J. Algebra Comb. Discrete Appl.
12(1) • 31–42

Received: 5 October 2024
Accepted: 8 November 2024

Journal of Algebra Combinatorics Discrete Structures and Applications

Separable additive quadratic residue codes over $\mathbb{Z}_2\mathbb{Z}_4$ and their applications

Research Article

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Abstract: This paper examines separable additive quadratic residue codes (QRCs) over $\mathbb{Z}_2\mathbb{Z}_4$ and their applications. The idempotent generators of these codes are obtained. Further, the properties of separable additive QRCs over $\mathbb{Z}_2\mathbb{Z}_4$ are studied including their idempotent generators. As applications, these codes are used to construct self-dual, self-orthogonal, additive complementary pair (ACP) codes, additive complementary dual (ACD) codes, and additive l-intersection pairs of codes over $\mathbb{Z}_2\mathbb{Z}_4$.

2010 MSC: 94B15, 94B60

Keywords: Separable additive quadratic residue codes, Idempotent generators, Self-dual codes, Additive complementary pair codes, Additive l-intersection pairs of codes

1. Introduction

Quadratic residue codes (QRC) over the finite field $\mathbb{F}_l$ are cyclic codes of prime length p where l is another prime which is a quadratic residue mod p. QRCs have been extensively studied because they have a rate close to $\frac{1}{2}$ and in many cases have a large minimum distance [15]. Over $\mathbb{Z}_2 = \mathbb{F}_2 = \{0, 1\}$ and $\mathbb{Z}_3 = \mathbb{F}_3 = \{0, 1, 2\}$, we have the binary $[7, 4, 3]$ Hamming code, and the binary $[23, 12, 7]$ and ternary $[11, 6, 5]$ Golay codes as examples of QRCs [14].

QRCs over the finite ring $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ were introduced in [13]. In [17], important properties of QRCs over $\mathbb{Z}_4$ such as idempotent generators, duals, and extended codes were studied.

Additive codes of length $n = \alpha + 2\beta$ over the mixed alphabet $\mathbb{Z}_2\mathbb{Z}_4$ were introduced in [6] and have subsequently received significant attention. A $\mathbb{Z}_2\mathbb{Z}_4$-additive code C is defined to be a subgroup of $\mathbb{Z}_2^\alpha\mathbb{Z}_4^\beta$

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https://doi.org/10.13069/jacodesmath.v12i1.353 ISSN 2148-838X

where $\alpha + 2\beta = n$ [6]. Note that if $\alpha = 0$, then C is a quaternary linear code over $\mathbb{Z}_4$ and if $\beta = 0$, then C is a binary linear code. In [16], it was shown that additive codes of length $n = \alpha + 2\beta$ over $\mathbb{Z}_2\mathbb{Z}_4$ have applications in steganography. Moreover, in [1], binary linear codes with good parameters were constructed as images of $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. Double cyclic codes over the mixed alphabet $\mathbb{Z}_2\mathbb{Z}_2$ were studied in [7]. In [11], double QRCs over $\mathbb{Z}_2\mathbb{Z}_2$ were examined and important properties of these codes such as idempotent generators, and self-dual and extended codes, were investigated. In this paper, we introduce the class of separable additive QRCs over $\mathbb{Z}_2\mathbb{Z}_4$. The generating polynomials of these codes are presented and their idempotent generators are given. It is shown that additive complementary dual (ACD) codes and self-orthogonal codes can be constructed as applications of separable QRCs over $\mathbb{Z}_2\mathbb{Z}_4$. We also give examples of self-orthogonal codes and ACD codes over $\mathbb{Z}_2\mathbb{Z}_4$ generated from separable QRCs over $\mathbb{Z}_2\mathbb{Z}_4$.

2. Preliminaries

To make the paper self-contained, this section presents the necessary definitions and required prior results. The reader is referred to [6] for more details about $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and [10] and [17] for more details about binary and quaternary QRCs over $\mathbb{Z}_2 = \{0, 1\}$ and $\mathbb{Z}_4 = \{0, 1, 2, 3\}$, respectively.

2.1. $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

Consider the finite rings $\mathbb{Z}_2 = \{0, 1\}$ and $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. Let $n = \alpha + 2\beta$ where α is odd and $R_{\alpha,\beta} = \mathbb{Z}_2[X]/\langle X^\alpha - 1\rangle \times \mathbb{Z}_4[X]/\langle X^\beta - 1\rangle$. A subset C of $\mathbb{Z}_2^\alpha$ is called a linear code of length α if C is a subspace of $\mathbb{Z}_2^\alpha$. A subset C of $\mathbb{Z}_4^\beta$ is called a linear code of length β if C is a subgroup of $\mathbb{Z}_4^\beta$. If C is a linear code over $\mathbb{Z}_2$ or $\mathbb{Z}_4$, then the hull of C is the linear code $H = \text{hull}(C) = C \cap C^\perp$, where $C^\perp$ is the Euclidian dual of C [2]. A subset $\mathcal{C}$ of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is called a $\mathbb{Z}_2\mathbb{Z}_4$-additive code if $\mathcal{C}$ is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, i.e. $\mathcal{C}$ is isomorphic to $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$ [6]. $\mathcal{C}$ is called a separable $\mathbb{Z}_2\mathbb{Z}_4$-additive code if $\mathcal{C} = C_X \times C_Y$, where C_X is a binary linear code and C_Y is a quaternary linear code [6]. Note that if $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, then $|C| = 2^r$ for some nonnegative integer r. The next two definitions introduce the dual of an additive code $\mathcal{C}$ over $\mathbb{Z}_2\mathbb{Z}_4$.

Definition 2.1. *[6] Let $u = (a_0a_1 \ldots a_{\alpha-1}|b_0b_1 \ldots b_{\beta-1})$ and $v = (d_0d_1 \ldots d_{\alpha-1}|e_0e_1 \ldots e_{\beta-1}) \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. The inner product $u \cdot v$ is defined as*

$$u \cdot v = \left[2\sum_{i=0}^{\alpha-1} a_i d_i + \sum_{i=0}^{\beta-1} b_i e_i\right] \bmod 4.$$

Definition 2.2. *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. Then the dual of $\mathcal{C}$ is the code*

$$\mathcal{C}^\perp = \left\{v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta : u \cdot v = 0 \ \forall u \in C\right\}.$$

In [5], it was shown that if C is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, then $\mathcal{C}^\perp$ is also a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, and if $\mathcal{C} = C_X \times C_Y$ is a separable $\mathbb{Z}_2\mathbb{Z}_4$-additive code, then $\mathcal{C}^\perp = C_X^\perp \times C_Y^\perp$.

Linear complementary pair (LCP) of codes and linear complementary dual (LCD) codes over finite fields were introduced in [12]. Subsequently, they have been studied extensively because of their applications in numerous areas such as cryptography and secret sharing [8, 12].

Definition 2.3. *[12] Let (C, D) be a pair of binary linear codes of length n. Then the pair (C, D) is called an LCP of codes if $C + D = \mathbb{Z}_2^n$ and $C \cap D = \{0\}$. If $D = C^\perp$, then C is called an LCD code.*

In [9], the definition of LCD codes was generalized to linear l-intersection codes over finite fields.

Definition 2.4. *Let (C, D) be a pair of binary linear codes of length n. Then the pair (C, D) is called a linear l-intersection pair of codes if $dim(C \cap D) = l$.*

In [3], the above concepts were generalized from finite fields to finite principal ideal rings. In [4], the concept of LCD codes over finite fields was generalized to ACD codes over $\mathbb{Z}_2\mathbb{Z}_4$.

Definition 2.5. *[4] Let (C, D) be a pair of additive codes over $\mathbb{Z}_2\mathbb{Z}_4$.*

1. *The pair (C, D) is called an additive complementary pair (ACP) of codes if $C + D = \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and $C \cap D = \{0\}$. If $D = C^\perp$, then C is called an additive complementary dual (ACD) code.*
2. *The pair (C, D) is called an additive l-intersection pair of codes if $|C \cap D| = 2^l$.*

Definition 2.6. *Two $\mathbb{Z}_2\mathbb{Z}_4$-additive codes C_1 and C_2 are (permutation) equivalent if there is a permutation of coordinates which sends C_1 to C_2.*

We now give the definition of additive cyclic codes over $\mathbb{Z}_2\mathbb{Z}_4$.

Definition 2.7. *[1] A subset $\mathcal{C}$ of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is called a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code if*

1. *$\mathcal{C}$ is an additive code, and*
2. *if $(a_0 a_1 \dots a_{\alpha-1} | b_0 b_1 \dots b_{\beta-1}) \in \mathcal{C}$, then*

$$(a_{\alpha-1} a_0 \dots a_{\alpha-2} | b_{\beta-1} b_0 \dots b_{\beta-2}) \in \mathcal{C}.$$

For an element $c = (a_0 a_1 \dots a_{\alpha-1} | b_0 b_1 \dots b_{\beta-1}) \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, define the polynomial

$$c(X) = \left(a_0 + a_1 X + \dots + a_{\alpha-1} X^{\alpha-1} | b_0 + b_1 X + \dots + b_{\beta-1} X^{\beta-1}\right),$$

in $R_{\alpha,\beta}$. This gives a one-to-one correspondence between elements in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and elements in $R_{\alpha,\beta}$. We know that $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes are identified as $\mathbb{Z}_4[X]$-submodules of $R_{\alpha,\beta}$ [1]. The structure of additive cyclic codes is given in the following theorem.

Theorem 2.8. *[5] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code of length $n = \alpha + \beta$ and type $(\alpha, \beta, \gamma, \delta, k)$. Then $C = \langle (b|0), (l|fh + 2f) \rangle$ where $fhg = X^\beta - 1, \gamma = \alpha - \deg(b) - \deg(h), \delta = \deg(g)$, $k = \alpha - \deg(\gcd(lg, b))$, and $|C| = 2^{\alpha - \deg(b)} 4^{\deg(g)} 2^{\deg(h)}$. If $l = 0$, then C is a separable $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code.*

2.2. Binary and quaternary quadratic residue codes

Let p and q be two prime numbers satisfying $p \equiv \pm 1 \bmod 8$ and $q \equiv \pm 1 \bmod 8$, and let $\phi : \mathbb{Z}_2[X] \to \mathbb{Z}_4[X]$ be the Hensel mapping. Further, let Q_p be the set of quadratic residue elements mod p and N_p be the set of non-residue elements mod p. It is known that $(X^p - 1) = (X - 1) f(X) h(X) \bmod 2$, where $f(X) = \prod_{r \in QR} (X - w^r)$ and $h(X) = \prod_{r \in NQR} (X - w^r)$. Similarly, $(X^q - 1) = (X - 1) g(X) k(X) \bmod 2$ and $X^q - 1 = (X - 1) g_4(X) k_4(X) \bmod 4$, where $g_4(X) = \phi(g(X)) = \phi\left(\prod_{r \in QR} (X - w^r)\right)$ and $k_4(X) = \phi(k(X)) = \phi\left(\prod_{r \in NQR} (X - w^r)\right)$.

The binary QRCs are the following four binary cyclic codes of prime length p over $\mathbb{F}_2$ where 2 is a quadratic residue modulo p

$$\begin{aligned} Q &= \langle f(X) \rangle, \\ N &= \langle h(X) \rangle, \\ Q' &= \langle (X - 1) f(X) \rangle, \\ N' &= \langle (X - 1) h(X) \rangle. \end{aligned}$$

From [17], it is known that $Q^{\perp} = Q'$ and $N^{\perp} = N'$ if $p \equiv -1 \bmod 8$, and $Q^{\perp} = N'$ and $N^{\perp} = Q'$ if $p \equiv 1 \bmod 8$. Q and N are equivalent codes and Q' and N' are also equivalent codes.

The quaternary QRCs are the following four cyclic codes of prime length q over $\mathbb{Z}_4$

$$\begin{aligned} Q_4 &= \langle g_4(X) \rangle, \\ N_4 &= \langle k_4(X) \rangle, \\ Q_4' &= \langle (X-1)\, g_4(X) \rangle, \\ N_4' &= \langle (X-1)\, k_4(X) \rangle. \end{aligned}$$

In [17], it was proven that $Q_4^{\perp} = Q_4'$ and $N_4^{\perp} = N_4'$ if $p \equiv -1 \bmod 8$ and $Q_4^{\perp} = N_4'$ and $N_4^{\perp} = Q_4'$ if $p \equiv 1 \bmod 8$. Q_4 and N_4 are equivalent codes and Q_4' is equivalent to N_4'.

Let $e_1(X) = \sum_{r \in Q_p} X^r, e_2(X) = \sum_{r \in N_p} X^r, e_1'(X) = \sum_{r \in Q_q} X^r$, and $e_2'(X) = \sum_{r \in N_q} X^r$. Further, let $j_2(X) = 1 + X + X^2 + \ldots + X^{p-1} \in \mathbb{Z}_2[X] / \langle X^p - 1 \rangle$ with corresponding codeword $\mathbf{1} = (1, 1, \ldots, 1)$ in $\mathbb{Z}_2^p$, and $j_4(X) = 1 + X + X^2 + \ldots + X^{q-1} \in \mathbb{Z}_4[X] / \langle X^q - 1 \rangle$ with corresponding codeword $\mathbf{1} = (1, 1, \ldots, 1)$ in $\mathbb{Z}_4^q$. Note that $e_1(X) + e_2(X) + j_2(X) = 1$ in $\mathbb{Z}_2[X]$ and $j_4(X) - e_1'(X) - e_2'(X) = 1$ in $\mathbb{Z}_4[X]$. The following two lemmas give the idempotent generators for binary and quaternary QRCs.

Lemma 2.9. *[10] Suppose that $p \equiv \pm 1 \bmod 8$. Then the idempotent generators for the binary QRCs are as follows*

	$p = -1 \bmod 8$	$p = 1 \bmod 8$
1.	$Q = \langle e_1(X) \rangle$	$Q = \langle 1 + e_2(X) \rangle$
2.	$N = \langle e_2(X) \rangle$	$N = \langle 1 + e_1(X) \rangle$
3.	$Q' = \langle 1 + e_2(X) \rangle$	$Q' = \langle e_1(X) \rangle$
4.	$N' = \langle 1 + e_1(X) \rangle$	$N' = \langle e_2(X) \rangle$

Lemma 2.10. *[17] Suppose that $p \equiv \pm 1 \bmod 8$. Then the idempotent generators for the quaternary QRCs are as follows*

	$q = -1 + 8l, l$ *odd*	$q = -1 + 8l, l$ *even*
1.	$Q_4 = \langle e_1'(X) + 2e_2'(X) \rangle$	$Q_4 = \langle 3e_1'(X)) \rangle$
2.	$N_4 = \langle 2e_1'(X) + e_2'(X) \rangle$	$N_4 = \langle 3e_2'(X)) \rangle$
3.	$Q_4' = \langle 1 + 2e_1'(X) + 3e_2'(X) \rangle$	$Q_4' = \langle 1 + e_2'(X)) \rangle$
4.	$N_4' = \langle 1 + 3e_1'(X) + 2e_2'(X) \rangle$	$N_4 = \langle 1 + e_1'(X)) \rangle$

	$q = 1 + 8l, l$ *odd*	$q = 1 + 8l, l$ *even*
1.	$Q_4 = \langle 1 + 3e_2'(X) + 2e_1'(X) \rangle$	$Q_4 = \langle 1 + e_2'(X) \rangle$
2.	$N_4 = \langle 1 + 3e_1'(X) + 2e_2'(X) \rangle$	$N_4 = \langle 1 + e_1'(X) \rangle$
3.	$Q_4' = \langle 2e_2'(X) + e_1'(X) \rangle$	$Q_4' = \langle 3e_1'(X) \rangle$
4.	$N_4' = \langle e_2'(X) + 2e_1'(X) \rangle$	$N_4' = \langle 3e_2'(X) \rangle$

3. Separable $\mathbb{Z}_2\mathbb{Z}_4$-additive QRCs and their properties

In this section, we define and study the properties of separable $\mathbb{Z}_2\mathbb{Z}_4$-additive QRCs.

Definition 3.1. *The separable $\mathbb{Z}_2\mathbb{Z}_4$-additive QRCs are*

1. $L_1 = \langle (f(X) | 0), (0 | g_4(X)) \rangle$,

2. $L_2 = \langle (h(X)|0), (0|k_4(X)) \rangle$,

3. $L_1' = \langle ((X-1) f(X)|0), (0|(X-1) g_4(X)) \rangle$,

4. $L_2' = \langle ((X-1) h(X)|0), (0|(X-1) k_4(X)) \rangle$.

We are interested in finding the idempotent generators for these separable $\mathbb{Z}_2\mathbb{Z}_4$-additive QRCs. Let $\mathcal{C} = C_X \times C_Y$ be a separable $\mathbb{Z}_2\mathbb{Z}_4$-additive code of length n. The next lemma gives the idempotent generators for C based on the idempotent generators for the codes C_X and C_Y.

Lemma 3.2. *Let $\mathcal{C} = C_X \times C_Y$ be a separable $\mathbb{Z}_2\mathbb{Z}_4$-additive code of length n. Suppose that the binary and quaternary cyclic codes C_X and C_Y have idempotent generators s_1 and s_2, respectively. Then $C = \langle (s_1|0), (0|s_2) \rangle$.*

Proof. Let $c = (c_1|c_2) \in \mathcal{C} = C_X \times C_Y$ so then $c_1 \in C_X = \langle s_1 \rangle$ and $c_2 \in C_Y = \langle s_2 \rangle$. Hence, $c_1 = q_1 s_1$ and $c_2 = q_2 s_2$ so $c = (c_1|c_2) = q_1(s_1|0) + q_2(0|s_2) \Rightarrow C \subseteq \langle (s_1|0), (0|s_2) \rangle$. Now suppose that $c = (c_1|c_2) \in \langle (s_1|0), (0|s_2) \rangle$. Then $c_1 = q_1 s_1$, $c_2 = q_2 s_2$ and $c = q_1(s_1|0) + q_2(0|s_2) \in C_X \times C_Y$, and hence $\mathcal{C} = \langle (s_1|0), (0|s_2) \rangle$. □

As an application of Lemmas 3.2, 2.9, and 2.10, Propositions 3.3 and 3.4 present the idempotent generators for all separable $\mathbb{Z}_2\mathbb{Z}_4$-additive QRCs.

Proposition 3.3. *Suppose that $p \equiv 1 \bmod 8$ and $q \equiv 1 \bmod 8$. Then we have the following*

	$q-1=8l, l$ *odd*	$q-1=8l, l$ *even*
1.	$L_1 = \langle (1+e_2(X)\|0), (0\|1+3e_2'(X)+2e_1'(X)) \rangle$	$L_1 = \langle (1+e_2(X)\|0), (0\|1+e_2'(X)) \rangle$
2.	$L_2 = \langle (1+e_1(X)\|0), (0\|1+3e_1'(X)+2e_2'(X)) \rangle$	$L_2 = \langle (1+e_1(X)\|0), (0\|1+e_1'(X)) \rangle$
3.	$L_1' = \langle (e_1(X)\|0), (0\|2e_2'(X)+e_1'(X)) \rangle$	$L_1' = \langle (e_1(X)\|0), (0\|3e_1'(X)) \rangle$
4.	$L_2' = \langle (e_2(X)\|0), (0\|e_2'(X)+2e_1'(X)) \rangle$	$L_2' = \langle (e_2(X)\|0), (0\|3e_2'(X)) \rangle$

Proof. The proof follows from Lemmas 3.2, 2.9, and 2.10. □

Proposition 3.4. *Suppose that $p \equiv -1 \bmod 8$ and $q \equiv -1 \bmod 8$. Then we have the following*

	$q+1=8l, l$ *odd*	$q+1=8l, l$ *even*
1.	$L_1 = \langle (e_1(X)\|0), (0\|e_1'(X)+2e_2'(X)) \rangle$	$L_1 = \langle (e_1(X)\|0), (0\|3e_1'(X)) \rangle$
2.	$L_2 = \langle (e_2(X)\|0), (0\|2e_1'(X)+e_2'(X)) \rangle$	$L_2 = \langle (e_2(X)\|0), (0\|3e_2'(X)) \rangle$
3.	$L_1' = \langle (1+e_2(X)\|0), (0\|1+2e_1'(X)+3e_2'(X)) \rangle$	$L_1' = \langle (1+e_2(X)\|0), (0\|1+e_2'(X)) \rangle$
4.	$L_2' = \langle (1+e_1(X)\|0), (0\|1+3e_1'(X)+2e_2'(X)) \rangle$	$L_2' = \langle (1+e_1(X)\|0), (0\|1+e_1'(X)) \rangle$

Proof. The proof follows from Lemmas 3.2, 2.9, and 2.10. □

The next theorem presents some properties of separable $\mathbb{Z}_2\mathbb{Z}_4$-additive QRCs.

Theorem 3.5. *Suppose that $p \equiv -1 \bmod 8$ and $q \equiv -1 \bmod 8$. Then*

1. *L_1 and L_1' are permutation equivalent to L_2 and L_2', respectively,*

2. $|L_1| = 2^{\frac{p+1}{2}} 4^{\frac{q+1}{2}} = |L_2|$,

3. $|L_1'| = 2^{\frac{p-1}{2}} 4^{\frac{q-1}{2}} = |L_2'|$.

Proof. Part 1 follows from the fact that Q and N are equivalent, Q' and N' are equivalent, Q_4 and N_4 are equivalent, and Q'_4 is equivalent to N'_4.

For Part 2, $|L_1| = |Q|\,|Q_4| = 2^{\frac{p+1}{2}}4^{\frac{q+1}{2}}$ and $|L_2| = |N|\,|N_4| = 2^{\frac{p+1}{2}}4^{\frac{q+1}{2}}$.

The proof of Part 3 is similar to that of Part 2. □

We also have the following theorem.

Theorem 3.6. *Suppose that* $p \equiv 1 \bmod 8$ *and* $q \equiv 1 \bmod 8$. *Then*

1. L_1 *and* L'_1 *are permutation equivalent to* L_2 *and* L'_2*, respectively,*
2. $|L_1| = 2^{\frac{p+1}{2}}4^{\frac{q+1}{2}} = |L_2|$,
3. $|L'_1| = 2^{\frac{p-1}{2}}4^{\frac{q-1}{2}} = |L'_2|$.

Proof. The proof is similar to that of Theorem 3.5. □

4. Classification of separable additive QRCs over $\mathbb{Z}_2\mathbb{Z}_4$

In this section, we provide a classification of separable additive QRCs over $\mathbb{Z}_2\mathbb{Z}_4$. Some applications are also given. Recall the following theorems from [10] and [13].

Theorem 4.1. *[10] Let* C_i *be a cyclic code of length* n *over* $\mathbb{F}_q$ *with idempotent generators* $f_i(X)$, $i = 1, 2$. *Then* $C_1 \cap C_2$ *and* $C_1 + C_2$ *have idempotent generators* $f_1(X)f_2(X)$ *and* $f_1(X) + f_2(X) - f_1(X)f_2(X)$*, respectively.*

Theorem 4.2. *[13] Let* C_i *be a cyclic code of length* n *over* $\mathbb{Z}_4$ *with idempotent generators* $f_i(X)$, $i = 1, 2$. *Then* $C_1 \cap C_2$ *and* $C_1 + C_2$ *have idempotent generators* $f_1(X)f_2(X)$ *and* $f_1(X) + f_2(X) - f_1(X)f_2(X)$*, respectively.*

Theorem 4.3. *Suppose that* $p \equiv -1 \bmod 8$ *and* $q \equiv -1 \bmod 8$. *Then* $L_1^{\perp} = L'_1$, $L_2^{\perp} = L'_2$, *and* L'_1 *and* L'_2 *are self-orthogonal.*

Proof. Suppose that $p \equiv -1 \bmod 8$ and $q \equiv -1 \bmod 8$. Since $L_1 = \langle (f(X)|0), (0|g_4(X)) \rangle = Q \times Q_4$ is a separable additive code over $\mathbb{Z}_2\mathbb{Z}_4$, then

$$L_1^{\perp} = Q^{\perp} \times Q_4^{\perp} = Q' \times Q'_4 = \langle ((X-1)f(X)|0), (0|(X-1)g_4(X)) \rangle = L'_1.$$

Similarly, we have $L_2 = \langle (h(X)|0), (0|k_4(X)) \rangle = N \times N_4$ and $L_2^{\perp} = N^{\perp} \times N_4^{\perp} = N' \times N'_4 = \langle ((X-1)h(X)|0), (0|(X-1)k_4(X)) \rangle = L'_2$. Note that

$$L'_1 = \langle ((X-1)f(X)|0), (0|(X-1)g_4(X)) \rangle \subseteq \langle (f(X)|0), (0|g_4(X)) \rangle = L_1,$$

and

$$L'_2 = \langle ((X-1)h(X)|0), (0|(X-1)k_4(X)) \rangle \subseteq \langle (h(X)|0), (0|k_4(X)) \rangle = L_2.$$

Hence, L'_1 and L'_2 are self-orthogonal. □

Theorem 4.4. *Suppose that* $p \equiv -1 \bmod 8$ *and* $q \equiv -1 \bmod 8$. *Then*

1. $L_1 + L_2 = \mathbb{Z}_2^p\mathbb{Z}_4^q$,
2. $L_1 \cap L_2 = \langle (j_2(X)|0), (0|j_4(X)) \rangle$ *and the pair of codes* L_1 *and* L_2 *are a* 3*-intersection pair of codes,*

3. the codes L_1' and L_2' are a 0-intersection pair of codes and $L_1' + L_2' = \langle (1 + j_2(X)|0), (0|1 + j_4(X))\rangle$.

Proof. Suppose that $p \equiv -1 \bmod 8$ and $q \equiv -1 \bmod 8$. We will prove Parts 1 and 2. The proof of Part 3 is similar.

For Part 1, let $q = -1 + 8l$ where l is an odd integer. By Proposition 3.4, we have $L_1 = \langle (e_1(X)|0), (0|e_1'(X) + 2e_2'(X))\rangle$ and $L_2 = \langle (e_2(X)|0), (0|2e_1'(X) + e_2'(X))\rangle$. Since $(e_1'(X))^2 = e_1'(X) + 2e_2'(X)$, $(e_2'(X))^2 = e_2'(X) + 2e_1'(X)$ and $e_1'(X)e_2'(X) = e_1'(X) + e_2'(X) + 3$, we get

$$\begin{aligned}
(e_1'(X) + 2e_2'(X))(2e_1'(X) + e_2'(X)) &= 2(e_1'(X))^2 + e_1'(X)e_2'(X) + 4e_1'(X)e_2'(X) + 2(e_2'(X))^2 \\
&= 2(e_1'(X))^2 + e_1'(X)e_2'(X) + 2(e_2'(X))^2 \\
&= 2e_1'(X) + 4e_2'(X) + e_1'(X) + e_2'(X) + 3 + 4e_1'(X) \\
&\quad + 2e_2'(X) \\
&= 3e_1'(X) + 3e_2'(X) + 3 \\
&= -j_4(X).
\end{aligned}$$

By Theorems 4.1 and 4.2, we get that $L_1 + L_2 = \langle (w_1(X)|0), (0|w_2(X))\rangle$ where

$$\begin{aligned}
w_1(X) &= e_1(X) + e_2(X) - e_1(X)e_2(X) \\
&= e_1(X) + e_2(X) - 1 - e_1(X) - e_2(X) = 1,
\end{aligned}$$

and

$$\begin{aligned}
w_2(X) &= e_1'(X) + 2e_2'(X) + 2e_1'(X) + e_2'(X) - (e_1'(X) + 2e_2'(X))(2e_1'(X) + e_2'(X)) \\
&= -e_1'(X) - e_2'(X) + j_4(X) = 1.
\end{aligned}$$

Thus, $L_1 + L_2 = \mathbb{Z}_2^p\mathbb{Z}_4^q$. Again by Theorems 4.1 and 4.2, we have

$$\begin{aligned}
L_1 \cap L_2 &= \langle (e_1(X)e_2(X)|0), (0|(e_1'(X) + 2e_2'(X))(2e_1'(X) + e_2'(X))\rangle \\
&= \langle (j_2(X)|0), (0| - j_4(X))\rangle.
\end{aligned}$$

Thus, $L_1 \cap L_2 = \langle (j_2(X)|0), (0|j_4(X))\rangle$ and $|L_1 \cap L_2| = 2^1 4^1 = 2^3$, so L_1 and L_2 are a 3-intersection pair of codes.

For Part 2, let $q = -1 + 8l$ where l is an even integer. By Proposition 3.4, we have $L_1 = \langle (e_1(X)|0), (0|3e_1'(X))\rangle$ and $L_2 = \langle (e_2(X)|0), (0|3e_2'(X))\rangle$. Since $(e_1'(X))^2 = 3e_1'(X)$, $(e_2'(X))^2 = 3e_2'(X)$ and $e_1'(X)e_2'(X) = 3e_1'(X) + 3e_2'(X) + 3$, we get $L_1 + L_2 = \langle (w_3(X)|0), (0|w_4(X))\rangle$, where

$$\begin{aligned}
w_3(X) &= e_1(X) + e_2(X) - e_1(X)e_2(X) \\
&= e_1(X) + e_2(X) - 1 - e_1(X) - e_2(X) = 1,
\end{aligned}$$

and

$$\begin{aligned}
w_4(X) &= 3e_1'(X) + 3e_2'(X) - (3e_1'(X)3e_2'(X)) \\
&= 3e_1'(X) + 3e_2'(X) + j_4(X) \\
&= 1.
\end{aligned}$$

Thus, $L_1 + L_2 = \langle (w_3(X)|0), (0|w_4(X))\rangle = \langle (1|0), (0|1)\rangle = \mathbb{Z}_2^p\mathbb{Z}_4^q$. Moreover, we have

$$\begin{aligned}
L_1 \cap L_2 &= \langle (e_1(X)e_2(X)|0), (0|e_1'(X)e_2'(X))\rangle \\
&= \langle (j_2(X)|0), (0|j_4(X))\rangle.
\end{aligned}$$

Hence, $L_1 \cap L_2 = \langle (j_2(X)|0), (0|j_4(X))\rangle$ and $|L_1 \cap L_2| = 2^1 4^1 = 2^3$. Therefore, L_1 and L_2 are a 3-intersection pair of codes. □

Similar to Theorems 4.3 and 4.4, we obtain the following theorem.

Theorem 4.5. *Suppose that* $p \equiv 1 \bmod 8$ *and* $q \equiv 1 \bmod 8$. *Then*

1. $L_1^{\perp} = L_2'$ *and* $L_2^{\perp} = L_1'$,
2. $L_1 + L_2 = \mathbb{Z}_2^p \mathbb{Z}_4^q$,
3. $L_1 \cap L_2 = \langle (j_2(X)|0), (0|j_4(X)) \rangle$ *and* L_1 *and* L_2 *are a* 3*-intersection pair of codes,*
4. *the codes* L_1' *and* L_2' *are a* 0*-intersection pair of codes and* $L_1' + L_2' = \langle (1 + j_2(X)|0), (0|1 + j_4(X)) \rangle$.

Proof. The proof is similar to that of Theorems 4.3 and 4.4 and so is omitted. □

Theorem 4.6. *Suppose that* $p \equiv -1 \bmod 8$ *and* $q + 1 = 8l$, *where* l *is even. Then*

1. $Hull(L_1) = Hull(L_1') = \langle (1 + e_2(X)|0), (0|1 + e_2'(X)) \rangle$,
2. $Hull(L_2) = Hull(L_2') = \langle (1 + e_1(X)|0), (0|1 + e_1'(X)) \rangle$.

Proof. Suppose that $p \equiv -1 \bmod 8$ and $q = -1 + 8l$ where l is even. Then by Proposition 3.4 and Theorem 4.3, we have $L_1 = \langle (e_1(X)|0), (0|3e_1'(X)) \rangle$, $L_1^{\perp} = L_1' = \langle (1 + e_2(X)|0), (0|1 + e_2'(X)) \rangle$, $L_2 = \langle (e_2(X)|0), (0|3e_2'(X)) \rangle$, and $L_2^{\perp} = L_2' = \langle (1 + e_1(X)|0), (0|1 + e_1'(X)) \rangle$. Applying Theorem 4.1, we obtain for Part 1

$$\begin{aligned}
\text{Hull}(L_1) &= \langle (e_1(X)(1 + e_2(X))|0), (0|3e_1'(X)(1 + e_2'(X))) \rangle \\
&= \langle (e_1(X) + e_1(X)e_2(X)|0), (0|3e_1'(X) + 3e_1'(X)e_2'(X)) \rangle \\
&= \langle (e_1(X) + 1 + e_1(X) + e_2(X)|0), (0|3e_1'(X) + 3(3 + 3e_1'(X) + 3e_2'(X))) \rangle \\
&= \langle (1 + e_2(X)|0), (0|1 + e_2'(X)) \rangle,
\end{aligned}$$

and for Part 2

$$\begin{aligned}
\text{Hull}(L_2) &= \langle (e_2(X)(1 + e_1(X))|0), (0|3e_2'(X)(1 + e_1'(X))) \rangle \\
&= \langle (e_2(X) + e_1(X)e_2(X)|0), (0|3e_2'(X) + 3e_1'(X)e_2'(X)) \rangle \\
&= \langle (e_2(X) + 1 + e_1(X) + e_2(X)|0), (0|3e_2'(X) + 3(3 + 3e_1'(X) + 3e_2'(X))) \rangle \\
&= \langle (1 + e_1(X)|0), (0|1 + e_1'(X)) \rangle.
\end{aligned}$$

□

Theorem 4.7. *Suppose that* $p \equiv -1 \bmod 8$ *and* $q + 1 = 8l$ *where* l *is odd. Then*

1. $Hull(L_1) = Hull(L_1') = \langle (1 + e_2(X)|0), (0|1 + 2e_1'(X) + 3e_2'(X)) \rangle$,
2. $Hull(L_2) = Hull(L_2') = \langle (1 + e_1(X)|0), (0|1 + 3e_1'(X) + 2e_2'(X)) \rangle$.

Proof. The proof is similar to that of Theorem 4.6. □

Corollary 4.8. *Suppose that* $p \equiv -1 \bmod 8$ *and* $q \equiv -1 \bmod 8$. *Then the QRCs* L_1, L_2, L_1', *and* L_2' *are not ACD.*

Proof. By Theorems 4.6 and 4.7, $C \cap C^{\perp} \neq \{0\}$ for $C = L_1$, L_2, L_1', and L_2'. Therefore, these codes are not ACD. □

Theorem 4.9. *Suppose that* $p \equiv 1 \bmod 8$ *and* $q \equiv 1 \bmod 8$. *Then the QRCs* L_1, L_2, L_1', *and* L_2' *are ACD.*

Proof. We will prove that L_1 is an ACD code. The proof for the other codes is similar. Suppose that $p \equiv 1 \bmod 8$ and $q \equiv 1 \bmod 8$.

Case 1: Assume that $q-1=8l$, where l is an odd integer. Then by Proposition 3.3 and Theorem 4.5, we have that

$$L_1 = \langle (1+e_2(X)|0)\,, (0|1+3e_2'(X)+2e_1'(X))\rangle \text{ and } L_1^{\perp} = L_2' = \langle (e_2(X)|0)\,, (0|e_2'(X)+2e_1'(X))\rangle.$$

Note that

$$(1+e_2(X))\,(e_2(X)) = e_2\,(X) + e_2^2\,(X) = 0,$$

and since $e_1'(X)^2 = e_1'(X) + 2e_2'(X)$ and $e_2'(X)^2 = 2e_1'(X) + e_2'(X)$, we have

$$\begin{aligned}(1+3e_2'(X)+2e_1'(X))\,(e_2'(X)+2e_1'(X)) &= e_2'(X)+2e_1'(X)+\\ &\quad 3(e_2'(X))^2+2e_2'(X)e_1'(X)+2e_1'(X)e_2'(X)\\ &= e_2'(X)+4e_1'(X)+3e_2'(X)=0.\end{aligned}$$

Hence, by Theorems 4.1 and 4.2 we have

$$\begin{aligned}L_1 + L_1^{\perp} &= \langle (1+e_2(X)+e_2(X)-0|0)\,, (0|1+3e_2'(X)+2e_1'(X)+e_2'(X)+2e_1'(X)-0)\rangle\\ &= \langle (1|0)\,, (0|1)\rangle \text{ and } L_1 \cap L_1^{\perp} = \langle 0|0\rangle\,.\end{aligned}$$

Case 2: Assume that $q-1=8l$, where l is an even integer. Then we have

$$L_1 = \langle (1+e_2(X)|0)\,, (0|1+e_2'(X))\rangle \text{ and } L_1^{\perp} = L_2' = \langle (e_2(X)|0)\,, (0|3e_2'(X))\rangle.$$

Note that

$$\begin{aligned}(1+e_2(X))\,e_2(X) &= e_2(X)+(e_2(X))^2\\ &= e_2(X)+e_2(X)=0,\end{aligned}$$

and since $e_1'(X)^2 = 3e_1'(X)$ and $e_2'(X)^2 = 3e_2'(X)$, we have

$$\begin{aligned}(1+e_2'(X))\,3e_2'(X) &= 3e_2'(X)+3(e_2'(X))^2\\ &= 3e_2'(X)+e_2'(X)=0.\end{aligned}$$

Therefore

$$\begin{aligned}L_1 + L_1^{\perp} &= \langle (1+e_2(X)+e_2(X)-0|0)\,, (0|1+e_2'(X)+3e_2'(X)-0)\rangle\\ &= \langle (1|0)\,, (0|1)\rangle \text{ and } L_1 \cap L_1^{\perp} = \langle 0|0\rangle\,,\end{aligned}$$

so L_1 is ACD. □

In [17], supplementary quaternary QRCs were defined to be the $\mathbb{Z}_4$-linear codes obtained by supplementing the codes Q_4' and N_4' with the all 2 q-tuple $2(1)^q$. Define the following $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

$$S_Q(q) = \langle Q_4', 2(1)^q\rangle\,, \text{ and } S_N(q) = \langle N_4', 2(1)^q\rangle.$$

As an application of the codes S_Q and S_N, we construct $\mathbb{Z}_2\mathbb{Z}_4$-additive codes in the following lemma.

Lemma 4.10. *Let*

$$\begin{aligned} D_1 &= Q \times S_Q(q), \\ D_2 &= N \times S_Q(q), \\ D_3 &= Q' \times S_Q(q), \\ D_4 &= N' \times S_Q(q), \end{aligned}$$

$$\begin{aligned} C_1 &= Q \times S_N(q), \\ C_2 &= N \times S_N(q), \\ C_3 &= Q' \times S_N(q), \\ C_4 &= N' \times S_N(q). \end{aligned}$$

1. *If* $p \equiv -1 \bmod 8$ *and* $q \equiv -1 \bmod 8$, *then* $D_1^\perp = D_3$, $D_2^\perp = D_4$, $C_1^\perp = C_3$, *and* $C_2^\perp = C_4$. *In addition,* D_3, D_4, C_3, *and* C_4 *are self-orthogonal codes.*

2. *If* $p \equiv 1 \bmod 8$ *and* $q \equiv 1 \bmod 8$, *then* $D_1^\perp = C_4$, $D_2^\perp = C_3$, $C_1^\perp = D_4$, *and* $C_2^\perp = D_3$.

Proof. For $p \equiv -1 \bmod 8$, we have $Q^\perp = Q'$ and $N^\perp = N'$, and for $p = 1 \bmod 8$, we have $Q^\perp = N'$ and $N^\perp = Q'$. Further, by [17, Proposition 11.19], if $q = -1 \bmod 8$, then $S_Q(q)$ and $S_N(q)$ are self-dual codes and if $q \equiv 1 \bmod 8$, then $S_Q^\perp(q) = S_N(q)$ and $S_N^\perp(q) = S_Q(q)$. This completes the proof of Part 1.

Suppose that $p \equiv -1 \bmod 8$ and $q \equiv -1 \bmod 8$. Since $D_1 = Q \times S_Q(q)$ is a separable additive code over $\mathbb{Z}_2\mathbb{Z}_4$, then $D_1^\perp = Q^\perp \times S_Q^\perp(q) = Q' \times S_Q(q) = D_3$. Similarly, we have $D_2^\perp = D_4$, $C_1^\perp = C_3$, and $C_2^\perp = C_4$. Hence, D_3, D_4, C_3, and C_4 are self-orthogonal codes. Now suppose that $p \equiv 1 \bmod 8$ and $q \equiv 1 \bmod 8$. Since $D_1 = Q \times S_Q(q)$ is a separable additive code over $\mathbb{Z}_2\mathbb{Z}_4$, then $D_1^\perp = Q^\perp \times S_Q^\perp(q) = N' \times S_N(q) = C_4$. Similarly, we have $D_2^\perp = C_3$, $C_1^\perp = D_4$, and $C_2^\perp = D_3$. This completes the proof of Part 2. □

It is clear that the codes D_1 and D_2 are equivalent and the codes D_3 and D_4 are equivalent. Furthermore, C_1 and C_2 are equivalent codes and C_3 and C_4 are equivalent codes.

5. Examples

In this section, we provide applications of our results and construct separable $\mathbb{Z}_2\mathbb{Z}_4$-additive QRCs that are self-orthogonal and ACD codes.

Example 5.1. *Let* $p = q = 7$. *If* w *is a primitive 7th root of unity over* $\mathbb{Z}_2$, *then* $X^7 - 1 = (X-1) f(X) h(X) \bmod 2$ *where* $f(X) = \prod_{r \in QR} (X - w^r) = X^3 + X + 1$ *and* $h(X) = \prod_{r \in NQR} (X - w^r) = X^3 + X^2 + 1$. *We also have* $X^7 - 1 = (X-1) g_4(X) k_4(X) \bmod 4$ *where* $g_4(X) = \phi(f(X)) = \phi\left(\prod_{r \in QR} (X - w^r)\right) = X^3 + 2X^2 + X + 3$ *and* $k_4(X) = \phi(h(X)) = \phi\left(\prod_{r \in NQR} (X - w^r)\right) = X^3 + 3X^2 + 2X + 3$. *Based on this factorization, we get the codes*

$$\begin{aligned} L_1' &= \left\langle \left((X-1)\left(X^3+X+1\right)|0\right), \left(0|\,(X-1)\left(X^3+2X^2+X+3\right)\right) \right\rangle, \\ L_2' &= \left\langle \left((X-1)\left(X^3+X^2+1\right)|0\right), \left(0|\,(X-1)\left(X^3+3X^2+2X+3\right)\right) \right\rangle. \end{aligned}$$

Since $p = q = -1 \bmod 8$, *from Theorem 4.3 we have that* L_1' *and* L_2' *are self-orthogonal codes of length* $n = 14$.

Example 5.2. *Let* $p = q = 17$. *If* w *is a primitive* $17th$ *root of unity over* $\mathbb{Z}_2$, *then* $X^{17} - 1 = (X-1) f(X) h(X) \bmod 2$ *where* $f(X) = \prod_{r \in QR} (X - w^r) = X^8 + X^7 + X^6 + X^4 + X^2 + X + 1$ *and* $h(X) = \prod_{r \in NQR} (X - w^r) = X^8 + X^5 + X^4 + X^3 + 1$. *We also have* $X^{17} - 1 = (X-1) g_4(X) k_4(X) \bmod 4$ *where* $g_4(X) = X^8 + X^7 + 3X^6 + 3X^4 + 3X^2 + X + 1$ *and* $k_4(X) = X^8 + 2X^6 + 3X^5 + X^4 + 3X^3 + 2X^2 + 1$. *Based on this factorization, we get that the codes*

$$\begin{aligned} L_1 &= \left\langle \left(X^8 + X^7 + X^6 + X^4 + X^2 + X + 1|0\right), \left(0|X^8 + X^7 + 3X^6 + 3X^4 + 3X^2 + X + 1\right) \right\rangle, \\ L_2 &= \left\langle \left(X^8 + X^5 + X^4 + X^3 + 1|0\right), \left(0|X^8 + 2X^6 + 3X^5 + X^4 + 3X^3 + 2X^2 + 1\right) \right\rangle, \\ L_1' &= \left\langle \begin{array}{c} \left((X-1)\left(X^8 + X^7 + X^6 + X^4 + X^2 + X + 1\right)|0\right), \\ \left(0|(X-1)\left(X^8 + X^7 + 3X^6 + 3X^4 + 3X^2 + X + 1\right)\right) \end{array} \right\rangle, \\ L_2' &= \left\langle \begin{array}{c} \left((X-1)\left(X^8 + X^5 + X^4 + X^3 + 1\right)|0\right), \\ \left(0|(X-1)\left(X^8 + 2X^6 + 3X^5 + X^4 + 3X^3 + 2X^2 + 1\right)\right) \end{array} \right\rangle. \end{aligned}$$

Since $p = q = 1 \bmod 8$, *from Theorem 4.9 we have that* L_1, L_2, L_1', *and* L_2' *are ACD codes of length* $n = 34$.

Example 5.3. *Let* $p = 17$ *and* $q = 41$. *If* w *is a primitive* $17th$ *root of unity over* $\mathbb{Z}_2$, *then* $X^{17} - 1 = (X-1) f(X) h(X) \bmod 2$ *where* $f(X) = \prod_{r \in QR} (X - w^r) = X^8 + X^7 + X^6 + X^4 + X^2 + X + 1$ *and* $h(X) = \prod_{r \in NQR} (X - w^r) = X^8 + X^5 + X^4 + X^3 + 1$. *We also have* $X^{41} - 1 = (X-1) g_4(X) k_4(X) \bmod 4$ *where* $g_4(X) = X^{20} + 2X^{19} + 3X^{18} + 3X^{17} + x^{16} + 3X^{15} + x^{14} + 3X^{11} + 3X^{10} + 3X^9 + x^6 + 3X^5 + x^4 + 3X^3 + 3X^2 + 2X + 1$, *and* $k_4(X) = X^{20} + 3X^{19} + X^{17} + X^{16} + 2X^{15} + X^{14} + 2X^{13} + 3X^{11} + X^{10} + 3X^9 + 2X^7 + X^6 + 2X^5 + X^4 + X^3 + 3X + 1$. *Based on this factorization, we get the codes*

$$\begin{aligned} L_1 &= \langle (f(X)|0), (0|g_4(X)) \rangle, \\ L_2 &= \langle (h(X)|0), (0|k_4(X)) \rangle, \\ L_1' &= \langle ((X-1)(f(X))|0), (0|(X-1)(g_4(X))) \rangle, \\ L_2' &= \langle ((X-1)(h(X))|0), (0|(X-1)(k_4(X))) \rangle. \end{aligned}$$

Since $p = q \equiv 1 \bmod 8$, *from Theorem 4.9 we have that* L_1, L_2, L_1', *and* L_2' *are ACD codes of length* $n = 58$.

6. Conclusion

In this paper, we introduced the class of separable additive QRCs over $\mathbb{Z}_2\mathbb{Z}_4$. The main properties of these codes and their duals were presented including the generator polynomials and idempotent generators. It was shown that ACD codes and self-orthogonal codes can be constructed as applications of QRCs over $\mathbb{Z}_2\mathbb{Z}_4$. We also presented examples of ACD codes and self-orthogonal QRCs over $\mathbb{Z}_2\mathbb{Z}_4$.

For future work, it will be interesting to generalize the results given to non-separable additive QRCs over $\mathbb{Z}_2\mathbb{Z}_4$ and study the existence of self-orthogonal and ACD codes of this type. Another interesting research topic would be to study the applications of the self-orthogonal and ACD constructed from separable additive QRCs over $\mathbb{Z}_2\mathbb{Z}_4$ in areas such as cryptography and secret-sharing.

Acknowledgment: The authors extend their sincere thanks to the reviewers for their thorough reading and invaluable feedback which have significantly contributed to an improved paper.

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