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On Albertson spectral properties of graphs with self-loops

Research Article

KV Madhumitha, Sabitha D'Souza, Swati Nayak

Abstract: The Albertson irregularity measure is defined as $Alb(\Gamma) = \sum_{uv\in E(\Gamma)} |d(u) - d(v)|$. In this work, the concept of Albertson energy is extended from simple graphs to graphs with self-loops. Also the expression for the Albertson eigenvalues of a graph with self-loops are given. Some bounds on the Albertson energy of graphs with self-loops and the spread of $Alb(\Gamma_S)$ are obtained. In the last section, the Albertson energy of complete, complete bipartite, crown and thorn graphs with self-loops are computed.

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Keywords: Self-loops, Albertson matrix, Albertson energy

1. Introduction

Let $\Gamma = (V, E)$ be a finite, simple, undirected graph. The order and size of Γ are given by $|V| = n$ and $|E| = m$, respectively. The number of edges incident on the vertex v is the degree of a vertex v in a graph Γ, denoted by $\deg(u)$ or $d_\Gamma(u)$. The concept, energy of a graph, was coined by I. Gutman in 1978 as the sum of the absolute values of all the eigenvalues of a graph [10], denoted by $E(\Gamma)$. This definition is a general formula to calculate total π-electron energies of conjugated hydrocarbon molecules which was calculated by Erich Huckel in Huckel molecular orbital theory. In 2022, I. Gutman et al. broadened the idea of graph energy from simple graphs to graphs with self-loops [11]. To learn more about the extended adjacency matrix and graph energy with self-loops, readers can refer to [1–5, 7, 13–15, 17**?** –21].

The spread of the matrix A is given by $S(A) = \max\{|\gamma_k - \gamma_j| : i, j = 1, \dots, n\}$, where $\gamma_k' s$ are the eigenvalues of matrix A. Thorn graph [8] is the graph obtained from Γ by attaching p_i pendant vertices to the vertex v_i of Γ, for $i = 1, 2, \dots, n$. Figure [1], [2], [3], [4] represents thorn cycle, thorn complete, thorn wheel, and thorn star graphs, respectively.

KV Madhumitha, Sabitha D'Souza, Swati Nayak (Corresponding Author); Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India (email: madhumithakv467@gmail.com, sabitha.dsouza@manipal.edu, swati.nayak@manipal.edu).

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A graph is considered regular if every vertex in it has the same degree. Calculations are frequently made easier by regularity. An irregular graph is one that has at least two distinct vertex degrees, making it non-regular. This has led to the definition of multiple irregularity measures. One such measure was proposed by Albertson in 1997 [6], which is given by $Alb(\Gamma) = \sum_{uv(\Gamma)} |d(u) - d(v)|$. In [12] authors introduced a new measure known as sigma index, given by $\sigma(\Gamma) = \sum_{uv(\Gamma)}(d(u) - d(v))^2$.

The Albertson matrix [6] of a graph Γ is a square matrix $A = [a_{ij}]$ of order n given by

$$a_{ij} = \begin{cases} |d(u_i) - d(u_j)|, & \text{if } u_i \sim u_j \\ 0, & \text{if } u_i \nsim u_j \\ 0, & \text{if } u_i = u_j. \end{cases}$$

Let $S \subseteq V(\Gamma)$ and $|S| = \alpha$. Let Γ_S be the graph obtained from the simple graph Γ, by attaching a self-loop to each of its vertices belonging to S. Let $E(\Gamma_S)$ and $d_{\Gamma_S}(u)$ represent edge set of Γ_S and degree of vertex u in Γ_S, respectively.

In this paper, the Albertson matrix for a graph with self-loops is defined as $Alb(\Gamma_S) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} |d(u_i) - d(u_j)|, & \text{if } u_i \sim u_j \\ 0, & \text{if } u_i \nsim u_j \\ 1, & \text{if } u_i = u_j \text{ and } u_i \in S. \end{cases}$$

Let $\gamma_1(\Gamma_S) \geq \gamma_2(\Gamma_S) \geq \cdots \geq \gamma_n(\Gamma_S)$ be the eigenvalues of $Alb(\Gamma_S)$. Then, the Albertson energy of a graph with self-loops is given by

$$Alb_E(\Gamma_S) = \sum_{k=1}^{n} \left|\gamma_k(\Gamma_S) - \frac{\alpha}{n}\right|.$$

Let $s_k = \left|\gamma_k(\Gamma_S) - \frac{\alpha}{n}\right|$, $k = 1, 2, \ldots, n$ denote the auxiliary eigenvalues of $Alb(\Gamma_S)$.

Lemma 1.1. *[16] Let a_k and b_k, $1 \leq k \leq n$ are non-negative real numbers. Then*

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \frac{n^2}{4}(M_1M_2 - m_1m_2)^2.$$

$M_1 = \max_{1\leq k\leq n} a_k$, $M_2 = \max_{1\leq k\leq n} b_k$, $m_1 = \min_{1\leq k\leq n} a_k$, $m_2 = \min_{1\leq k\leq n} b_k$.

Lemma 1.2. *[16] Suppose a_k and b_k, $1 \leq k \leq n$ are positive real numbers, then*

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{k=1}^{n} a_k b_k\right)^2.$$

$M_1 = \max_{1\leq k\leq n} a_k$, $M_2 = \max_{1\leq k\leq n} b_k$, $m_1 = \min_{1\leq k\leq n} a_k$, $m_2 = \min_{1\leq k\leq n} b_k$.

Lemma 1.3. *[16] Let a_k and b_k, $1 \leq k \leq n$ are positive real numbers, then*

$$\left|n\sum_{k=1}^{n} a_k b_k - \sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k\right| \leq \beta(n)(A - a)(B - b),$$

where a, b, A and B are real constants i.e., for each k, $1 \leq k \leq n$, $a \leq a_k \leq A$ and $b \leq b_k \leq B$. Further $\beta(n) = n\left\lfloor\frac{n}{2}\right\rfloor\left(1 - \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right)$.

Lemma 1.4. *[16] Let $y = (y_k)$ and $b = (b_k)$, $k = 1, 2, \ldots, n$ be real number sequence such that $\sum_{k=1}^{n} |y_k| = 1$ and $\sum_{k=1}^{n} y_k = 0$. Then*

$$\left|\sum_{k=1}^{n} b_k y_k\right| \leq \frac{1}{2}(\max_{1\leq i\leq n}(b_k) - \min_{1\leq i\leq n}(b_k)).$$

Lemma 1.5. *[9] Let A be an $n \times n$ Hermitian matrix $(n \geq 3)$ with eigenvalues $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$. Define $M(A) = \{2||A||^{\ 2} - \frac{2}{n}(tr(A))^2\}^{\frac{1}{2}}$. Then $\sqrt{\frac{2}{n}}M(A) \leq S(A) \leq M(A)$, where $||A||$ is the Euclidean norm and $tr(A)$ denote the trace of A.*

Lemma 1.6. *[22] Suppose p, q are non-negative integers, and suppose A, B, C, D are respectively $p \times p$, $p \times q$, $q \times p$, and $q \times q$ matrices of complex numbers. Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a matrix of order $(p + q)$. If A is invertible, then $det(M) = det(A)det(D - CA^{-1}B)$.*

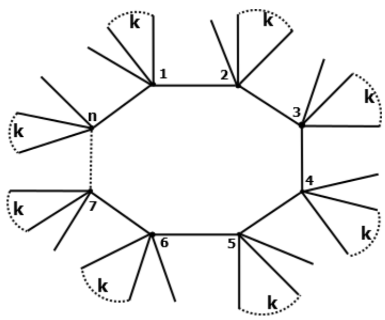

Figure 1. Thorn cycle

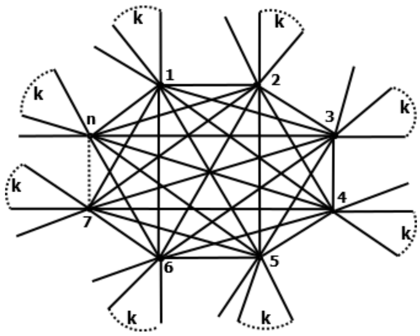

Figure 2. Thorn complete

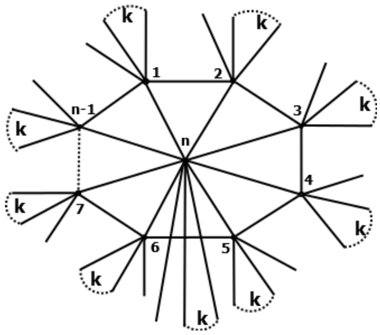

Figure 3. **Thorn wheel**

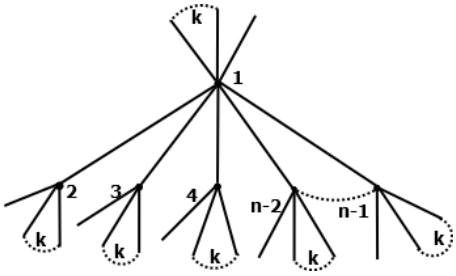

Figure 4. **Thorn star**

2. Properties

Theorem 2.1. *Let γ_k, $1 \leq k \leq n$ be the eigenvalues of $Alb(\Gamma_S)$. Then*

1. $\sum_{k=1}^{n} \gamma_k = \alpha$.

2. $\sum_{k=1}^{n} \gamma_k^2 = 2\sigma(\Gamma_S) + \alpha$.

Proof. 1. Sum of eigenvalues of $Alb(\Gamma_S)$ is equal to trace of $Alb(\Gamma_S)$, $\sum_{k=1}^{n} \gamma_k = \sum_{k=1}^{n} a_{kk} = |S| = \alpha$.

2. The sum of squares of eigenvalues of $Alb(\Gamma_S)$ is the trace of $[Alb(\Gamma_S)]^2$.

$$\begin{aligned}
\sum_{k=1}^{n} \gamma_k^2 &= \sum_{k=1}^{n} \sum_{k=1}^{n} a_{kj} a_{jk} \\
&= \sum_{k=1}^{n} a_{kk}^2 + \sum_{k \neq j} a_{kj} a_{jk} \\
&= |S| + 2\sigma(\Gamma_S) \\
\sum_{k=1}^{n} \gamma_k^2 &= \alpha + 2\sigma(\Gamma_S).
\end{aligned}$$

□

Theorem 2.2. *Let s_k, $1 \leq k \leq n$ be the auxiliary eigenvalues of $Alb(\Gamma_S)$. Then*

1. $\sum\limits_{k=1}^{n} s_k = 0$.
2. $\sum\limits_{k=1}^{n} s_k^2 = \alpha + 2\sigma(\Gamma_S) - \dfrac{\alpha^2}{n}$.

Proof. 1. We have,

$$\begin{aligned}\sum_{k=1}^{n} s_k &= \sum_{k=1}^{n}\left(\gamma_k(\Gamma_S) - \frac{|S|}{n}\right)\\ &= \sum_{k=1}^{n}\gamma_k(\Gamma_S) - \sum_{k=1}^{n}\frac{\alpha}{n}\\ &= 0.\end{aligned}$$

2. Also,

$$\begin{aligned}\sum_{k=1}^{n} s_k^2 &= \sum_{k=1}^{n}\left(\gamma_k(\Gamma_S) - \frac{|S|}{n}\right)^2\\ &= \sum_{k=1}^{n}\gamma_k(\Gamma_S)^2 + \sum_{k=1}^{n}\left(\frac{\alpha}{n}\right)^2 - 2\sum_{k=1}^{n}\gamma_i(\Gamma_S)\left(\frac{\alpha}{n}\right)\\ &= \alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n}.\end{aligned}$$

□

Theorem 2.3. *Let $\Gamma = (V, E)$ be a graph and $S \subseteq V(\Gamma)$. Let Γ_S be a graph with self-loops attached to all vertices of S. Then*

1. *If $\alpha = 0$, $Alb_E(\Gamma_S) = Alb_E(\Gamma)$.*
2. *If $\alpha = n$, $Alb_E(\Gamma_S) = Alb_E(\Gamma)$.*

Proof. 1. If $\alpha = 0$, then $\Gamma \simeq \Gamma_S$. Therefore, $Alb_E(\Gamma_S) = Alb_E(\Gamma)$.

2. If $\alpha = n$, then $\gamma_i(\Gamma_S) = \gamma_i(\Gamma) + 1$.

$$\begin{aligned}Alb_E(\Gamma_S) &= \sum_{i=1}^{n}|\gamma_i(\Gamma_S) - 1|\\ &= \sum_{i=1}^{n}|\gamma_i(\Gamma) + 1 - 1|\\ &= \sum_{i=1}^{n}|\gamma_i(\Gamma)|\\ &= Alb_E(\Gamma).\end{aligned}$$

□

3. Bounds

Theorem 3.1. *Let $s_1 \geq s_2 \geq \cdots \geq s_n$ be the auxiliary eigenvalues of $Alb(\Gamma_S)$. Then*

$$Alb_E(\Gamma_S) \geq \sqrt{n\left(\alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n}\right) - \frac{n^2}{4}(s_1 - s_n)^2}.$$

Equality holds if $\Gamma \cong (K_n)_S$ with $\alpha = n$.

Proof. On substituting $a_k = 1$ and $b_k = |s_k|$ in Lemma [1.1], we have

$$\sum_{k=1}^{n} 1 \sum_{k=1}^{n} |s_k|^2 - \left(\sum_{k=1}^{n} |s_k|\right)^2 \leq \frac{n^2}{4}(s_1 - s_n)^2$$

$$n\left(\alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n}\right) - (Alb_E(\Gamma_S))^2 \leq \frac{n^2}{4}(s_1 - s_n)^2$$

$$Alb_E(\Gamma_S) \geq \sqrt{n\left(\alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n}\right) - \frac{n^2}{4}(s_1 - s_n)^2}.$$

□

Theorem 3.2. *Let $s_1 \geq s_2 \geq \cdots \geq s_n$ be the auxiliary eigenvalues of $Alb(\Gamma_S)$. Then*

$$Alb_E(\Gamma_S) \geq \frac{2\sqrt{ns_1 s_n\left(\alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n}\right)}}{s_1 + s_n}.$$

Proof. On replacing $a_k = |s_k|$ and $b_k = 1$ in Lemma [1.2], we obtain

$$\sum_{k=1}^{n} |s_k|^2 \sum_{k=1}^{n} 1 \leq \frac{1}{4}\left(\sqrt{\frac{s_1}{s_n}} + \sqrt{\frac{s_n}{s_1}}\right)^2 \left(\sum_{k=1}^{n} |s_k|\right)^2$$

$$n\left(\alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n}\right) \leq \frac{1}{4}\frac{(s_1 + s_n)^2}{s_1 s_n}(Alb_E(\Gamma_S))^2$$

$$\frac{4(s_1 s_n)n\left(\alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n}\right)}{(s_1 + s_n)^2} \leq (Alb_E(\Gamma_S))^2$$

$$Alb_E(\Gamma_S) \geq \frac{2\sqrt{ns_1 s_n\left(\alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n}\right)}}{s_1 + s_n}.$$

□

Theorem 3.3. *Let $s_1 \geq s_2 \geq \cdots \geq s_n$ be the auxiliary eigenvalues of $Alb(\Gamma_S)$. Then*

$$Alb_E(\Gamma_S) \geq \sqrt{n\left(\alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n}\right) - \beta(n)(|s_1| - |s_n|)^2}.$$

Equality holds if $\Gamma \cong (K_n)_S$ with $\alpha = n$.

Proof. On substituting $a_k = |s_k| = b_k$, $a = |s_n| = b$ and $A = |s_1| = B$ in Lemma [1.3], we have

$$\left| n\sum_{k=1}^{n} |s_k|^2 - \left(\sum_{k=1}^{n} |s_k|\right)^2 \right| \leq \beta(n)(|s_1| - |s_n|)^2$$
$$\left| n\left(\alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n}\right) - (Alb_E(\Gamma_S))^2 \right| \leq \beta(n)(|s_1| - |s_n|)^2$$
$$Alb_E(\Gamma_S) \geq \sqrt{n\left(\alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n}\right) - \beta(n)(|s_1| - |s_n|)^2}.$$

□

Theorem 3.4. *Let $S(Alb(\Gamma_S))$ denote the spread of $Alb(\Gamma_S)$. Then*

$$\sqrt{\frac{2}{n}}\left(2\sum_{k=1}^{n} s_k^2\right)^{\frac{1}{2}} \leq S(Alb(\Gamma_S)) \leq \left(2\sum_{k=1}^{n} s_k^2\right)^{\frac{1}{2}}.$$

Left inequality is sharp if Γ is a $r-$regular graph on n vertices with consecutive $\frac{n}{2}$ self-loops.

Proof. We have $||A||^2 = \sigma(\Gamma_S) + \alpha$ and $[tr(Alb(\Gamma_S))]^2 = \alpha^2$.

On substituting this in Lemma [1.5], we have

$$\sqrt{\frac{2}{n}}\left(2(\sigma(\Gamma_S) + \alpha) - \frac{2}{n}\alpha^2\right)^{\frac{1}{2}} \leq S(Alb(\Gamma_S)) \leq \left(2(\sigma(\Gamma_S) + \alpha) - \frac{2}{n}\alpha^2\right)^{\frac{1}{2}}$$
$$\sqrt{\frac{2}{n}}\left(2\left(\sigma(\Gamma_S) + \alpha - \frac{\alpha^2}{n}\right)\right)^{\frac{1}{2}} \leq S(Alb(\Gamma_S)) \leq \left(2\left(\sigma(\Gamma_S) + \alpha - \frac{\alpha^2}{n}\right)\right)^{\frac{1}{2}}$$
$$\sqrt{\frac{2}{n}}\left(2\sum_{k=1}^{n} s_k^2\right)^{\frac{1}{2}} \leq S(Alb(\Gamma_S)) \leq \left(2\sum_{k=1}^{n} s_k^2\right)^{\frac{1}{2}}.$$

□

Theorem 3.5. *Let $S(Alb(\Gamma_S))$ denote the spread of $Alb(\Gamma_S)$. Then*

$$Alb(\Gamma_S) \geq \frac{2\alpha + 4\sigma(\Gamma_S) - \frac{4\alpha^2}{n}}{S(Alb(\Gamma_S))}.$$

Proof. On setting $b_k = \gamma_k$ and $y_k = \dfrac{\gamma_k - \frac{\alpha}{n}}{\sum_{k=1}^{n}\left|\gamma_k - \frac{\alpha}{n}\right|}$, for all $k = 1, 2, \ldots, n$ in Lemma [1.4], we obtain

$$\left|\sum_{k=1}^{n} b_k y_k\right| = \left|\sum_{k=1}^{n} \frac{\gamma_k^2 - \frac{\alpha}{n}\gamma_k}{\sum_{k=1}^{n}|\gamma_k - \frac{\alpha}{n}|}\right| \leq \frac{1}{2}\left(\max_{1\leq k\leq n}\gamma_k - \min_{1\leq k\leq n}\gamma_k\right) = \frac{1}{2}S(Alb(\Gamma_S)).$$

$$\left|\sum_{k=1}^{n}\frac{\gamma_k^2-\frac{\alpha}{n}\gamma_k}{\sum_{k=1}^{n}|\gamma_k-\frac{\alpha}{n}|}\right| \geq \frac{\sum_{k=1}^{n}\gamma_k^2}{\sum_{k=1}^{n}|\gamma_k-\frac{\alpha}{n}|}-\frac{\left|\frac{\alpha}{n}\sum_{k=1}^{n}\gamma_k\right|}{\sum_{k=1}^{n}|\gamma_k-\frac{\alpha}{n}|}=\frac{\alpha+2\sigma(\Gamma_S)-\frac{\alpha^2}{n}-\frac{\alpha^2}{n}}{Alb_E(\Gamma_S)}.$$

We have

$$\frac{1}{2}S(Alb(\Gamma_S)) \geq \frac{\alpha+2\sigma(\Gamma_S)-2\frac{\alpha^2}{n}}{Alb_E(\Gamma_S)}.$$

$$\implies Alb_E(\Gamma_S) \geq \frac{2\alpha+4\sigma(\Gamma_S)-\frac{4\alpha^2}{n}}{S(Alb(\Gamma_S))}.$$

□

4. Albertson energy of some graphs with self-loops

Theorem 4.1. *For complete graph K_n with $\alpha \geq 1$ self-loops,*

$$Alb_E(K_n)_S = \frac{2(\alpha n-\alpha^2)-n}{n}+\sqrt{1+16\alpha(n-\alpha)}.$$

Proof. For complete graph K_n with $\alpha \geq 1$ self-loops, we have

$$Alb(K_n)_S = \begin{bmatrix} I_\alpha & 2J_{\alpha\times(n-\alpha)} \\ 2J_{(n-\alpha)\times\alpha} & 0_{(n-\alpha)} \end{bmatrix}_n,$$

where J is all 1's matrix.

Let $W = \begin{bmatrix} Y \\ Z \end{bmatrix}$ be an eigenvector of order n, such that vector Y be of order α and vector Z be of order $n-\alpha$. Let $\gamma(\Gamma_S)$ be a eigenvalue of $Alb(K_n)_S$. Then,

$$[Alb(K_n)_S-\gamma I]\begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} (1-\gamma I)Y+(2J)Z \\ (2J)Y-\gamma IZ \end{bmatrix}_n. \tag{1}$$

Case 1. Let $Y = Y_j = e_1 - e_j$, $2 \leq j \leq \alpha$ and $Z = 0_{n-\alpha\times 1}$. Using Equation [1], $[1-\gamma I]Y_j + 0 = (1-\gamma)Y_j$ then, $\gamma = 1$ is the eigenvalue with multiplicity of at least $\alpha-1$ since there are $\alpha-1$ independent vectors of the form Y_j.

Case 2. Let $Y = 0_k$ and $Z = Z_j$, $j = 2,3,\cdots,n-\alpha$. Using Equation [1], $0-\gamma IZ_j = -\gamma Z_j$ then, $\gamma = 0$ is the eigenvalue with multiplicity of at least $n-\alpha-1$ since there are $n-\alpha-1$ independent vectors of the form Z_j.

Case 3. Let $Y = \dfrac{2(n-\alpha)}{\gamma-1}I_\alpha$ and $Z = I_{n-\alpha}$. Here, γ denotes root of the equation, $\gamma^2-\gamma-4(n-\alpha)\alpha = 0$.

From Equation [1],

$$\begin{aligned}(2J)Y - \gamma IZ &= 2J_{(n-\alpha)\times\alpha}\frac{2(n-\alpha)}{\gamma-1}1_\alpha - \gamma I_{n-\alpha}1_{n-\alpha}\\ &= \left\{\frac{4(n-\alpha)\alpha}{\gamma-1} - \gamma\right\}1_{n-\alpha}\\ &= \left\{\frac{\gamma^2-\gamma-4(n-\alpha)\alpha}{\gamma-1}\right\}1_{n-\alpha}.\end{aligned}$$

So, $\gamma_1 = \dfrac{1+\sqrt{1+16(n-\alpha)\alpha}}{2}$ and $\gamma_2 = \dfrac{1+\sqrt{1-16(n-\alpha)\alpha}}{2}$ are the eigenvalues both with multiplicity of at least one.

The spectrum of $Alb(K_n)_S$ is given by,

$$\begin{pmatrix} 1 & 0 & \gamma_1 & \gamma_2 \\ \alpha-1 & n-\alpha-1 & 1 & 1 \end{pmatrix}$$

where, $\gamma_1 = \dfrac{1+\sqrt{1+16(n-\alpha)\alpha}}{2}$, $\gamma_2 = \dfrac{1-\sqrt{1+16(n-\alpha)\alpha}}{2}$.

The Albertson characteristic polynomial of $Alb(K_n)_S$ is given by,

$$\gamma^{n-\alpha-1}(\gamma-1)^{\alpha-1}\{\gamma^2-\gamma-4(n-\alpha)\alpha\}.$$

$$\begin{aligned}Alb_E(K_n)_S &= (\alpha-1)\left|1-\frac{\alpha}{n}\right| + (n-\alpha-1)\left|0-\frac{\alpha}{n}\right|\\ &+ \left|\frac{1+\sqrt{1+16(n-\alpha)\alpha}}{2} - \frac{\alpha}{n}\right| + \left|\frac{1-\sqrt{1+16(n-\alpha)\alpha}}{2} - \frac{\alpha}{n}\right|\\ &= (\alpha-1)\left(\frac{n-\alpha}{n}\right) + (n-\alpha-1)\left(\frac{\alpha}{n}\right) + \sqrt{1+16\alpha(n-\alpha)}\\ &= \frac{2(\alpha n-\alpha^2)-n}{n} + \sqrt{1+16\alpha(n-\alpha)}.\end{aligned}$$

□

Theorem 4.2. *For complete bipartite graph $K_{p,q}$,*

$$Alb_E(K_{p,q})_S = \frac{2pq-p-q}{p+q} + \sqrt{1+4pq(q-p+2)^2}.$$

Proof. For complete bipartite graph $K_{p,q}$ with $\alpha = p$ self-loops, we have

$$[Alb(K_{p,q})_S - \gamma I] = \begin{bmatrix} [1-\gamma]I_p & (q-p+2)J_{p\times q} \\ (q-p+2)J_{q\times p} & -\gamma I_q \end{bmatrix}_{p+q},$$

where J is matrix of all 1's.

Since block $A = [1-\gamma]I_p$ is invertible, by Lemma [1.6], we have

$$|Alb(K_{p,q})_S - \gamma I| = |[1-\gamma]I_p||-\gamma I_q - (q-p+2)J_{q\times p}\frac{1}{1-\gamma}I_p(q-p+2)J_{p\times q}|.$$

On simplifying, we obtain the characteristic polynomial of $Alb(K_{p,q})_S$ is given by

$$\gamma^{q-1}(\gamma-1)^{p-1}\{\gamma^2-\gamma-pq(q-p+2)^2\}.$$

The spectrum of $Alb(K_{p,q})_S$ is given by

$$\begin{pmatrix} 0 & 1 & \gamma_1 & \gamma_2 \\ q-1 & p-1 & 1 & 1 \end{pmatrix}$$

where, $\gamma_1 = \dfrac{1+\sqrt{1+4pq(q-p+2)^2}}{2}$ and $\gamma_2 = \dfrac{1-\sqrt{1+4pq(q-p+2)^2}}{2}$.

$$\begin{aligned} Alb_E(K_{p,q})_S &= (q-1)\left|0-\frac{\alpha}{p+q}\right| + (p-1)\left|1-\frac{\alpha}{p+q}\right| \\ &+ \left|\frac{1+\sqrt{1+4pq(q-p+2)^2}}{2} - \frac{\alpha}{p+q}\right| \\ &+ \left|\frac{1+\sqrt{1+4pq(q-p+2)^2}}{2} - \frac{\alpha}{p+q}\right| \\ &= (q-1)\left(\frac{p}{p+q}\right) + (p-1)\left(\frac{p}{p+q}\right) \\ &+ \sqrt{1+4pq(q-p+2)^2} \\ &= \frac{2pq-p-q}{p+q} + \sqrt{1+4pq(q-p+2)^2}. \end{aligned}$$

□

Corollary 4.3. *For star graph,* $Alb_E(K_{1,q-1})_S = \dfrac{q-1}{q+1} + \sqrt{1+4q(q+1)^2}$.

Proof. On substituting $p = 1$ in Theorem [4.2], we obtain the required result. □

Theorem 4.4. *For crown graph,*

$$Alb_E(S_n^0)_S = (n-1)(4.0616) + \sqrt{1+16(n-1)^2}.$$

Proof. Let S_n^0 be crown graph of order $2n$ and let $S = \{1, 2, \dots, n\}$. Then,

$$Alb(S_n^0)_S = \begin{bmatrix} I_n & 2(J-I)_n \\ 2(J-I)_n & 0_n \end{bmatrix}_{2n},$$

is the Albertson matrix of $(S_n^0)_S$. The result is proved by showing

$Alb[(S_n^0)_S]Z = \gamma Z$ for certain vector Z and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue γ is same, as $Alb(S_n^0)_S$ is real and symmetric.

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with $Alb(S_n^0)_S$.

Consider,

$$[Alb(S_n^0)_S - \gamma I] \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} (1-\gamma)IX + 2(J-I)Y \\ 2(J-I)X - \gamma IY \end{bmatrix}_{2n}. \tag{2}$$

Case 1. Let $X = \dfrac{-2Y_j}{\gamma - 1}$, and $Y = Y_j = e_1 - e_j$, $j = 3, \dots, 2n$.

Using Equation [2], we have $\frac{1+\sqrt{17}}{2}$ and $\frac{1-\sqrt{17}}{2}$ are the eigenvalues both with multiplicity of at least $2n-2$ since there are $2n-2$ independent vectors of the form Y_j.

Case 2. Let $X = \frac{2(n-1)}{1-\gamma}1_n$ and $Y = 1_n$, where γ is any root of the Equation,

$$\gamma^2 - \gamma - 4(n-1)^2 = 0.$$

Using Equation [2],

$$\begin{aligned}
2(J-I)X - \gamma IY &= 2(J-I)\frac{2(n-1)}{1-\gamma}1_n - \gamma I 1_n \\
&= \left\{\frac{4(n-1)(1-n)}{\gamma-1} - \gamma\right\}1_n \\
&= \left\{\frac{4(n-1)^2}{1-\gamma} - \gamma\right\}1_n \\
&= \left\{\frac{\gamma^2-\gamma-4(n-1)^2}{\gamma-1}\right\}1_n.
\end{aligned}$$

So, $\gamma = \frac{1+\sqrt{1+16(n-1)^2}}{2}$ and $\gamma = \frac{1-\sqrt{1+16(n-1)^2}}{2}$ are the eigenvalues both with multiplicity of at least 1.

Thus, the spectrum of $Alb(S_n^0)_S$ is given by,

$$\begin{pmatrix} \frac{1+\sqrt{17}}{2} & \frac{1-\sqrt{17}}{2} & \gamma_1 & \gamma_2 \\ n-1 & n-1 & 1 & 1 \end{pmatrix},$$

where $\gamma_1 = \frac{1+\sqrt{1+16(n-1)^2}}{2}$, $\gamma_2 = \frac{1-\sqrt{1+16(n-1)^2}}{2}$.

$$\begin{aligned}
Alb_E(S_n^0)_S &= (n-1)\left|\frac{1+\sqrt{17}}{2} - \frac{1}{2}\right| + (n-1)\left|\frac{1-\sqrt{17}}{2} - \frac{1}{2}\right| \\
&\quad + \left|\frac{1+\sqrt{1+16(n-1)^2}}{2} - \frac{1}{2}\right| + \left|\frac{1-\sqrt{1+16(n-1)^2}}{2} - \frac{1}{2}\right| \\
&= (n-1)(4.0616) + \sqrt{1+16(n-1)^2}.
\end{aligned}$$

□

Theorem 4.5. *Let Γ be thorn graph of order $n(\alpha+1)$. Let Γ_S be the graph obtained by attaching α self-loops to all the pendant vertices of Γ. Then, whenever Γ is regular,* $Alb_E(\Gamma_S) = \frac{(\alpha-1)(n\alpha+n-\alpha)}{\alpha+1} + \sqrt{1+4\alpha N}$, *where* $N = d_\Gamma(v) + \alpha - 3$.

Proof. We have,

$$[Alb(\Gamma_S) - \gamma I] = \begin{bmatrix} [(Alb(\Gamma) - \gamma I)_n]_1 & [(NI)_n]_{1\times\alpha} \\ [(NI)_n]_{\alpha\times 1} & [(1-\gamma I)_n]_\alpha \end{bmatrix}_{\alpha+1}, \quad (3)$$

where $N = d_\Gamma(v) + \alpha - 3$.

On applying row operation $R_i \longrightarrow R_i - R_{i+1}$, $2 \leq i \leq \alpha$ and column operations $C_i \longrightarrow C_i + C_{i-1} + \cdots + C_2$, $2 \leq i \leq k$, in Equation [3], we get

$$|Alb(\Gamma_S) - \gamma I| = (\gamma - 1)^{n(\alpha-1)} \begin{vmatrix} [Alb(\Gamma) - \gamma I]_n & [\alpha N I]_n \\ [NI]_n & [1-\gamma I]_n \end{vmatrix} \tag{4}$$

Whenever Γ is regular, the Albertson characteristic polynomial of thorn graph with self-loops is given by $(\gamma-1)^{n(\alpha-1)}\{\gamma^2 - \gamma - \alpha N^2\}^n$.

Thus, the spectrum of $Alb(\Gamma_S)$ is given by,

$$\begin{pmatrix} 1 & \gamma_1 & \gamma_2 \\ n(\alpha-1) & n & n \end{pmatrix},$$

where $\gamma_1 = \dfrac{1+\sqrt{1+4\alpha N}}{2}$, $\gamma_2 = \dfrac{1-\sqrt{1+4\alpha N}}{2}$.

$$\begin{aligned} Alb_E(\Gamma_S) &= n(\alpha-1)\left|1 - \frac{\alpha}{n(\alpha+1)}\right| + \left|\frac{1+\sqrt{1+4\alpha N}}{2} - \frac{\alpha}{n(\alpha+1)}\right| \\ &\quad + \left|\frac{1-\sqrt{1+4\alpha N}}{2} - \frac{\alpha}{n(\alpha+1)}\right| \\ &= \frac{(\alpha-1)(n\alpha+n-\alpha)}{\alpha+1} + \sqrt{1+4\alpha N}. \end{aligned}$$

□

Corollary 4.6. *For thorn complete graph,*

$$Alb_E(K_{n(\alpha+1)})_S = \frac{(\alpha-1)(n\alpha+n-\alpha)}{\alpha+1} + n\sqrt{1+4\alpha(n+\alpha-4)^2}.$$

Proof. On substituting $Alb(K_n)_S = 0$ and $N = n+\alpha-4$ in Equation [4], we have the Albertson characteristic polynomial of $(K_{n(\alpha+1)})_S$ as $(\gamma-1)^{n(\alpha-1)}\{\gamma^2 - \gamma - \alpha(n+\alpha-4)^2\}^n$.

$$\begin{aligned} Alb_E(K_{n(\alpha+1)})_S &= n(\alpha-1)\left|1 - \frac{\alpha}{n(\alpha+1)}\right| \\ &\quad + n\left|\frac{1+\sqrt{1+4\alpha(n+\alpha-4)^2}}{2} - \frac{\alpha}{n(\alpha+1)}\right| \\ &\quad + \left|\frac{1-\sqrt{1+4\alpha(n+\alpha-4)^2}}{2} - \frac{\alpha}{n(\alpha+1)}\right| \\ &= \frac{(\alpha-1)(n\alpha+n-\alpha)}{\alpha+1} + n\sqrt{1+4\alpha(n+\alpha-4)^2}. \end{aligned}$$

□

Corollary 4.7. *For thorn cycle,*

$$Alb_E(C_{n(\alpha+1)})_S = \frac{(\alpha-1)(n\alpha+n-\alpha)}{\alpha+1} + n\sqrt{1+4\alpha(\alpha-1)^2}.$$

Proof. On substituting $Alb(C_n)_S = 0$ and $N = \alpha - 1$ in Equation [4], we have the Albertson characteristic polynomial of $(C_{n(\alpha+1)})_S$ is given by $(\gamma-1)^{n(\alpha-1)}\{\gamma^2 - \gamma - \alpha(\alpha-1)^2\}^n$.

$$
\begin{aligned}
Alb_E(C_{n(\alpha+1)})_S &= n(\alpha-1)\left|1-\frac{\alpha}{n(\alpha+1)}\right| \\
&\quad + n\left|\frac{1+\sqrt{1+4\alpha(\alpha-1)^2}}{2}-\frac{\alpha}{n(\alpha+1)}\right| \\
&\quad + \left|\frac{1-\sqrt{1+4\alpha(\alpha-1)^2}}{2}-\frac{\alpha}{n(\alpha+1)}\right| \\
&= \frac{(\alpha-1)(n\alpha+n-\alpha)}{\alpha+1}+n\sqrt{1+4\alpha(\alpha-1)^2}.
\end{aligned}
$$

□

Theorem 4.8. *The Albertson characteristic polynomial of thorn wheel graph $W_{n(\alpha+1)}$ with α self-loops is given by $(\gamma-1)^{n(\alpha-1)}(\gamma-\gamma^2+\alpha(n+\alpha-5)^2)^{n-2}\{\gamma^4-2\gamma^3+\gamma^2[1-\alpha(a^2+b^2)-n+1]+\gamma[\alpha(a^2+b^2)-2(n+1)]+\alpha^2a^2b^2-n+1\}$, where $a=n+\alpha-5$ and $b=n+\alpha-4$.*

Proof. From Theorem [4.5], we have

$$|Alb(W_{n(\alpha+1)})_S-\gamma I| = (\gamma-1)^{n(\alpha-1)}\begin{vmatrix}[Alb(\Gamma)-\gamma I]_n & [mNI]_n \\ [NI]_n & [1-\gamma I]_n\end{vmatrix}.$$

On expanding the above equation,

$$
\begin{aligned}
|Alb(W_{n(\alpha+1)})_S-\gamma I| &= (\gamma-1)^{n(\alpha-1)}\{Alb(W_n)_S(1-\gamma)I-(\gamma(1-\gamma))I-\alpha N^2 I\} \\
&= (\gamma-1)^{n(\alpha-1)}\begin{bmatrix}0_{n-1} & (1-\gamma)_{n-1\times 1} \\ (1-\gamma)_{1\times n-1} & 0_1\end{bmatrix}_n - \big[\gamma(1-\gamma)I\big]_n \\
&- \begin{bmatrix}(\alpha(n+\alpha-5)^2)I_{n-1} & 0_{n-1\times 1} \\ 0_{1\times n-1} & (\alpha(n+\alpha-4)^2)I_1\end{bmatrix}_n
\end{aligned}
$$

$$[Alb(W_{n(\alpha+1)})_S-\gamma I] = \begin{bmatrix}[\gamma^2-\gamma-\alpha a^2]I_{n-1} & 0_{n-1\times 1} \\ 0_{1\times n-1} & [\gamma^2-\gamma-\alpha b^2]_1\end{bmatrix}_n \tag{5}$$

On applying row operation $R_i \longrightarrow R_i - R_{i+1}$, $2\le i\le n-1$ and column operations $C_i \longrightarrow C_i + C_{i-1} + \cdots + C_1$, $1\le i\le n$, in Equation [5], we get

$[Alb(W_{n(\alpha+1)})_S-\gamma I] = (\gamma-\gamma^2+\alpha(n+\alpha-5)^2)^{n-2}\{(\gamma^2-\gamma-\alpha(n+\alpha-4)^2)(\gamma^2-\gamma-\alpha(n+\alpha-5)^2)-(1-\gamma)^2(n-1)\}$.

The Albertson characteristic polynomial of thorn wheel graph with self-loops is given by $(\gamma-1)^{n(\alpha-1)}(\gamma-\gamma^2+\alpha(n+\alpha-5)^2)^{n-2}\{\gamma^4-2\gamma^3+\gamma^2[1-\alpha(a^2+b^2)-n+1]+\gamma[\alpha(a^2+b^2)-2(n+1)]+\alpha^2a^2b^2-n+1\}$. □

Theorem 4.9. *The Albertson characteristic polynomial of thorn star graph $K_{1,q-1}$ with α self-loops is given by $(\gamma-1)^{q(\alpha-1)}(\gamma-\gamma^2+\alpha(q+\alpha-5)^2)^{q-2}\{\gamma^4-2\gamma^3+\gamma^2[1-\alpha(a^2+b^2)-q^3+5q^2-8q+4]+\gamma[\alpha(a+b)+2q^3-10q^2+16q-8]+\alpha a^2b^2-q^3+5q^2-8q+4\}$, where $a=q+\alpha-5$ and $b=q+\alpha-4$.*

Proof. From Theorem [4.5], we have

$$|Alb(K_{\alpha,\alpha(q-1)})_S-\gamma I| = (\gamma-1)^{q(\alpha-1)}\begin{vmatrix}(Alb(K_{1,q-1})-\gamma I)_q & \alpha NI_q \\ NI_q & (1-\gamma I)_q\end{vmatrix}.$$

On expanding the above equation,

$$\begin{aligned}[Alb(K_{\alpha,\alpha(q-1)})_S - \gamma I] &= (\gamma-1)^{q(\alpha-1)}\{Alb(K_{1,q-1})(1-\gamma)I - (\gamma(1-\gamma))I - \alpha N^2 I\} \\ &= \begin{bmatrix} 0_1 & [(q-2)(1-\gamma)]_{1\times q-1} \\ [(q-2)(1-\gamma)]_{q-1\times 1} & 0_{q-1} \end{bmatrix}_q \\ &- \left[\gamma(1-\gamma)I\right]_q - \begin{bmatrix} [\alpha(q+\alpha-4)^2]I_{q-1} & 0_{q-1\times 1} \\ 0_{1\times q-1} & [\alpha(q+\alpha-5)^2]_1 \end{bmatrix}_q \end{aligned}$$

$$[Alb(K_{\alpha,\alpha(q-1)})_S - \gamma I] = \begin{bmatrix} [\gamma^2-\gamma-\alpha b^2]_{q-1} & 0_{q-1\times 1} \\ 0_{1\times q-1} & [\gamma^2-\gamma-\alpha a^2]_1 \end{bmatrix}_q . \tag{6}$$

On applying row operation $R_i \longrightarrow R_i - R_{i+1}$, $2 \le i \le q-1$ and column operations $C_i \longrightarrow C_i + C_{i-1} + \cdots + C_2$, $2 \le i \le q$, in Equation [6] we get

$[Alb(K_{\alpha,\alpha(q-1)})_S - \gamma I] = (\gamma - \gamma^2 + \alpha(q+\alpha-5)^2)^{q-2}\{(\gamma^2-\gamma-\alpha(q+\alpha-4)^2)(\gamma^2-\gamma-\alpha(q+\alpha-5)^2) - (1-\gamma)^2(q-1)\}$.

The Albertson characteristic polynomial of thorn star with self-loops is given by

$(\gamma-1)^{q(\alpha-1)}(\gamma-\gamma^2+\alpha(q+\alpha-5)^2)^{q-2}\{\gamma^4 - 2\gamma^3 + \gamma^2[1-\alpha(a^2+b^2) - q^3 + 5q^2 - 8q + 4] + \gamma[\alpha(a+b) + 2q^3 - 10q^2 + 16q - 8] + \alpha a^2 b^2 - q^3 + 5q^2 - 8q + 4\}$. □

5. Conclusion

The concept of Albertson energy of a graph with self-loops has been defined. The set $S \subseteq V$ that we choose affects the Albertson energy. Some bounds on $Alb_E(G_S)$ and $S(Alb(G_S))$ have been obtained. Additionally, the Albertson energy of a few standard graphs with self-loops is calculated. Also, considered the thorn graph with self-loops attached to the pendant vertices and compute the Albertson energy of the thorn complete, thorn cycle with self-loops. Furthermore, an expression for the Albertson characteristic polynomial for the thorn wheel and thorn star with self-loops has been provided.

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References

[1] H. Abdo, N. Cohen, D. Dimitrov, Graphs with maximal irregularity, Filomat 28 (2014) 1315–1322.
[2] H. Abdo, D. Dimitrov, I. Gutman, Graph irregularity and its measures, Applied Mathematics and Computation 357 (2019) 317–324.
[3] M. Abdullah, B. Gebremichel, S. Hayat, J. H. Koolen, Distance-regular graphs with a few q-distance eigenvalues, Discrete Mathematics 347 (2024) 113926.
[4] A. Abiad, B. Brimkov, S. Hayat, A. P. Khramova, J. H. Koolen, Extending a conjecture of Graham and Lovász on the distance characteristic polynomial, Linear Algebra and its Applications 693 (2024) 63–82.
[5] S. Akbari, H. Al Menderj, M. H. Ang, J. Lim, Z. C. Ng, Some results on spectrum and energy of graphs with loops, Bulletin of the Malaysian Mathematical Sciences Society 46 (2023) 1–18.
[6] M. O. Albertson, The irregularity of a graph, Ars Combinatoria 46 (1997) 219–225.
[7] D. V. Anchan, S. D'Souza, H. J. Gowtham, P. G. Bhat, Sombor energy of a graph with self-Loops, MATCH Communications in Mathematical and in Computer Chemistry 90 (2023) 773-786.

[8] M. Azari, On the Gutman index of thorn graphs, Kragujevac Journal of Science 40 (2018) 33–48.
[9] E. R. Barnes, A. J. Hoffman, Bounds for the spectrum of normal matrices, Linear Algebra and its Applications 201 (1994) 79–90.
[10] I. Gutman, The energy of a graph, Berichte der Mathematisch-Statistischen Sektion im Forschungszentrum Graz 103 (1978) 1–22.
[11] I. Gutman, I. Redžepović, B. Furtula, A. Sahal, Energy of graphs with self-loops, MATCH Communications in Mathematical and in Computer Chemistry 87 (2022) 645–652.
[12] I. Gutman, M. Togan, A. Yurttas, A. S. Cevik, I. N. Cangul, Inverse problem for sigma index, MATCH Communications in Mathematical and in Computer Chemistry 79 (2018) 491–508.
[13] A. Jahanbani, Albertson energy and Albertson Estrada index of graphs, Journal of Linear and Topological Algebra 8 (2019) 11–24.
[14] J. H. Koolen, M. Abdullah, B. Gebremichel, S. Hayat, Distance-regular graphs with exactly one positive q-distance eigenvalue, Linear Algebra and its Applications 689 (2024) 230–246.
[15] J. Liua, Y. Chena, D. Dimitrovb, J. Chena, New bounds on the energy of graphs with self–Loops, MATCH Communications in Mathematical and in Computer Chemistry 91 (2024) 779–796.
[16] D. S. Mitrinovic, P. M. Vasic, Analytic inequalities. Berlin: Springer; (1970).
[17] K. M. Popat, K. R. Shingala, Some new results on energy of graphs with self-loops, Journal of Mathematical Chemistry 61 (2023) 1462–1469.
[18] B. R. Rakshith, K. C. Das, B. J. Manjunatha, Y. Shang, Relations between ordinary energy and energy of a self-loop graph, Heliyon 10 (2024) e27756.
[19] Z. Raza, The expected values of some indices in random phenylene chains, The European Physical Journal Plus 136 (2021) 1–5.
[20] Z. Raza, S. Akhter, R. Hasni, Maximal first Zagreb connection index of trees with given total domination number, Discrete Applied Mathematics 368 (2025) 82–90.
[21] Z. Raza, A. Ali, Bounds on the Zagreb indices for molecular (n, m)-graphs, J. International Journal of Quantum Chemistry 120 (2020) e26333.
[22] F. Zhang, The Schur complement and its applications, Springer Science and Business Media (2006).