1}, \mu_1^{-1}, \mu_2^{-1})$-constacyclic code over $\mathbb{Z}_p\mathcal{RS}$.*

Proof. The proof follows from Theorem 3.17. □

4. Gray maps on $\mathcal{R}$, $\mathcal{S}$ and $\mathbb{Z}_p\mathcal{RS}$

In this section, we define a few Gray maps and then study their images.

Define the Gray map

$$\phi_1 : \mathcal{R} \to \mathbb{Z}_p^2$$

by

$$\phi_1(1) = (1, 0)$$
$$\phi_1(u) = (1, \kappa),$$

where $\kappa \in \mathbb{Z}_p$ is such that $\kappa^2 \equiv -1 \pmod{p}$, and thus $\phi_1(a+ub) = (a+b, \kappa b)$ for all $a, b \in \mathbb{Z}_p$. Such an element κ will always exist as we consider only those rings $\mathbb{Z}_p$ in which $p-1$ is a quadratic residue. One can easily verify that the map ϕ_1 is $\mathbb{Z}_p$-linear and bijective.

The Lee weight of an element $x \in \mathcal{R}$ is defined as $wt_L(x) = wt_H(\phi_1(x))$, where $wt_H(y)$ denotes the Hamming weight of y and the Lee distance between two elements $x, y \in \mathcal{R}$ is defined as $d_L(x,y) = wt_L(x-y)$. The Gray map ϕ_1 can be extended to $\overline{\phi}_1 : \mathcal{R}^r \to \mathbb{Z}_p^{2r}$ by

$$\begin{aligned}&\overline{\phi}_1(a_0 + ub_0, a_1 + ub_1, \ldots, a_{r-1} + ub_{r-1})\\ &= (a_0 + b_0, a_1 + b_1, \ldots, a_{r-1} + b_{r-1} | \kappa b_0, \kappa b_1, \ldots, \kappa b_{r-1}),\end{aligned}$$

where $\kappa \in \mathbb{Z}_p$ is such that $\kappa^2 \equiv -1 \pmod{p}$.

The Lee weight of an element $x = (x_0, x_1, \ldots, x_{r-1}) \in \mathcal{R}^r$ is defined as $wt_L(x) = \sum_{i=0}^{r-1} wt_L(x_i)$ and the Lee distance between two elements $x, y \in \mathcal{R}^r$ is defined as $d_L(x,y) = wt_L(x-y)$.

From the above definition of $\overline{\phi}_1$, it can be observed that $\overline{\phi}_1$ is a distance preserving map from $\mathcal{R}^r$ (Lee weight) to $\mathbb{Z}_p^{2r}$ (Hamming weight). Further, if $\mathfrak{C}$ is an $\mathcal{R}$-additive code with parameters $[r,k,d]$ then $\overline{\phi}_1(\mathfrak{C})$ is a $[2r,k,d]$-code over $\mathbb{Z}_p$.

Theorem 4.1. *For any two elements $x, x' \in \mathcal{R}^r$,*

$$\overline{\phi}_1(x) \cdot \overline{\phi}_1(x') = \pi_1 \circ \phi_1(x \cdot x'),$$

where $\pi_1 : \mathbb{Z}_p^2 \to \mathbb{Z}_p$ is the projection map defined as $\pi_1(a,b) = a$ for $a, b \in \mathbb{Z}_p$.

Proof. Let $x = (x_0, x_1, \ldots, x_{r-1})$ and $x' = (x'_0, x'_1, \ldots, x'_{r-1})$, where $x_i = a_i + ub_i$, $x'_i = a'_i + ub'_i$ and $a_i, b_i, a'_i, b'_i \in \mathbb{Z}_p$ for $i = 0, 1, \ldots, r-1$. Now,

$$\begin{aligned}x \cdot x' &= \sum_{i=0}^{r-1} x_i x'_i\\ &= \sum_{i=0}^{r-1} \{a_i a'_i + u(a_i b'_i + a'_i b_i)\}\\ \Rightarrow\ \phi_1(x \cdot x') &= (a_0 a'_0 + a_0 b'_0 + a'_0 b_0 + \cdots + a_{r-1} a'_{r-1} + a_{r-1} b'_{r-1} + a'_{r-1} b_{r-1}\ ,\ \kappa(a_0 b'_0 + a'_0 b_0)\\ &\quad + \cdots + \kappa(a_{r-1} b'_{r-1} + a'_{r-1} b_{r-1})).\end{aligned}$$

Also,

$$\begin{aligned}\overline{\phi}_1(x) \cdot \overline{\phi}_1(x') &= (a_0 + b_0, a_1 + b_1, \ldots, a_{r-1} + b_{r-1} | \kappa b_0, \kappa b_1, \ldots, \kappa b_{r-1})\\ &\quad \cdot (a'_0 + b'_0, a'_1 + b'_1, \ldots, a'_{r-1} + b'_{r-1} | \kappa b'_0, \kappa b'_1, \ldots, \kappa b'_{r-1})\\ &= (a_0 a'_0 + a_0 b'_0 + a'_0 b_0 + \cdots + a_{r-1} a'_{r-1} + a_{r-1} b'_{r-1} + a'_{r-1} b_{r-1}).\end{aligned}$$

Thus,

$$\overline{\phi}_1(x) \cdot \overline{\phi}_1(x') = \pi_1 \circ \phi_1(x \cdot x').$$

□

The above theorem shows that the images of two orthogonal elements in $\mathcal{R}^r$ under the map $\overline{\phi}_1$ are also orthogonal. Now, we have the following result.

Corollary 4.2. *Let $\mathfrak{C}$ be an $\mathcal{R}$-additive code of length r. Then*

$$\overline{\phi}_1(\mathfrak{C}^{\perp}) = \overline{\phi}_1(\mathfrak{C})^{\perp}.$$

Proof. From Theorem 4.1, we have $\overline{\phi}_1(\mathfrak{C}^\perp) \subseteq \overline{\phi}_1(\mathfrak{C})^\perp$. Now, if we can show that $|\overline{\phi}_1(\mathfrak{C})^\perp| = |\overline{\phi}_1(\mathfrak{C}^\perp)|$ then we are done. We know that $\overline{\phi}_1$ is an isomorphism and hence $|\overline{\phi}_1(\mathfrak{C})| = |\mathfrak{C}|$, $|\overline{\phi}_1(\mathfrak{C}^\perp)| = |\mathfrak{C}^\perp|$. Also, $|\mathbb{Z}_p^{2r}| = p^{2r}$. Thus, $|\overline{\phi}_1(\mathfrak{C})^\perp| = \frac{p^{2r}}{|\overline{\phi}_1(\mathfrak{C})|} = \frac{p^{2r}}{|\mathfrak{C}|} = |\mathfrak{C}^\perp| = |\overline{\phi}_1(\mathfrak{C}^\perp)|$. □

We define another Gray map

$$\phi_2 : \mathcal{S} \to \mathbb{Z}_p^3$$

by

$$\begin{aligned}\phi_2(1) &= (1,0,0)\\ \phi_2(u) &= (1,\kappa,1)\\ \phi_2(u^2) &= (1,\kappa,0),\end{aligned}$$

where $\kappa \in \mathbb{Z}_p$ is such that $\kappa^2 \equiv -1 \pmod p$, and thus $\phi_2(a+ub+u^2d) = (a+b+d, \kappa(b+d), b)$ for all $a,b,d \in \mathbb{Z}_p$. One can easily verify that the map ϕ_2 is $\mathbb{Z}_p$-linear and bijective.

The Lee weight of an element $x \in \mathcal{S}$ is defined as $wt_L(x) = wt_H(\phi_2(x))$, where $wt_H(y)$ denotes the Hamming weight of y and the Lee distance between two elements $x, y \in \mathcal{S}$ is defined as $d_L(x,y) = wt_L(x-y)$. The Gray map ϕ_2 can be extended to $\overline{\phi}_2 : \mathcal{S}^s \to \mathbb{Z}_p^{3s}$ by

$$\begin{aligned}&\overline{\phi}_2(a_0+ub_0+u^2d_0, a_1+ub_1+u^2d_1, \ldots, a_{s-1}+ub_{s-1}+u^2d_{s-1})\\ =\ &(a_0+b_0+d_0, a_1+b_1+d_1, \ldots, a_{s-1}+b_{s-1}+d_{s-1}|\kappa(b_0+d_0), \kappa(b_1+d_1),\\ &\ldots, \kappa(b_{s-1}+d_{s-1})|b_0, b_1, \ldots, b_{s-1}),\end{aligned}$$

where $\kappa \in \mathbb{Z}_p$ is such that $\kappa^2 \equiv -1 \pmod p$.

The Lee weight of an element $x = (x_0, x_1, \ldots, x_{s-1}) \in \mathcal{S}^s$ is defined as $wt_L(x) = \sum_{i=0}^{s-1} wt_L(x_i)$ and the Lee distance between two elements $x, y \in \mathcal{S}^s$ is defined as $d_L(x,y) = wt_L(x-y)$.

From the above definition of $\overline{\phi}_2$, it can be observed that $\overline{\phi}_2$ is a distance preserving map from $\mathcal{S}^s$ (Lee weight) to $\mathbb{Z}_p^{3s}$ (Hamming weight). Further, if $\mathfrak{C}$ is an $\mathcal{S}$-additive code with parameters $[s,k,d]$ then $\overline{\phi}_2(\mathfrak{C})$ is a $[3s,k,d]$-code over $\mathbb{Z}_p$.

Now, similar to the Theorem 4.1 and Corollary 4.2, we have the following two results.

Theorem 4.3. *For any two elements $x, x' \in \mathcal{S}^s$,*

$$\overline{\phi}_2(x) \cdot \overline{\phi}_2(x') = \hat{\pi}_1 \circ \phi_2(x \cdot x'),$$

where $\hat{\pi}_1 : \mathbb{Z}_p^3 \to \mathbb{Z}_p$ is the projection map defined as $\hat{\pi}_1(a,b,d) = a$ for $a,b,d \in \mathbb{Z}_p$. In particular, the images of two orthogonal elements in $\mathcal{S}^s$ under the map $\overline{\phi}_2$ are also orthogonal.

Proof. Similar to the proof of Theorem 4.1. □

Corollary 4.4. *Let $\mathfrak{C}$ be an $\mathcal{S}$-additive code of length s. Then*

$$\overline{\phi}_2(\mathfrak{C}^\perp) = \overline{\phi}_2(\mathfrak{C})^\perp.$$

Proof. Similar to the proof of Corollary 4.2. □

Now, we define a Gray map from $\mathbb{Z}_p\mathcal{RS}$ to $\mathbb{Z}_p^6$ by using the Gray maps ϕ_1 and ϕ_2. Define

$$\phi : \mathbb{Z}_p\mathcal{RS} \to \mathbb{Z}_p^6$$

by

$$\phi(x,y,z) = (x, \phi_1(y), \phi_2(z)) \quad \text{for all } x \in \mathbb{Z}_p,\ y \in \mathcal{R},\ z \in \mathcal{S}.$$

Obviously, the map ϕ is also $\mathbb{Z}_p$-linear and bijective. The Lee weight of an element $\alpha = (x,y,z) \in \mathbb{Z}_p\mathcal{RS}$ is defined as $wt_L(\alpha) = wt_H(x)+wt_L(y)+wt_L(z)$. The Lee distance between two elements $\alpha = (x,y,z)$, $\alpha' = (x',y',z') \in \mathbb{Z}_p\mathcal{RS}$, is defined as $d_L(\alpha,\alpha') = wt_L(\alpha-\alpha')$.

The map ϕ can be extended to $\overline{\phi} : \mathbb{Z}_p^q\mathcal{R}^r\mathcal{S}^s \to \mathbb{Z}_p^{q+2r+3s}$ by

$$\overline{\phi}(x,y,z) = (x, \overline{\phi}_1(y), \overline{\phi}_2(z)),$$

where $x \in \mathbb{Z}_p^q,\ y \in \mathcal{R}^r,\ z \in \mathcal{S}^s$.

The Lee weight of $\alpha = (x_0,x_1,\ldots,x_{q-1}|y_0,y_1,\ldots,y_{r-1}|z_0,z_1,\ldots,z_{s-1}) \in \mathbb{Z}_p^q\mathcal{R}^r\mathcal{S}^s$ is $wt_L(\alpha) = \sum_{i=0}^{q-1} wt_H(x_i) + \sum_{i=0}^{r-1} wt_L(y_i) + \sum_{i=0}^{s-1} wt_L(z_i)$ and the Lee distance between $\alpha,\alpha' \in \mathbb{Z}_p^q\mathcal{R}^r\mathcal{S}^s$ is $d_L(\alpha,\alpha') = wt_L(\alpha-\alpha')$.

We observe that $\overline{\phi}$ is also a distance preserving map from $\mathbb{Z}_p^q\mathcal{R}^r\mathcal{S}^s$ (Lee weight) to $\mathbb{Z}_p^{q+2r+3s}$ (Hamming weight). Further, if $\mathfrak{C}$ is a $\mathbb{Z}_p^q\mathcal{R}^r\mathcal{S}^s$-additive code of block length (q,r,s), having p^k codewords with minimum distance d then $\overline{\phi}(\mathfrak{C})$ is a $[q+2r+3s,k,d]$-code over $\mathbb{Z}_p$.

Theorem 4.5. *For any two elements $\alpha = (x,y,z)$, $\alpha' = (x',y',z') \in \mathbb{Z}_p^q\mathcal{R}^r\mathcal{S}^s$, we have*

$$\overline{\phi}(\alpha)\cdot\overline{\phi}(\alpha') = x\cdot x' + \pi_1\circ\phi_1(y\cdot y') + \hat{\pi}_1\circ\phi_2(z\cdot z'),$$

where $x,x' \in \mathbb{Z}_p^q,\ y,y' \in \mathcal{R}^r,\ z,z' \in \mathcal{S}^s$.

Proof. We know that $\overline{\phi}(\alpha) = (x,\overline{\phi}_1(y),\overline{\phi}_2(z))$ and $\overline{\phi}(\alpha') = (x',\overline{\phi}_1(y'),\overline{\phi}_2(z'))$. Thus

$$\overline{\phi}(\alpha)\cdot\overline{\phi}(\alpha') = x\cdot x' + \overline{\phi}_1(y)\cdot\overline{\phi}_1(y') + \overline{\phi}_2(z)\cdot\overline{\phi}_2(z').$$

Now, the proof follows from Theorem 4.1 and Theorem 4.3. □

Considering the inner product which we defined in Definition 3.16, we get from Theorem 4.5 that the images of two orthogonal elements in $\mathbb{Z}_p^q\mathcal{R}^r\mathcal{S}^s$ under the map $\overline{\phi}$ are also orthogonal and we have the following corollary.

Corollary 4.6. *If $\mathfrak{C}$ is a $\mathbb{Z}_p\mathcal{RS}$-additive code of block length (q,r,s) then*

$$\overline{\phi}(\mathfrak{C}^\perp) = \overline{\phi}(\mathfrak{C})^\perp.$$

Proof. Similar to the proof of the Corollary 4.2. □

4.1. Results on Gray images of additive codes

Now, we will give a few results related to the Gray images of additive cyclic and additive constacyclic codes.

Lemma 4.7. *Let σ_1, θ_2, $\theta_{\mu_1,2}$ and T_{μ_1} are respectively the cyclic shift operator, the 2-quasi-cyclic shift operator, the $(\mu_1,2)$-quasi-twisted shift operator and the μ_1-constacyclic shift operator on $\mathcal{R}^r$ with $\mu_1 \in \mathbb{Z}_p^*$. Then*

1. $\overline{\phi}_1\circ\sigma_1 = \theta_2\circ\overline{\phi}_1$;

2. $\overline{\phi}_1 \circ T_{\mu_1} = \theta_{\mu_1,2} \circ \overline{\phi}_1$.

Proof. Let $x = (x_0, x_1, \ldots, x_{r-1}) \in \mathcal{R}^r$, where $x_i = a_i + ub_i$ with $a_i, b_i \in \mathbb{Z}_p$ for $i = 0, 1, \ldots, r-1$.

1. We have

$$\begin{aligned}\overline{\phi}_1 \circ \sigma_1(x) &= \overline{\phi}_1(x_{r-1}, x_0, \ldots, x_{r-2}) \\ &= (a_{r-1} + b_{r-1}, a_0 + b_0, \ldots, a_{r-2} + b_{r-2} | \kappa b_{r-1}, \kappa b_0, \ldots, \kappa b_{r-2}),\end{aligned}$$

and also

$$\begin{aligned}\theta_2 \circ \overline{\phi}_1(x) &= \theta_2(a_0 + b_0, a_1 + b_1, \ldots, a_{r-1} + b_{r-1} | \kappa b_0, \kappa b_1, \ldots, \kappa b_{r-1}) \\ &= (a_{r-1} + b_{r-1}, a_0 + b_0, \ldots, a_{r-2} + b_{r-2} | \kappa b_{r-1}, \kappa b_0, \ldots, \kappa b_{r-2}).\end{aligned}$$

Thus, $\overline{\phi}_1 \circ \sigma_1 = \theta_2 \circ \overline{\phi}_1$.

2. We have

$$\begin{aligned}\overline{\phi}_1 \circ T_{\mu_1}(x) &= \overline{\phi}_1(\mu_1 x_{r-1}, x_0, \ldots, x_{r-2}) \\ &= (\mu_1(a_{r-1} + b_{r-1}), a_0 + b_0, \ldots, a_{r-2} + b_{r-2} | \kappa\mu_1 b_{r-1}, \kappa b_0, \ldots, \kappa b_{r-2}),\end{aligned}$$

and

$$\begin{aligned}\theta_{\mu_1,2} \circ \overline{\phi}_1(x) &= \theta_{\mu_1,2}(a_0 + b_0, a_1 + b_1, \ldots, a_{r-1} + b_{r-1} | \kappa b_0, \kappa b_1, \ldots, \kappa b_{r-1}) \\ &= (\mu_1(a_{r-1} + b_{r-1}), a_0 + b_0, \ldots, a_{r-2} + b_{r-2} | \kappa\mu_1 b_{r-1}, \kappa b_0, \ldots, \kappa b_{r-2}).\end{aligned}$$

Thus, $\overline{\phi}_1 \circ T_{\mu_1} = \theta_{\mu_1,2} \circ \overline{\phi}_1$.

□

Using the above lemma, we have the following theorem on the Gray image of cyclic and constacyclic codes over $\mathcal{R}$.

Theorem 4.8. *Let $\mu_1 \in \mathbb{Z}_p^*$. Then we have the following:*

1. *If $\mathfrak{C}$ is a cyclic code of length r over $\mathcal{R}$ then $\overline{\phi}_1(\mathfrak{C})$ is a 2-quasi-cyclic code of length $2r$ over $\mathbb{Z}_p$.*
2. *If $\mathfrak{C}$ is a μ_1-constacyclic code of length r over $\mathcal{R}$ then $\overline{\phi}_1(\mathfrak{C})$ is a $(\mu_1, 2)$-quasi-twisted code of length $2r$ over $\mathbb{Z}_p$.*

Proof. 1. Suppose $\mathfrak{C}$ is a cyclic code of length r over $\mathcal{R}$. Then $\sigma_1(\mathfrak{C}) = \mathfrak{C}$. Now from Lemma 4.7, we have

$$\theta_2(\overline{\phi}_1(\mathfrak{C})) = \overline{\phi}_1(\sigma_1(\mathfrak{C})) = \overline{\phi}_1(\mathfrak{C}),$$

and this implies that $\overline{\phi}_1(\mathfrak{C})$ is a 2-quasi-cyclic code of length $2r$ over $\mathbb{Z}_p$.

2. If $\mathfrak{C}$ is a μ_1-constacyclic code of length r over $\mathcal{R}$ then $T_{\mu_1}(\mathfrak{C}) = \mathfrak{C}$. Now from Lemma 4.7, we have

$$\theta_{\mu_1,2}(\overline{\phi}_1(\mathfrak{C})) = \overline{\phi}_1(T_{\mu_1}(\mathfrak{C})) = \overline{\phi}_1(\mathfrak{C}),$$

which implies that $\overline{\phi}_1(\mathfrak{C})$ is a $(\mu_1, 2)$-quasi-twisted code of length $2r$ over $\mathbb{Z}_p$.

□

Lemma 4.9. *Let σ_2, θ_3, $\theta_{\mu_2,3}$ and T_{μ_2} be the cyclic shift operator, the 2-quasi-cyclic shift operator, the $(\mu_2, 3)$-quasi-twisted shift operator and the μ_2-constacyclic shift operator, respectively on $\mathcal{S}^s$ with $\mu_2 \in \mathbb{Z}_p^*$. Then*

1. $\overline{\phi}_2 \circ \sigma_2 = \theta_3 \circ \overline{\phi}_2$;
2. $\overline{\phi}_2 \circ T_{\mu_2} = \theta_{\mu_2,3} \circ \overline{\phi}_2$.

Proof. Let $x = (x_0, x_1, \ldots, x_{s-1}) \in \mathcal{S}^s$ where $x_i = a_i + ub_i + u^2 d_i$ with $a_i, b_i, d_i \in \mathbb{Z}_p$ for $i = 0, 1, \ldots, s-1$.

1. We have

$$\begin{aligned}\overline{\phi}_2 \circ \sigma_2(x) &= \overline{\phi}_2(x_{s-1}, x_0, \ldots, x_{s-2}) \\ &= (a_{s-1} + b_{s-1} + d_{s-1}, a_0 + b_0 + d_0, \ldots, a_{s-2} + b_{s-2} + d_{s-2} | \kappa(b_{s-1} + d_{s-1}), \\ &\quad \kappa(b_0 + d_0), \ldots, \kappa(b_{s-2} + d_{s-2}) | b_{s-1}, b_0, \ldots, b_{s-2}).\end{aligned}$$

   Also,

$$\begin{aligned}\theta_3 \circ \overline{\phi}_2(x) &= \theta_3(a_0 + b_0 + d_0, a_1 + b_1 + d_1, \ldots, a_{s-1} + b_{s-1} + d_{s-1} | \kappa(b_0 + d_0), \kappa(b_1 + d_1), \\ &\quad \ldots, \kappa(b_{s-1} + d_{s-1}) | b_0, b_1, \ldots, b_{s-1}) \\ &= (a_{s-1} + b_{s-1} + d_{s-1}, a_0 + b_0 + d_0, \ldots, a_{s-2} + b_{s-2} + d_{s-2} | \kappa(b_{s-1} + d_{s-1}), \\ &\quad \kappa(b_0 + d_0), \ldots, \kappa(b_{s-2} + d_{s-2}) | b_{s-1}, b_0, \ldots, b_{s-2}).\end{aligned}$$

   Thus, $\overline{\phi}_2 \circ \sigma_2 = \theta_3 \circ \overline{\phi}_2$.

2. We have

$$\begin{aligned}&\overline{\phi}_2 \circ T_{\mu_2}(x) \\ &= \overline{\phi}_2(\mu_2 x_{s-1}, x_0, \ldots, x_{s-2}) \\ &= (\mu_2(a_{s-1} + b_{s-1} + d_{s-1}), a_0 + b_0 + d_0, \ldots, a_{s-2} + b_{s-2} + d_{s-2} | \kappa\mu_2(b_{s-1} + d_{s-1}), \kappa(b_0 + d_0), \\ &\quad \ldots, \kappa(b_{s-2} + d_{s-2}) | \mu_2 b_{s-1}, b_0, \ldots, b_{s-2}),\end{aligned}$$

   and

$$\begin{aligned}&\theta_{\mu_2,3} \circ \overline{\phi}_2(x) \\ &= \theta_{\mu_2,3}(a_0 + b_0 + d_0, a_1 + b_1 + d_1, \ldots, a_{s-1} + b_{s-1} + d_{s-1} | \kappa(b_0 + d_0), \kappa(b_1 + d_1), \\ &\quad \ldots, \kappa(b_{s-1} + d_{s-1}) | b_0, b_1, \ldots, b_{s-1}) \\ &= (\mu_2(a_{s-1} + b_{s-1} + d_{s-1}), a_0 + b_0 + d_0, \ldots, a_{s-2} + b_{s-2} + d_{s-2} | \kappa\mu_2(b_{s-1} + d_{s-1}), \kappa(b_0 + d_0), \\ &\quad \ldots, \kappa(b_{s-2} + d_{s-2}) | \mu_2 b_{s-1}, b_0, \ldots, b_{s-2}).\end{aligned}$$

   Thus, $\overline{\phi}_2 \circ T_{\mu_2} = \theta_{\mu_2,3} \circ \overline{\phi}_2$.

□

Now, we have the following theorem on the Gray image of cyclic and constacyclic codes over $\mathcal{S}$.

Theorem 4.10. *Let $\mu_2 \in \mathbb{Z}_p^*$. Then we have the following:*

1. *If $\mathfrak{C}$ is a cyclic code of length s over $\mathcal{S}$, then $\overline{\phi}_2(\mathfrak{C})$ is a 3-quasi-cyclic code of length $3s$ over $\mathbb{Z}_p$.*
2. *If $\mathfrak{C}$ is a μ_2-constacyclic code of length s over $\mathcal{S}$, then $\overline{\phi}_2(\mathfrak{C})$ is a $(\mu_2, 3)$-quasi-twisted code of length $3s$ over $\mathbb{Z}_p$.*

Proof. 1. If $\mathfrak{C}$ is a cyclic code of length s over $\mathcal{S}$, then $\sigma_2(\mathfrak{C}) = \mathfrak{C}$. Now from Lemma 4.9, we have

$$\theta_3(\overline{\phi}_2(\mathfrak{C})) = \overline{\phi}_2(\sigma_2(\mathfrak{C})) = \overline{\phi}_2(\mathfrak{C}),$$

and this implies that $\overline{\phi}_2(\mathfrak{C})$ is a 3-quasi-cyclic code of length $3s$ over $\mathbb{Z}_p$.

2. Assume that $\mathfrak{C}$ is a μ_2-constacyclic code of length s over $\mathcal{S}$. Then $T_{\mu_2}(\mathfrak{C}) = \mathfrak{C}$. Now from Lemma 4.9, we have

$$\theta_{\mu_2,3}(\overline{\phi}_2(\mathfrak{C})) = \overline{\phi}_2(T_{\mu_2}(\mathfrak{C})) = \overline{\phi}_2(\mathfrak{C}),$$

which implies that $\overline{\phi}_2(\mathfrak{C})$ is a $(\mu_2, 3)$-quasi-twisted code of length $3s$ over $\mathbb{Z}_p$.

□

The next result is on the Gray image of a constacyclic code over $\mathbb{Z}_p\mathcal{RS}$.

Theorem 4.11. *Let $\mu_0, \mu_1, \mu_2 \in \mathbb{Z}_p^*$ and suppose $\mathfrak{C}$ is a (μ_0, μ_1, μ_2)-constacyclic code of block length (q, r, s) over $\mathbb{Z}_p\mathcal{RS}$. Then $\overline{\phi}(\mathfrak{C})$ is a generalized $(\mu_0, \mu_1, \mu_1, \mu_2, \mu_2, \mu_2)$-quasi-twisted code of block length (q, r, r, s, s, s).*

Proof. Take $w \in \overline{\phi}(\mathfrak{C})$. Then there exists $v \in \mathfrak{C}$ such that $w = \overline{\phi}(v)$. Let

$$v = (x_0, x_1, \ldots, x_{q-1}|y_0, y_1, \ldots, y_{r-1}|z_0, z_1, \ldots, z_{s-1}),$$

where $y_i = a_i + ub_i$, $z_i = a'_i + ub'_i + u^2d'_i$ and $x_i, a_i, b_i, a'_i, b'_i, d'_i \in \mathbb{Z}_p$. Then

$$\begin{aligned}\overline{\phi}(v) = (&x_0, x_1, \ldots, x_{q-1}|a_0 + b_0, a_1 + b_1, \ldots, a_{r-1} + b_{r-1}|\kappa b_0, \kappa b_1, \ldots, \kappa b_{r-1}|a'_0 + b'_0 + d'_0,\\ &a'_1 + b'_1 + d'_1, \ldots, a'_{s-1} + b'_{s-1}d'_{s-1}|\kappa(b'_0 + d'_0), \kappa(b'_1 + d'_1), \ldots, \kappa(b'_{s-1} + d'_{s-1})|b'_0, b'_1, \ldots, b'_{s-1}).\end{aligned}$$

Let T_{μ_0,μ_1,μ_2} be the (μ_0, μ_1, μ_2)-constacyclic shift operator on $\mathbb{Z}_p^q\mathcal{R}^r\mathcal{S}^s$. Then we have

$$\begin{aligned}&\overline{\phi} \circ T_{\mu_0,\mu_1,\mu_2}(v)\\ &= \overline{\phi}(\mu_0 x_{q-1}, x_0, \ldots, x_{q-2}|\mu_1 y_{r-1}, y_0, \ldots, y_{r-2}|\mu_2 z_{s-1}, z_0, \ldots, z_{s-2})\\ &= (\mu_0 x_{q-1}, x_0, \ldots, x_{q-2}|\mu_1(a_{r-1} + b_{r-1}), a_0 + b_0, \ldots, a_{r-2} + b_{r-2}|\kappa\mu_1 b_{r-1}, \kappa b_0, \ldots, \kappa b_{r-2}|\\ &\quad \mu_2(a'_{s-1} + b'_{s-1} + d'_{s-1}), a'_0 + b'_0 + d'_0, \ldots, a'_{s-2} + b'_{s-2} + d'_{s-2}|\kappa\mu_2(b'_{s-1} + d'_{s-1}), \kappa(b'_0 + d'_0),\\ &\quad \ldots, \kappa(b'_{s-2} + d'_{s-2})|\mu_2 b'_{s-1}, b'_0, \ldots, b'_{s-2}).\end{aligned}$$

Since $\mathfrak{C}$ is a (μ_0, μ_1, μ_2)-constacyclic code, $\overline{\phi} \circ T_{\mu_0,\mu_1,\mu_2}(v) \in \overline{\phi}(\mathfrak{C})$ and we observe that $\overline{\phi} \circ T_{\mu_0,\mu_1,\mu_2}(v)$ is the generalized $(\mu_0, \mu_1, \mu_1, \mu_2, \mu_2, \mu_2)$-quasi-twisted shift of $\overline{\phi}(v)$. Therefore, $\overline{\phi}(\mathfrak{C})$ is a generalized $(\mu_0, \mu_1, \mu_1, \mu_2, \mu_2, \mu_2)$-quasi-twisted code of block length (q, r, r, s, s, s). □

5. The weight enumerator and MacWilliams identities

Weight enumerators play an important role in calculating the probability of an incorrect message being received undetected by the receiver. It involves the number of codewords of a particular weight, which can be easily obtained once the weight enumerator is known. Further, one can express the weight enumerator of the dual code in terms of the weight enumerator of the code itself. MacWilliams identities provide exactly this. In this section, we study different weight enumerators, such as complete weight enumerator, symmetrized weight enumerator, etc., and establish the MacWilliams identities.

5.1. The complete weight enumerator and the Hamming weight enumerator

First, we arrange the members of each of $\mathbb{Z}_p$, $\mathcal{R}$ and $\mathcal{S}$, respectively, in a particular order.

The members of $\mathbb{Z}_p$ are arranged in increasing order by considering the members as mere integers, i.e., $\{0 < 1 < 2 < \cdots < p-1\}$.

For any two members $a_0 + a_1u, b_0 + b_1u \in \mathcal{R}$, we say $a_0 + a_1u < b_0 + b_1u$ if one of the following two conditions hold.

1. $a_1 < b_1$;
2. $a_1 = b_1$, $a_0 < b_0$.

For any two members $a_0 + a_1u + a_2u^2, b_0 + b_1u + b_2u^2 \in \mathcal{S}$, we say $a_0 + a_1u + a_2u^2 < b_0 + b_1u + b_2u^2$ if one of the following conditions hold.

1. $a_2 < b_2$;
2. $a_2 = b_2$, $a_1 < b_1$;
3. $a_2 = b_2$, $a_1 = b_1$, $a_0 < b_0$.

Using the above ordering of the members of each of $\mathbb{Z}_p$, $\mathcal{R}$ and $\mathcal{S}$, now we can order the members of $\mathbb{Z}_p\mathcal{RS}$. For any two members $\alpha = (\alpha_0, \alpha_1, \alpha_2), \beta = (\beta_0, \beta_1, \beta_2) \in \mathbb{Z}_p\mathcal{RS}$, where $\alpha_0, \beta_0 \in \mathbb{Z}_p$, $\alpha_1, \beta_1 \in \mathcal{R}$, and $\alpha_2, \beta_2 \in \mathcal{S}$, we say $\alpha < \beta$ if one of the following conditions hold.

1. $\alpha_0 < \beta_0$;
2. $\alpha_0 = \beta_0$, $\alpha_1 < \beta_1$;
3. $\alpha_0 = \beta_0$, $\alpha_1 = \beta_1$, $\alpha_2 < \beta_2$.

We know that any integer i with $0 \le i \le p^6 - 1$, can be written as $i = \sum_{j=0}^{5} a_j p^j$, where $0 \le a_j \le p - 1$ for $j = 0, 1, \ldots, 5$. If we consider all the members of $\mathbb{Z}_p\mathcal{RS}$ as $\mathbb{Z}_p\mathcal{RS} = \{f_0 < f_1 < f_2 < \cdots < f_{p^6-1}\}$, then using the ordering of the members of $\mathbb{Z}_p\mathcal{RS}$, we have

$$f_i = (a_5, a_4u + a_3, a_2u^2 + a_1u + a_0),$$

where $i = \sum_{j=0}^{5} a_j p^j$.

Now, we are ready to define the complete weight enumerator of a $\mathbb{Z}_p\mathcal{RS}$-additive code $\mathfrak{C}$.

Definition 5.1. *Let $\mathfrak{C}$ be a $\mathbb{Z}_p\mathcal{RS}$-additive code of length n. Then the complete weight enumerator of $\mathfrak{C}$ is denoted by $\mathcal{W}_{\mathbf{C}}^{(\mathfrak{C})}$ and is defined by*

$$\mathcal{W}_{\mathbf{C}}^{(\mathfrak{C})}(x_1, x_2, \ldots, x_{p^6}) = \sum_{c \in \mathfrak{C}} \prod_{i=1}^{p^6} x_i^{w_{f_{i-1}}(c)},$$

where for each $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathfrak{C}$, $w_{f_i}(c) = |\{j : c_j = f_i, 0 \le j \le n-1\}|$ for $i = 1, 2, \ldots, p^6$.

From the above definition, it is evident that $\mathcal{W}_{\mathbf{C}}^{(\mathfrak{C})}(x_1, x_2, \ldots, x_{p^6})$ is a homogeneous polynomial. The total degree of each monomial in $\mathcal{W}_{\mathbf{C}}^{(\mathfrak{C})}(x_1, x_2, \ldots, x_{p^6})$ is n. Also, we observe that $\mathcal{W}_{\mathbf{C}}^{(\mathfrak{C})}(1, 1, \ldots, 1) = |\mathfrak{C}|$.

Now, we will investigate the MacWilliams identity corresponding to the complete weight enumerator. First, we define a generating character on $\mathbb{Z}_p\mathcal{RS}$.

Definition 5.2. *Define $\chi : \mathbb{Z}_p\mathcal{RS} \to \mathbb{C}^*$ by*

$$\chi((a, a' + ub', a'' + ub'' + u^2d'')) = (-1)^{a+a'+b'+a''+b''+d''}.$$

It can be easily verified that χ-image of any non-zero ideal is always non-trivial and hence by Lemma 2.7, χ is a generating character on $\mathbb{Z}_p\mathcal{RS}$. Moreover, an element $a + ub + u^2d \in \mathcal{S}$ can be seen as an element of $\mathbb{Z}_p\mathcal{RS}$ as $(0, 0, a + ub + u^2d)$, and we consider $\chi(a + ub + u^2d) = \chi((0, 0, a + ub + u^2d))$.

Suppose $P = [p_{ij}]$ is a matrix of order p^6 with $p_{ij} = \chi(< f_{i-1}, f_{j-1} >)$ where χ is the generating character defined above and $f_{i-1}, f_{j-1} \in \mathbb{Z}_p\mathcal{RS}$ for all $i, j = 1, 2, \ldots, p^6$. It is to be noted that since $f_{i-1}f_{j-1} = f_{j-1}f_{i-1}$, the matrix P is symmetric. We find the matrix P for two different cases, $p = 2$ and $p \neq 2$.

First, we consider the case when $p = 2$. Here, we have

$$P = \begin{bmatrix} B & B \\ B & -B \end{bmatrix},$$

where

$$B = \begin{bmatrix} A & A & A & A \\ A & -A & -A & A \\ A & -A & A & -A \\ A & A & -A & -A \end{bmatrix}$$

is a matrix of order p^5 and A is a matrix of order p^3, given by

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

The matrix A is actually the submatrix of P, consisting of the first p^3 rows and the first p^3 columns. Similarly, the matrix B is a submatrix of P, consisting of the first p^5 rows and the first p^5 columns. In other words, the matrix A is generated by the first p^3 members of $\mathbb{Z}_p\mathcal{RS}$ whereas the matrix B is generated by the first p^5 members of $\mathbb{Z}_p\mathcal{RS}$.

Now, we discuss the case when $p \neq 2$. As we did in the previous case for $p = 2$, here as well, we describe the matrix P in terms of its submatrices B and A. Since P is symmetric, its ith row is the same as its ith column. First, we write P in p rows and p columns, and each one of their entries is given in terms of the submatrix B. In this way, the first two rows of P are, respectively, $(B \;\; B \;\; B \;\; \cdots \;\; B)$ and $(B \;\; -B \;\; B \;\; -B \;\; B \;\; \cdots \;\; B)$. Therefore, all the entries in the first row are B, and the second row has B and $-B$ alternatively. Since p is odd, and the matrix P has p submatrices in each of its rows, the last entry in the second row of P is B. The penultimate row is $(B \;\; \overbrace{-B \;\; -B \;\; \cdots \;\; -B}^{\frac{p-1}{2} \text{ entries}} \;\; \overbrace{B \;\; B \;\; \cdots \;\; B}^{\frac{p-1}{2} \text{ entries}})$, i.e., the first entry is B, and among the remaining $p-1$ entries, the first $\frac{p-1}{2}$ entries are $-B$ and the next $\frac{p-1}{2}$ entries are B. The last row is $(B \;\; B \;\; -B \;\; B \;\; -B \;\; B \;\; \cdots \;\; -B)$, i.e., the first two entries in the last row are B each, and then $-B$ and B appear alternatively. The appearance of the in-between rows varies as we consider different primes p. Similarly, we can find the matrices B and A. The matrix B has p^2 rows and p^2 columns, among which the first row is $(A \;\; A \;\; A \;\; \cdots \;\; A)$. The second row is

$$(\overbrace{A \;\; -A \;\; A \;\; -A \;\; \cdots \;\; A}^{p \text{ entries}} \;\; \underbrace{-A \;\; A \;\; -A \;\; A \;\; \cdots \;\; -A}_{p \text{ entries}} \;\; \cdots \;\; \overbrace{A \;\; -A \;\; A \;\; -A \;\; \cdots \;\; A}^{p \text{ entries}}),$$

where the first p entries and the immediate next p entries appear alternatively. The appearance of the remaining rows varies as we take different values of the prime p. The matrix A has p^3 rows and p^3 columns, among which the first row is $(1 \;\; 1 \;\; 1 \;\; \cdots \;\; 1)$. The second row is

$$(\overbrace{1 \;\; -1 \;\; 1 \;\; -1 \;\; \cdots \;\; 1}^{p \text{ entries}} \;\; \underbrace{-1 \;\; 1 \;\; -1 \;\; 1 \;\; \cdots \;\; -1}_{p \text{ entries}} \;\; \cdots \;\; \overbrace{1 \;\; -1 \;\; 1 \;\; -1 \;\; \cdots \;\; 1}^{p \text{ entries}}),$$

where the first p entries and the immediate next p entries appear alternatively. The appearance of the remaining rows varies with the different values of p.

The next Lemma follows from [4, Lemma 3.2].

Lemma 5.3. *Let $\mathfrak{C}$ be a $\mathbb{Z}_p\mathcal{RS}$-additive code of length n and $\mathfrak{C}^\perp$ be its dual. Let*

$$\hat{f}(z) = \sum_{v \in \mathbb{Z}_p^n \mathcal{R}^n \mathcal{S}^n} \chi(< z, v >) f(v).$$

Then

$$\sum_{v \in \mathfrak{C}^\perp} f(v) = \frac{1}{|\mathfrak{C}|} \sum_{u \in \mathfrak{C}} \hat{f}(u).$$

Now, we find the MacWilliams identity with respect to the complete weight enumerator.

Theorem 5.4. *If $\mathfrak{C}$ is a $\mathbb{Z}_p\mathcal{RS}$-additive code of length n then*

$$\mathcal{W}_{\mathbf{C}}^{(\mathfrak{C}^\perp)}(x_1, x_2, \ldots, x_{p^6}) = \frac{1}{|\mathfrak{C}|} \mathcal{W}_{\mathbf{C}}^{(\mathfrak{C})}(P \cdot (x_1, x_2, \ldots, x_{p^6})^T),$$

where $(x_1, x_2, \ldots, x_{p^6})^T$ denotes the transpose of $(x_1, x_2, \ldots, x_{p^6})$.

Proof. Now, using Lemma 5.3 for $f(v) = \prod_{i=1}^{p^6} x_i^{w_{f_{i-1}}(v)}$, the proof follows from [27, Corollary 8.2]. □

Example 5.5. *Consider the $\mathbb{Z}_2\mathcal{RS}$-additive code of length 2,*

$$\mathfrak{C}_2 = \langle \{(1, 0, 1 + u^2; 0, u, 0), (0, 1 + u, 0; 1, 0, 1 + u)\} \rangle.$$

Then $\mathfrak{C}_2$ is a linear code over $\mathbb{Z}_2$ of dimension 6 and

$$\begin{aligned} B_1 = \{&(1, 0, 1 + u^2; 0, u, 0), (0, 0, u; 0, 0, 0), (0, 0, u^2; 0, 0, 0), (0, 1 + u, 0; 1, 0, 1 + u), \\ &(0, u, 0; 0, 0, u + u^2), (0, 0, 0; 0, 0, u^2)\} \end{aligned}$$

forms a $\mathbb{Z}_2$-basis of $\mathfrak{C}_2$. Thus the dual code ${\mathfrak{C}_2}^\perp$ is also a linear code over $\mathbb{Z}_2$ of dimension 6 and

$$\begin{aligned} B_2 = \{&(0, 0, 0; 0, u, 0), (1, 0, 0; 0, 1, 0), (0, u, 0; 0, 0, u^2), (0, u, 0; 1, 0, 0), \\ &(0, 1 + u, 0; 0, 0, u + u^2), (1, 0, u^2; 0, 0, 0)\} \end{aligned}$$

is a $\mathbb{Z}_2$-basis of $\mathfrak{C}_2^\perp$. Now, according to Definition 5.1, the complete weight enumerator of $\mathfrak{C}_2$ is given by

$$\begin{aligned} &\mathcal{W}_{\mathbf{C}}^{(\mathfrak{C}_2)}(x_1, x_2, \ldots, x_{64}) \\ &= x_1^2 + x_{38}x_{17} + x_3x_1 + x_{40}x_{17} + 2x_5x_1 + x_{34}x_{17} + x_7x_1 + x_{36}x_{17} + x_{25}x_{36} + x_{62}x_{52} + x_{27}x_{36} + \\ &\quad x_{64}x_{52} + x_{29}x_{36} + x_{58}x_{52} + x_{31}x_{36} + x_{60}x_{52} + x_{17}x_7 + x_{54}x_{23} + x_{19}x_7 + x_{56}x_{23} + x_{21}x_7 + x_{50}x_{23} \\ &\quad + x_{23}x_7 + x_{52}x_{23} + x_9x_{38} + x_{46}x_{54} + x_{11}x_{38} + x_{48}x_{54} + x_{13}x_{38} + x_{42}x_{54} + x_{15}x_{38} + x_{44}x_{54} + \\ &\quad x_{38}x_{21} + x_3x_5 + x_{40}x_{21} + x_5^2 + x_{34}x_{21} + x_7x_5 + x_{36}x_{21} + x_{25}x_{40} + x_{62}x_{56} + x_{27}x_{40} + x_{64}x_{56} + \\ &\quad x_{29}x_{40} + x_{58}x_{56} + x_{31}x_{40} + x_{60}x_{56} + x_{17}x_3 + x_{54}x_{19} + x_{19}x_3 + x_{56}x_{19} + x_{21}x_3 + x_{50}x_{19} + x_{23}x_3 \\ &\quad + x_{52}x_{19} + x_9x_{34} + x_{46}x_{50} + x_{11}x_{34} + x_{48}x_{50} + x_{13}x_{34} + x_{42}x_{50} + x_{15}x_{34} + x_{44}x_{50}. \end{aligned}$$

Similarly, the complete weight enumerator of $\mathfrak{C}_2^{\perp}$ *is given by*

$$\begin{aligned}
&\mathcal{W}_{\mathbf{C}}^{(\mathfrak{C}_2^{\perp})}(x_1, x_2, \ldots, x_{64}) \\
&= x_1^2 + 2x_{37}x_1 + x_{25}x_7 + x_{61}x_7 + x_{17}x_{33} + x_{53}x_{33} + x_9x_{39} + x_{45}x_{39} + x_{17}x_5 + x_{53}x_5 + x_9x_3 + \\
&\quad x_{45}x_3 + x_{37}^2 + x_{25}x_{35} + x_{61}x_{35} + x_{33}x_9 + x_5x_9 + x_{57}x_{15} + x_{29}x_{15} + x_{49}x_{41} + x_{21}x_{41} + x_{41}x_{47} + \\
&\quad x_{13}x_{47} + x_{49}x_{13} + x_{21}x_{13} + x_{41}x_{11} + x_{13}x_{11} + x_{33}x_{45} + x_5x_{45} + x_{57}x_{43} + x_{29}x_{43} + x_1x_{17} + \\
&\quad x_{37}x_{17} + x_{25}x_{23} + x_{61}x_{23} + x_{17}x_{49} + x_{53}x_{49} + x_9x_{55} + x_{45}x_{55} + x_{17}x_{21} + x_{53}x_{21} + x_9x_{19} + \\
&\quad x_{45}x_{19} + x_1x_{53} + x_{37}x_{53} + x_{25}x_{51} + x_{61}x_{51} + x_{33}x_{25} + x_5x_{25} + x_{57}x_{31} + x_{29}x_{31} + x_{49}x_{57} + x_{21}x_{57} \\
&\quad + x_{41}x_{63} + x_{13}x_{63} + x_{49}x_{29} + x_{21}x_{29} + x_{41}x_{27} + x_{13}x_{27} + x_{33}x_{61} + x_5x_{61} + x_{57}x_{59} + x_{29}x_{59}.
\end{aligned}$$

Next, we define the Hamming weight enumerator and find the corresponding MacWilliams identity.

Definition 5.6. *Let* $\mathfrak{C}$ *be a* $\mathbb{Z}_p\mathcal{RS}$*-additive code of length* n*. Then the Hamming weight enumerator of* $\mathfrak{C}$ *is denoted by* $\mathcal{W}_{\mathbf{H}}^{(\mathfrak{C})}$ *and is defined by*

$$\mathcal{W}_{\mathbf{H}}^{(\mathfrak{C})}(x, y) = \sum_{c \in \mathfrak{C}} x^{n - wt_H(c)} y^{wt_H(c)} \ ,$$

where $wt_H(c)$ *is the Hamming weight of* c*.*

Like the complete weight enumerator, the Hamming weight enumerator is also a homogeneous polynomial of degree n. Further, we observe that

$$\mathcal{W}_{\mathbf{H}}^{(\mathfrak{C})}(x, y) = \mathcal{W}_{\mathbf{C}}^{(\mathfrak{C})}(x, y, y, \ldots, y),$$

which gives us a relation between the complete weight enumerator and the Hamming weight enumerator.

Theorem 5.7. *If* $\mathfrak{C}$ *is a* $\mathbb{Z}_p\mathcal{RS}$*-additive code of length* n*, then*

$$\mathcal{W}_{\mathbf{H}}^{(\mathfrak{C}^{\perp})}(x, y) = \frac{1}{|\mathfrak{C}|} \mathcal{W}_{\mathbf{H}}^{(\mathfrak{C})}(x + (p^6 - 1)y, x - y).$$

Proof. The proof follows from Theorem 5.4 and uses the relation between the complete weight enumerator and the Hamming weight enumerator. □

Example 5.8. *We consider the same code* $\mathfrak{C}_2$ *as in Example 5.5. Then by using the complete weight enumerator of* $\mathfrak{C}_2$*, found in Example 5.5, the Hamming weight enumerator of* $\mathfrak{C}_2$ *is*

$$\mathcal{W}_{\mathbf{H}}^{(\mathfrak{C}_2)}(x, y) = x^2 + 4xy + 59y^2,$$

and by using Theorem 5.7, the Hamming weight enumerator of $\mathfrak{C}_2^{\perp}$ *is given by*

$$\begin{aligned}
\mathcal{W}_{\mathbf{H}}^{(\mathfrak{C}_2^{\perp})}(x, y) &= \frac{1}{64} \mathcal{W}_{\mathbf{H}}^{(\mathfrak{C}_2)}(x + 63y, x - y) \\
&= \frac{1}{64} [(x + 63y)^2 + 4(x + 63y)(x - y) + 59(x - y)^2] \\
&= x^2 + 4xy + 59y^2.
\end{aligned}$$

5.2. The symmetrized weight enumerator and the Lee weight enumerator

First, we find the value of $wt_L(f_i)$, the Lee weight of f_i, $i = 0, 1, 2, \ldots, p^6 - 1$, where f_i are the members of $\mathbb{Z}_p\mathcal{RS}$ in the same order as we have considered earlier. Let $i = \sum_{j=0}^{5} a_j p^j$, where $0 \leq a_j \leq p - 1$ for $j = 0, 1, \ldots, 5$. Then $0 \leq i \leq p^6 - 1$, and the Lee weight of f_i is given by

$$\begin{aligned}
wt_L(f_i) &= wt_H(a_5) + wt_L(a_3 + ua_4) + wt_L(a_0 + ua_1 + u^2a_2) \\
&= wt_H(a_5) + wt_H(a_3 + a_4, \kappa a_4) + wt_H(a_0 + a_1 + a_2, \kappa(a_1 + a_2), a_1).
\end{aligned}$$

Thus, the Lee weights of the elements of $\mathbb{Z}_p\mathcal{RS}$ can vary from 0 to 6.

Now, we define the symmetrized weight enumerator.

Definition 5.9. *Let $\mathfrak{C}$ be a $\mathbb{Z}_p\mathcal{RS}$-additive code of length n. Then the symmetrized weight enumerator of $\mathfrak{C}$ is denoted by $\mathcal{W}_{\mathbf{S}}^{(\mathfrak{C})}$ and is defined by*

$$\mathcal{W}_{\mathbf{S}}^{(\mathfrak{C})}(W_0, W_1, W_2, \ldots, W_6) = \mathcal{W}_{\mathbf{C}}^{(\mathfrak{C})}\left(W_{wt_L(f_0)}, W_{wt_L(f_1)}, W_{wt_L(f_2)}, \ldots, W_{wt_L(f_{p^6-1})}\right),$$

where W_i represents the variable corresponding to the Lee weight i.

From the above definition, we have

$$\mathcal{W}_{\mathbf{S}}^{(\mathfrak{C})}(W_0, W_1, W_2, \ldots, W_6) = \sum_{c\in\mathfrak{C}} W_0^{n_0(c)} W_1^{n_1(c)} W_2^{n_2(c)} \cdots W_6^{n_6(c)}\ , \tag{1}$$

where $n_i(c) = |\{j : wt_L(c_j) = i,\ 0 \le j \le n-1\}|$ for $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathfrak{C}$ and $i = 0, 1, \ldots, 6$.

Remark 5.10. *Interestingly, we observe that the Lee weight of an element in the same position, say f_i, which is in the $(i+1)$-th position, can be different for different choices of p. For example,*

$$wt_L(f_{p^2+p}) = \begin{cases} 1 \text{ for } p = 2 \\ 3 \text{ otherwise} \end{cases}, \quad wt_L(f_{p^2+p+1}) = \begin{cases} 2 \text{ for } p \in \{2,3\} \\ 3 \text{ otherwise} \end{cases},$$

$$wt_L(f_{p^5-1}) = \begin{cases} 3 \text{ for } p = 2 \\ 4 \text{ for } p = 3 \\ 5 \text{ otherwise} \end{cases}, \quad \text{and so on.}$$

This makes the task of characterizing the Lee weights of all the p^6 elements of $\mathbb{Z}_p\mathcal{RS}$ for all primes p in general, very complicated. Below, we find the Lee weights of the elements of $\mathbb{Z}_p\mathcal{RS}$ for two different primes, $p = 2$ and $p = 3$, in Table 1 and Table 2, respectively.

Table 1. Lee weights of the members of $\mathbb{Z}_2\mathcal{RS}$

Members of $\mathbb{Z}_2\mathcal{RS}$	Lee weights
f_0	0
$f_1, f_5, f_6, f_8, f_{24}, f_{32}$	1
$f_3, f_4, f_7, f_9, f_{13}, f_{14}, f_{16}, f_{25}, f_{29}, f_{30}, f_{33}, f_{37}, f_{38}, f_{40}, f_{56}$	2
$f_2, f_{11}, f_{12}, f_{15}, f_{17}, f_{21}, f_{22}, f_{27}, f_{28}, f_{31}, f_{35}, f_{36}, f_{39}, f_{41}, f_{45}, f_{46}, f_{48}, f_{57}, f_{61}, f_{62}$	3
$f_{10}, f_{19}, f_{20}, f_{23}, f_{26}, f_{34}, f_{43}, f_{44}, f_{47}, f_{49}, f_{53}, f_{54}, f_{59}, f_{60}, f_{63}$	4
$f_{18}, f_{42}, f_{51}, f_{52}, f_{55}, f_{58},$	5
f_{50}	6

Due to the presence of a large number of elements, precisely $3^6 = 729$, in $\mathbb{Z}_3\mathcal{RS}$, we take a different approach to present the Lee weights of $\mathbb{Z}_3\mathcal{RS}$ in Table 2. Keeping the ascending order, we break the set $\mathbb{Z}_3\mathcal{RS}$ in 3^3 number of sets, each one of the sets containing 3^3 number of elements. Let $L^{(3)}$ represents the sequence of Lee weights of the first 27 elements of $\mathbb{Z}_3\mathcal{RS}$, i.e., $wt_L(f_0), wt_L(f_1), \ldots, wt_L(f_{26})$. Then, we have

$$L^{(3)} := 0, 1, 1, 3, 3, 2, 3, 2, 3, 2, 2, 1, 3, 2, 3, 1, 2, 2, 2, 1, 2, 1, 2, 2, 3, 3, 2.$$

For any positive integer i, let $L_{+i}^{(3)}$ represents the sequence

$$L_{+i}^{(3)} := 0+i, 1+i, 1+i, 3+i, 3+i, 2+i, 3+i, 2+i, 3+i, 2+i, 2+i, 1+i, 3+i, 2+i, 3+i,$$
$$1+i, 2+i, 2+i, 2+i, 1+i, 2+i, 1+i, 2+i, 2+i, 3+i, 3+i, 2+i.$$

Now, from Table 2, one can obtain the Lee weights of the elements of $\mathbb{Z}_3\mathcal{RS}$.

Table 2. **Lee weights of the members of $\mathbb{Z}_3\mathcal{RS}$**

Members of $\mathbb{Z}_3\mathcal{RS}$	Lee weights
$f_0, f_1, f_2, \ldots, f_{26}$	$L^{(3)}$
$f_{27}, f_{28}, f_{29}, \ldots, f_{53}$	$L_{+1}^{(3)}$
$f_{54}, f_{55}, f_{56}, \ldots, f_{80}$	$L_{+1}^{(3)}$
$f_{81}, f_{82}, f_{83}, \ldots, f_{107}$	$L_{+2}^{(3)}$
$f_{108}, f_{109}, f_{110}, \ldots, f_{134}$	$L_{+2}^{(3)}$
$f_{135}, f_{136}, f_{137}, \ldots, f_{161}$	$L_{+1}^{(3)}$
$f_{162}, f_{163}, f_{164}, \ldots, f_{188}$	$L_{+2}^{(3)}$
$f_{189}, f_{190}, f_{191}, \ldots, f_{215}$	$L_{+1}^{(3)}$
$f_{216}, f_{217}, f_{218}, \ldots, f_{242}$	$L_{+2}^{(3)}$
$f_{243}, f_{244}, f_{245}, \ldots, f_{269}$	$L_{+1}^{(3)}$
$f_{270}, f_{271}, f_{272}, \ldots, f_{296}$	$L_{+2}^{(3)}$
$f_{297}, f_{298}, f_{299}, \ldots, f_{323}$	$L_{+2}^{(3)}$
$f_{324}, f_{325}, f_{326}, \ldots, f_{350}$	$L_{+3}^{(3)}$
$f_{351}, f_{352}, f_{353}, \ldots, f_{377}$	$L_{+3}^{(3)}$
$f_{378}, f_{379}, f_{380}, \ldots, f_{404}$	$L_{+2}^{(3)}$
$f_{405}, f_{406}, f_{407}, \ldots, f_{431}$	$L_{+2}^{(3)}$
$f_{432}, f_{433}, f_{434}, \ldots, f_{458}$	$L_{+2}^{(3)}$
$f_{459}, f_{460}, f_{461}, \ldots, f_{485}$	$L_{+3}^{(3)}$
$f_{486}, f_{487}, f_{488}, \ldots, f_{512}$	$L_{+1}^{(3)}$
$f_{513}, f_{514}, f_{515}, \ldots, f_{539}$	$L_{+2}^{(3)}$
$f_{540}, f_{541}, f_{542}, \ldots, f_{566}$	$L_{+2}^{(3)}$
$f_{567}, f_{568}, f_{569}, \ldots, f_{593}$	$L_{+3}^{(3)}$
$f_{594}, f_{595}, f_{596}, \ldots, f_{620}$	$L_{+3}^{(3)}$
$f_{621}, f_{622}, f_{623}, \ldots, f_{647}$	$L_{+2}^{(3)}$
$f_{648}, f_{649}, f_{650}, \ldots, f_{674}$	$L_{+2}^{(3)}$
$f_{675}, f_{676}, f_{677}, \ldots, f_{701}$	$L_{+2}^{(3)}$
$f_{702}, f_{703}, f_{704}, \ldots, f_{728}$	$L_{+3}^{(3)}$

Using the same notations as used in Definition 5.9, we find the MacWilliams identity for the symmetrized weight enumerator.

Theorem 5.11. *If $\mathfrak{C}$ is a $\mathbb{Z}_p\mathcal{RS}$-additive code of length n, then*

$$\mathcal{W}_{\mathbf{S}}^{(\mathfrak{C}^{\perp})}(W_0, W_1, W_2, \ldots, W_6) = \frac{1}{|\mathfrak{C}|}\mathcal{W}_{\mathbf{S}}^{(\mathfrak{C})}(Q \cdot (W_0, W_1, W_2, \ldots, W_6)^T),$$

where Q is a matrix of order 7, and it is the coefficient matrix of the linearly independent non-zero rows of the matrix

$$P \cdot \left(W_{wt_L(f_0)}, W_{wt_L(f_1)}, W_{wt_L(f_2)}, \ldots, W_{wt_L(f_{p^6-1})}\right)^T.$$

Proof. The proof follows from Theorem 5.4 and Definition 5.9. □

Example 5.12. *For the code $\mathfrak{C}_2$, defined in Example 5.5, the symmetrized weight enumerator of $\mathfrak{C}_2$ is given by*

$$\begin{aligned}&\mathcal{W}_{\mathbf{S}}^{(\mathfrak{C}_2)}(W_0, W_1, W_2, \ldots, W_6)\\ &= W_0^2 + 5W_2^2 + W_3W_0 + 8W_3W_2 + 2W_2W_0 + W_1W_0 + 3W_1W_3 + 7W_3W_5 + 11W_4W_3\\ &\quad + 6W_4W_5 + 3W_3^2 + 4W_2W_1 + W_5W_1 + W_4W_1 + 4W_4W_2 + 4W_4^2 + 2W_5^2,\end{aligned}$$

and the symmetrized weight enumerator of $\mathfrak{C}_2^{\perp}$ *is given by*

$$\begin{aligned}&\mathcal{W}_{\mathbf{S}}^{(\mathfrak{C}_2^{\perp})}(W_0, W_1, W_2, \ldots, W_6)\\ &= W_0^2 + 2W_3W_0 + 3W_1^2 + 5W_4W_1 + 4W_2W_1 + 2W_5W_1 + 8W_4W_2 + 3W_2^2 + 3W_5W_2 + 2W_1W_3 +\\ &\quad 6W_4W_3 + 5W_3^2 + 2W_4^2 + 8W_3W_2 + W_0W_2 + 4W_3W_5 + 2W_5W_4 + W_0W_5 + W_1W_6 + W_4W_6.\end{aligned}$$

Next, we define the Lee weight enumerator.

Definition 5.13. *Let* $\mathfrak{C}$ *be a* $\mathbb{Z}_p\mathcal{RS}$*-additive code of length* n*. Then the Lee weight enumerator of* $\mathfrak{C}$ *is denoted by* $\mathcal{W}_{\mathbf{L}}^{(\mathfrak{C})}$ *and is defined by*

$$\mathcal{W}_{\mathbf{L}}^{(\mathfrak{C})}(x, y) = \mathcal{W}_{\mathbf{H}}^{(\overline{\phi}(\mathfrak{C}))}(x, y) = \sum_{c\in\mathfrak{C}} x^{6n-wt_L(c)}y^{wt_L(c)}.$$

The following result gives a relation between the Lee weight enumerator and the symmetrized weight enumerator.

Theorem 5.14. *Let* $\mathfrak{C}$ *be a* $\mathbb{Z}_p\mathcal{RS}$*-additive code of length* n*. Then*

$$\mathcal{W}_{\mathbf{L}}^{(\mathfrak{C})}(x, y) = \mathcal{W}_{\mathbf{S}}^{(\mathfrak{C})}(x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6).$$

Proof. Take $c \in \mathfrak{C}$. We have $wt_L(c) = n_1(c) + 2n_2(c) + 3n_3(c) + \cdots + 6n_6(c)$ and $n = n_0(c) + n_1(c) + n_2(c) + \cdots + n_6(c)$. Then

$$\begin{aligned}\mathcal{W}_{\mathbf{L}}^{(\mathfrak{C})}(x, y) &= \sum_{c\in\mathfrak{C}} x^{6n-wt_L(c)}y^{wt_L(c)}\\ &= \sum_{c\in\mathfrak{C}} x^{6n_0(c)+5n_1(c)+4n_2(c)+3n_3(c)+2n_4(c)+n_5(c)}\ y^{n_1(c)+2n_2(c)+3n_3(c)+\cdots+6n_6(c)}\\ &= \sum_{c\in\mathfrak{C}} (x^6)^{n_0(c)}(x^5y)^{n_1(c)}(x^4y^2)^{n_2(c)}(x^3y^3)^{n_3(c)}(x^2y^4)^{n_4(c)}(xy^5)^{n_5(c)}(y^6)^{n_6(c)}\\ &= \mathcal{W}_{\mathbf{S}}^{(\mathfrak{C})}(x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6). \qquad \text{(From Eq. (1))}\end{aligned}$$

□

Using Theorem 5.11 and Theorem 5.14, we find the MacWilliams identity corresponding to the Lee weight enumerator.

Theorem 5.15. *If* $\mathfrak{C}$ *is a* $\mathbb{Z}_p\mathcal{RS}$*-additive code of length* n*, then*

$$\mathcal{W}_{\mathbf{L}}^{(\mathfrak{C}^{\perp})}(x, y) = \frac{1}{|\mathfrak{C}|}\mathcal{W}_{\mathbf{L}}^{(\mathfrak{C})}(x + y, x - y).$$

Example 5.16. *Again, we consider the code* $\mathfrak{C}_2$*, defined in Example 5.5. Then using Theorem 5.14 and Example 5.12, the Lee weight enumerator of* $\mathfrak{C}_2$ *is*

$$\begin{aligned}&\mathcal{W}_{\mathbf{L}}^{(\mathfrak{C}_2)}(x, y)\\ &= x^{12} + 5x^8y^4 + x^9y^3 + 8x^7y^5 + 2x^{10}y^2 + x^{11}y + 3x^8y^4 + 7x^4y^8 + 11x^5y^7 + 6x^3y^9 + 3x^6y^6 + 4x^9y^3\\ &\quad + x^6y^6 + x^7y^5 + 4x^6y^6 + 4x^4y^8 + 2x^2y^{10}\\ &= x^{12} + x^{11}y + 2x^{10}y^2 + 5x^9y^3 + 8x^8y^4 + 9x^7y^5 + 8x^6y^6 + 11x^5y^7 + 11x^4y^8 + 6x^3y^9 + 2x^2y^{10}.\end{aligned}$$

Now, using the above expression and Theorem 5.15, the Lee weight enumerator of $\mathfrak{C}_2^{\perp}$ *is given by*

$$\begin{aligned}\mathcal{W}_{\mathbf{L}}^{(\mathfrak{C}_2^{\perp})}(x,y) &= \frac{1}{64}\mathcal{W}_{\mathbf{L}}^{(\mathfrak{C}_2)}(x+y,x-y)\\ &= \frac{1}{64}\Big[(x+y)^{12}+(x+y)^{11}(x-y)+2(x+y)^{10}(x-y)^2+5(x+y)^9(x-y)^3\\ &\quad +8(x+y)^8(x-y)^4+9(x+y)^7(x-y)^5+8(x+y)^6(x-y)^6+11(x+y)^5(x-y)^7\\ &\quad +11(x+y)^4(x-y)^8+6(x+y)^3(x-y)^9+2(x+y)^2(x-y)^{10}\Big]\\ &= x^{12}+4x^{10}y^2+6x^9y^3+5x^8y^4+14x^7y^5+15x^6y^6+10x^5y^7+6x^4y^8+2x^3y^9+x^2y^{10}.\end{aligned}$$

6. Conclusion

We have considered the Frobenius rings $\mathcal{R}$, $\mathcal{S}$, $\mathcal{RS}$, and $\mathbb{Z}_p\mathcal{RS}$, and studied the additive constacyclic codes over these rings. By defining suitable inner products, we have determined the generators of the constacyclic codes and their duals. Later, we have defined Gray maps on $\mathcal{R}$, $\mathcal{S}$, and $\mathbb{Z}_p\mathcal{RS}$, and studied the images under these maps. We have established a few results on the Gray images of additive cyclic and additive constacyclic codes. Weight enumerators have a significant role in calculating the probability of an incorrect message being received undetected by the receiver. Motivated by the importance of the weight enumerators, we have defined several weight enumerators, such as the complete weight enumerator, the Hamming weight enumerator, the symmetrized weight enumerator, and the Lee weight enumerator, and obtained the MacWilliams identities corresponding to each of these weight enumerators for the $\mathbb{Z}_p\mathcal{RS}$-additive codes.

There is a matrix P which appears in Theorem 5.4, in the MacWilliams identity for the complete weight enumerator. We have given the exact form of the matrix P for $p = 2$. However, for odd primes, we have provided only partially the form of P as it becomes quite challenging to present a general form of P for all odd primes. Thus, a complete general form of P for $p \neq 2$ is still open. Further, one may also consider more general rings such as $\mathbb{Z}_p \times \frac{\mathbb{Z}_p[u]}{\langle u^2\rangle} \times \cdots \times \frac{\mathbb{Z}_p[u]}{\langle u^k\rangle}$ with $k > 3$, as the code alphabet to study the additive constacyclic codes and the various weight enumerators.

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