



Role coloring of various families of snark graphs

Research Article

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Abstract: Social networks are an integral part of everyday life. A social network can be envisioned as a graph, where the vertices represent individuals and the edges indicate relationships between them, enabling the study of their behavior. This gives rise to the decision problem known as k -role coloring. A k -role coloring of a graph G is an assignment of exactly k colors to its vertices such that any two vertices receiving the same color have identical sets of colors in their neighborhoods. A k -role coloring is trivial when $k = 1$ or $k = n$, and the problem is NP-complete on general graphs for all $k \geq 2$. In this paper, we study the class of snark graphs. A snark is a special class of 3-regular graphs, and we determine the k -role colorings of snarks for $k \leq 3$.

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1. Introduction

All graphs considered in this paper are finite and undirected. A graph G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$ of edges, together with an incidence function $\psi(G)$. An edge with end vertices u and v is denoted by uv . A graph G is simple if it contains no loops or multiple edges. A loop is an edge that connects a vertex to itself. The collection of all vertices in the graph G that are adjacent to a vertex v (for every $v \in V(G)$) is called the (open) neighborhood of v , denoted by $N_G(v)$ or simply $N(v)$ when the graph G is fixed. The degree of a vertex v , denoted by $\deg(v)$, is the number of edges incident to v . Let $\delta(G)$ be the minimum degree of a graph G , $\Delta(G)$ the

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maximum degree of a graph G . For a vertex coloring α , let $\alpha(v)$ represent the color of the vertex v and $\alpha(N(v)) = \{\alpha(u) : u \in N(v)\}$. A dominating set of G is a subset $D \subseteq V$ such that every vertex in $V \setminus D$ is adjacent to at least one vertex in D . An independent set of G is a subset $I \subseteq V$ such that no two vertices in I are adjacent. An independent dominating set of G is a subset $ID \subseteq V$ that is both a dominating set and an independent set. A gadget is a subgraph used as a building block in the construction of snark graphs. In our construction, each subgraph H_i serves as a gadget. We follow John Adrian Bondy and Uppaluri Siva Ramachandra Murty [2] for the standard graph terminologies and notations.

Graph coloring is a significant and well-studied area of research. Among the various coloring techniques, role coloring was introduced by Martin G. Everett and Steve Borgatti in 1991 [6]. For a graph G , a k -role coloring of G is an assignment of exactly k colors to its vertices such that any two vertices receiving the same color have identical sets of colors in their neighborhoods. That is, k -role coloring is a surjective map $\alpha: V(G) \rightarrow \{1, 2, \dots, k\}$ such that for all $u, v \in V(G)$, if $\alpha(u) = \alpha(v)$ then $\alpha(N(u)) = \alpha(N(v))$ [13]. A graph G is said to be k -role colorable if there exists a partition of its vertices into k color classes (roles) such that vertices in the same class have identical color neighborhoods. In general, every graph has two trivial role colorings, corresponding to $k = 1$ or $k = n$. The role graph R is defined as the graph with $V(R) = \{1, 2, \dots, k\}$ and $E(R) = \{(\alpha(u), \alpha(v)) | (u, v) \in E(G)\}$, where $|V(R)| \leq |V(G)|$. A role graph R is bipartite if it is loopless and does not contain any odd cycle. Consequently, role graphs such as R_2 and R_3 shown in Fig. 1 that contain a loop are not considered bipartite. Furthermore, for all $v \in V(G)$, $\deg_G(v) \geq \deg_R(\alpha(v))$ [13]. From now on all graphs are assumed to be simple, except that role graphs R may contain loops. Observe that, if the graph G is connected, then the role graph R is also connected (the converse is not true in general). Also, if a role graph R is bipartite, then so is G . Fig. 1 illustrates all possible role graphs corresponding to the 2-role coloring of connected graphs.

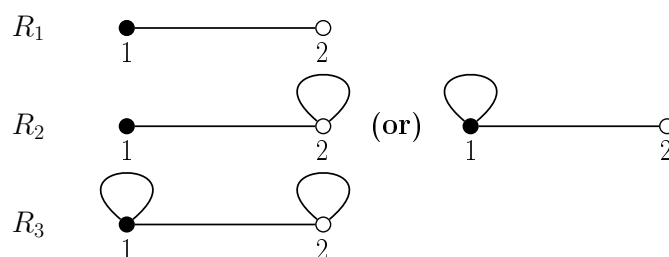


Figure 1. Role graphs on two vertices

Fig. 2 illustrates some of the possible role graphs on three vertices considered in this paper. All snark graphs considered in this paper admit a 3-role coloring in one of the suggested ways. In particular, if a graph G has a 3-role coloring with role graph $R_{3,3}$, then each color class is an independent dominating set in G .

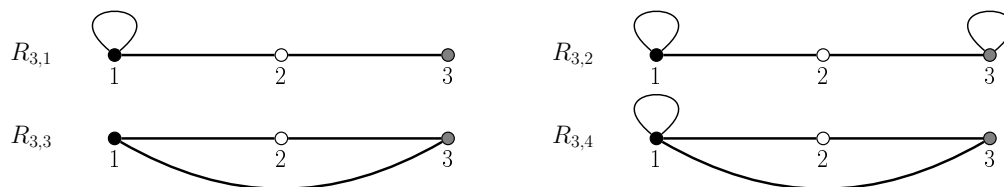


Figure 2. Role graphs on three vertices

1.1. Previous work

Fred S. Roberts and Li Sheng [16] showed that the problem of determining whether a graph G has a 2-role coloring is NP-complete. Jiri Fiala and Daniel Paulusma [8] analyzed the complete complexity classification of the R -role coloring problem. For certain graph classes, including chordal graphs, trees, and cographs, the problem can be solved efficiently, as shown in [14, 17]. Characterization results for 3-role colorings in split graphs, an interesting class of graphs in which 2-role coloring is always possible, were obtained in [5]. The role coloring of hereditary graph classes was studied by Puck Rombach and Christopher Purcell [15]. A study by Sukanya Pandey and Vibha Sahlot [13] demonstrated that k -role coloring on bipartite graphs is NP-complete for $k \geq 3$. Finally, the complexity of k -role assignment restricted to Cartesian products was analyzed by Diane Castonguay et al. [3], who also showed that Cartesian products are always 2-role colorable.

1.2. Snarks

Snarks are a special class of graphs that arise in important problems in graph theory, particularly those related to coloring. They first gained significance in connection with the Four-Color Problem, one of the most famous problems in mathematics. A snark is a cubic, bridgeless, non-Hamiltonian, 4-edge-chromatic graph. The first snark, discovered by Petersen in 1898, later became known as the Petersen graph. The second snark, discovered by Blanuša in 1964, is called the Blanuša snark [1]. The Szekeres snark was discovered by George Szekeres in 1973 [18], and the Watkins snark, with 50 vertices, was discovered by John J. Watkins in 1989 [20]. In 1975, Rufus Isaacs introduced the flower snark and the double-star snark [11]. For a comprehensive analysis of snark graphs, we refer the reader to the survey papers [4, 20, 21].

1.3. Motivation

k -role coloring is trivial when $k = 1$ or $k = n$, and it is NP-complete on general graphs when $k \geq 2$. In a connected bipartite graph with at least two vertices, the problem is trivial when $k = 2$ and NP-complete when $k \geq 3$. The complexity of k -role coloring of regular graphs is unknown. Even for 3-regular graphs, the problem of k -role coloring has not been investigated. Since snarks are a unique type of 3-regular graph, we were motivated to investigate the role coloring of snark families. We present our results on k -role coloring of snarks when $k \leq 3$ in this work.

The paper is organized as follows. In Section 2, we present our findings on the k -role coloring of generalized snark families. Section 3 discusses the results for a few specific snark graphs. Finally, Section 4 presents the conclusion and outlines directions for future work.

2. k -role coloring of generalized snarks

2.1. Flower snark

The flower snark J_n was introduced by Rufus Isaacs [11] in the year 1975 and the construction is as follows: Construct n copies of the star graph on four vertices. Let u_i denote the central vertex of the i -th star, and let v_i , w_i , and x_i denote its leaves. This results in a disconnected graph on $4n$ vertices with $3n$ edges ($u_i v_i$, $u_i w_i$ and $u_i x_i$) for $1 \leq i \leq n$. Construct the n -cycle $(v_1 \dots v_n)$ and the $2n$ -cycle $(w_1 \dots w_n x_1 \dots x_n)$. This adds n and $2n$ edges respectively. Thus the flower snark J_n , is a cubic graph with $4n$ vertices and $6n$ edges, where n should be odd. Fig. 3 shows an illustration of the flower snark J_5 .

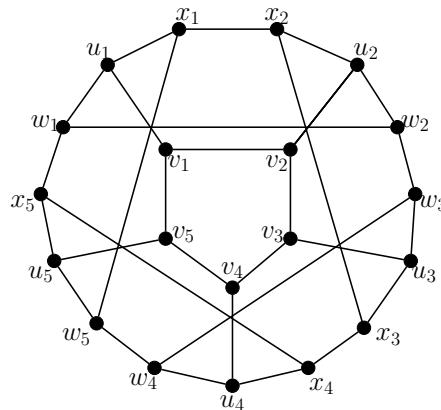


Figure 3. Flower snark J_5 .

Theorem 2.1. Let J_n be a flower snark on $4n$ vertices and $6n$ edges, where $n \geq 3$ is odd. Then J_n is k -role colorable for $k \leq 3$.

Proof. Let $\{u_i, v_i, w_i, x_i\}$ be the vertices of the stars S_i in J_n , where $1 \leq i \leq n$. Let u_i be the central vertex of a star S_i . Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$ be the vertex set of an n -cycle, and let $V(C_{2n}) = \{w_1, w_2, \dots, w_n, x_1, x_2, \dots, x_n\}$ be the vertex set of a $2n$ -cycle.

Case 1:

Let us consider the case $k = 2$. Define a mapping $\alpha : V(J_n) \rightarrow \{1, 2\}$ for all $1 \leq i \leq n$ as follows: $\alpha(u_i) = 1$, $\alpha(v_i) = \alpha(w_i) = \alpha(x_i) = 2$. Here $\alpha(N(u_i)) = \{2\}$ and $\alpha(N(v_i)) = \alpha(N(w_i)) = \alpha(N(x_i)) = \{1, 2\}$. Hence it is a valid 2-role coloring with role graph R_2 .

Case 2:

Let us consider the case $k = 3$. Define a mapping $\alpha : V(J_n) \rightarrow \{1, 2, 3\}$ for all $1 \leq i \leq n$ as follows: $\alpha(u_i) = 2$, $\alpha(v_i) = 1$, $\alpha(w_i) = \alpha(x_i) = 3$. We now show that α is a valid 3-role coloring. The vertex v_i is adjacent to two vertices in C_n and to the vertex u_i . Since $\alpha(u_i) = 2$, it follows that $\alpha(N(v_i)) = \{1, 2\}$. Next, take $u_i \in V(S_i)$, where $N(u_i) = \{v_i, w_i, x_i\}$. Since $\alpha(v_i) = 1$, $\alpha(w_i) = \alpha(x_i) = 3$, it follows that $\alpha(N(u_i)) = \{1, 3\}$. Finally, for all $w_i, x_i \in V(S_i)$, we note that $w_i, x_i \in V(C_{2n})$ and $\alpha(u_i) = 2$. Thus, $\alpha(N(w_i)) = \alpha(N(x_i)) = \{2, 3\}$. Hence α is a valid 3-role coloring with role graph $R_{3,2}$, illustrated in Fig. 4. $\square$

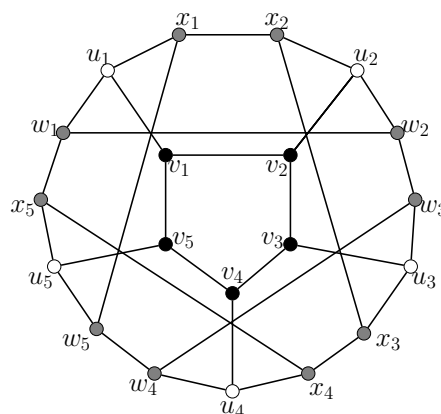


Figure 4. 3-role coloring of flower snark J_5 .

2.2. Goldberg snark

The Goldberg snarks are a family of graphs introduced by Goldberg [10] in the year 1981. These graphs are defined recursively using a basic gadget H_i . The vertex set and edge set of H_i are given by $V(H_i) = \{u_i, v_i, x_i, y_i, z_i, w_i, s_i, t_i\}$, $E(H_i) = \{u_i v_i, x_i y_i, x_i z_i, y_i w_i, z_i v_i, z_i t_i, v_i w_i, w_i s_i, s_i t_i\}$. The basic gadget H_i is illustrated in Fig. 5(a). For each gadget H_i , we add a set of link edges $E_{i,i+1} = \{t_i s_{i+1}, y_i x_{i+1}, u_i u_{i+1}\}$, $1 \leq i \leq n$, indices taken modulo n . The resulting graph is called the link graph L_i . For odd $n \geq 3$, the graph G_n is obtained by joining the gadgets $H_1, H_2, \dots, H_n$ using the link edges. Its vertex set is $V(G_n) = V(H_1) \cup V(H_2) \cup \dots \cup V(H_n)$, $|V(G_n)| = 8n$. In G_n , we obtain an n -cycle $(u_1 u_2 \dots u_n)$ and two $2n$ -cycles $(s_1 t_1 s_2 t_2 \dots s_n t_n)$ and $(x_1 y_1 x_2 y_2 \dots x_n y_n)$. The Goldberg snark G_5 , shown in Fig. 5(b) is obtained by adding link edges to the gadgets H_1, H_2, H_3, H_4 and H_5 .

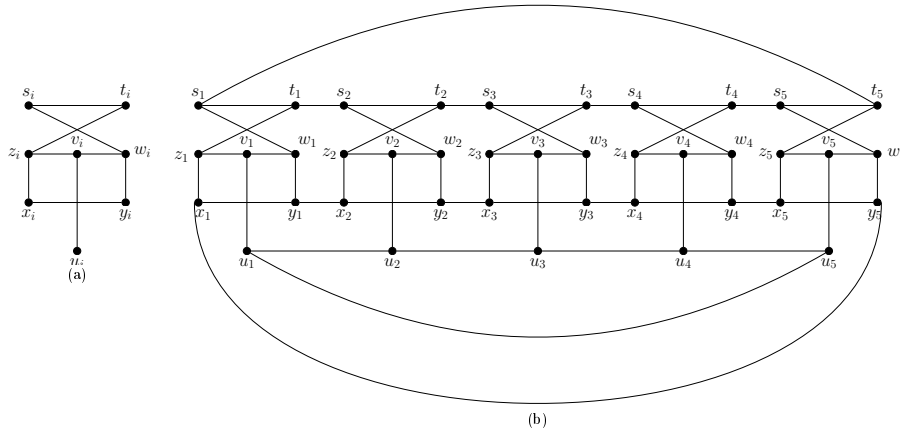


Figure 5. (a) Gadget H_i , (b) Goldberg snark G_5

Theorem 2.2. Let G_n be a Goldberg snark on $8n$ vertices, where $n \geq 3$ is odd. Then G_n is k -role colorable for $k \leq 3$.

Proof. Let G_n be a Goldberg snark constructed from a basic gadget H_i by taking n copies of H_i . Let $\{u_i, v_i, x_i, y_i, z_i, w_i, s_i, t_i\}$ be the vertices of G_n for all $1 \leq i \leq n$.

Case 1:

Let $k = 2$. Define $\alpha : V(G_n) \rightarrow \{1, 2\}$ by $\alpha(s_i) = \alpha(x_i) = \alpha(u_i) = \alpha(z_i) = \alpha(w_i) = 1$ and $\alpha(t_i) = \alpha(y_i) = \alpha(v_i) = 2$, for all $1 \leq i \leq n$. Let $V_1 = \{t_i, v_i, y_i\}$ and $V_2 = \{s_i, x_i, u_i, z_i, w_i\}$. The set V_1 is independent, and hence $\alpha(N(a)) = \{1\}$ for all $a \in V_1$. Now let $b \in V_2$. Each vertex in V_2 has at least one neighbor in V_1 and at least one neighbor in V_2 . Therefore, $\alpha(N(b)) = \{1, 2\}$ for all $b \in V_2$. Hence it is a valid 2-role coloring with role graph R_2 .

Case 2:

Let $k = 3$, we define $\alpha : V(G_n) \rightarrow \{1, 2, 3\}$ as follows, $\alpha(s_i) = \alpha(x_i) = 1$,

$$\alpha(y_i) = \alpha(t_i) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 3, & \text{if } i \text{ is even} \end{cases}$$

$$\alpha(w_i) = \alpha(z_i) = \begin{cases} 3, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$$

$$\alpha(v_i) = \begin{cases} 2, & \text{if } i = n \\ 1, & \text{otherwise} \end{cases}$$

$$\alpha(u_i) = \begin{cases} 1, & \text{if } i = n \\ 2, & \text{if } i \text{ is odd and } i \neq n \\ 3, & \text{if } i \text{ is even} \end{cases}$$

Now we prove that α is a valid 3-role coloring with role graph $R_{3,3}$, as shown in Fig. 6. It suffices to show that each color class forms an independent dominating set. Consider the subset $V_1 = \{s_i, v_i, x_i, u_n, s_n, x_n\}$. Here, the vertices s_i, v_i, x_i dominate all other vertices in gadgets $H_1, \dots, H_{n-1}$, and s_n, x_n, u_n dominate the remaining vertices in gadget H_n . Consider the subset $V_2 = \{t_i, u_i, y_i, z_i, w_i, v_n\}$. In this set, t_i, u_i, y_i are included from gadgets with odd indices, while z_i, w_i are included from gadgets with even indices. These vertices dominate all other vertices in their respective gadgets, except the u_i vertices in even-indexed gadgets, which belong to the cycle and are dominated by the adjacent vertex in a cycle. Additionally, t_n, y_n, v_n dominate all remaining vertices in H_n . Consider the subset $V_3 = \{z_i, w_i, u_i, t_i, y_i\}$. Here, z_i, w_i are included from odd-indexed gadgets, and u_i, t_i, y_i from even-indexed gadgets. These vertices dominate all other vertices in their respective gadgets, except the u_i vertices in odd-indexed gadgets, which are dominated by the cycle. By construction, V_1 , V_2 , and V_3 are disjoint dominating sets, and each is independent. Therefore, α is a valid 3-role coloring. $\square$

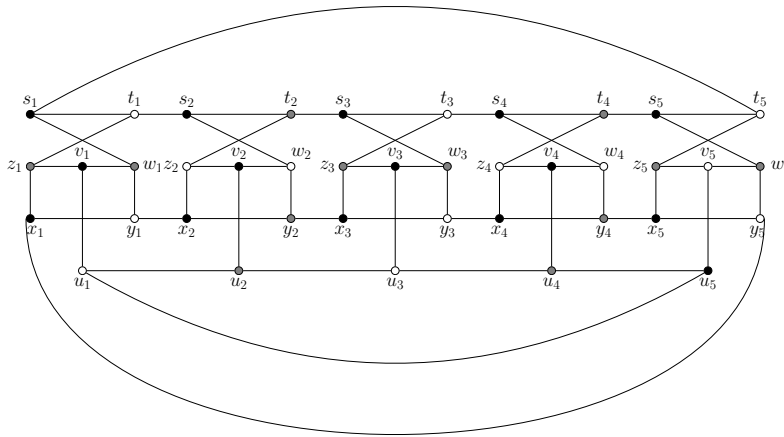


Figure 6. 3-role coloring of Goldberg snark G_5

2.3. Loupekin snark

Loupekin snarks were originally defined by F. Loupekin and first presented by Isaacs [7, 12]. The family LP_1 forms an infinite class of Loupekin snarks, constructed as follows. Let $n \geq 3$ be an odd integer. An n - LP_1 snark G is built from n subgraphs, called blocks, each obtained from the Petersen graph P . Let $R_1, R_2, \dots, R_n$ be n copies of P . For each $i \in \{1, \dots, n\}$, the gadget H_i is formed from R_i by deleting the vertices of an arbitrary path $P_3 \subset R_i$. The resulting gadget H_i has five vertices of degree 2, denoted x_i, u_i, w_i, v_i, y_i called the border vertices (see Fig. 7). For each $i \in \{1, \dots, n\}$, the border vertices x_i and y_i of H_i are joined to the border vertices v_{i+1} and u_{i+1} of H_{i+1} , respectively, by new edges, called linking edges, where indices are taken modulo n (that is, $H_{n+1} = H_1$). This creates a cyclic

chain of blocks (see Fig. 8). Next, choose an odd integer q with $1 \leq q \leq n$. Select q distinct vertices from $\{w_1, w_2, \dots, w_n\}$ and partition them into triples. For each triple $\{w_i, w_j, w_\ell\}$, add a new vertex $h_{i,j,\ell}$, called a star vertex, and join it to w_i, w_j , and w_ℓ by three new edges. Since n is odd, the remaining $n - q$ vertices among $\{w_1, \dots, w_n\}$ form an even set. If $n - q > 0$, pair these vertices arbitrarily and join each pair w_i and w_j by a new edge. This completes the construction of an n -LP₁ snark as shown in Fig. 9.

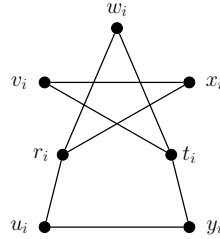


Figure 7. Basic gadget H_i

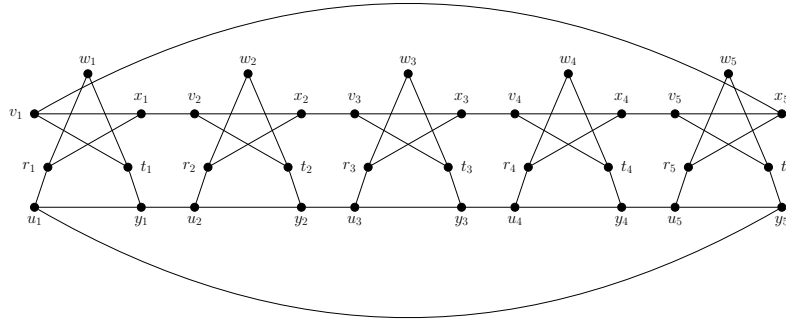


Figure 8. Graph with linked edges

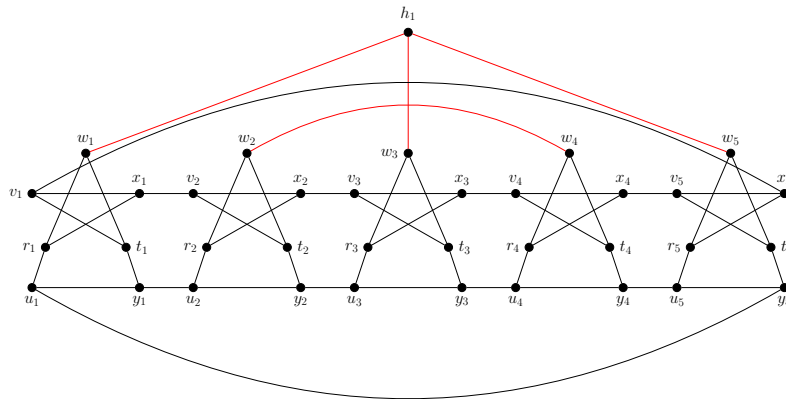


Figure 9. 5-LP₁ snark with star vertex- h_1 , star edges- w_1h_1, w_3h_1, w_5h_1 and an edge joining the remaining pair (w_2, w_4)

Theorem 2.3. Let $n\text{-LP}_1$ be an infinite family of Loupekin snark, where $n \geq 3$ is odd. Then $n\text{-LP}_1$ is k -role colorable for $k \leq 3$.

Proof. Let $V(H_i) = \{w_i, u_i, v_i, r_i, t_i, x_i, y_i, h_s\}$, $1 \leq i \leq n$. For each star vertex h_s ($1 \leq s \leq q$), let $N(h_s) = \{w_j, w_k, w_\ell\}$ be the three border vertices adjacent to h_s . Define $W_1 = \bigcup_{s=1}^q N(h_s)$, that is, W_1 is the set of all vertices w_i adjacent to a star vertex. For each star vertex h_s , fix exactly one vertex from $N(h_s)$. Define $W'_1 := \{w : w \text{ is the neighbor fixed by } h_s \text{ for } 1 \leq s \leq q\}$. The remaining vertices in $\{w_1, \dots, w_n\} \setminus W_1$ are paired by construction and joined by edges. For each such edge $w_i w_j$, assign one endpoint to W_2 and the other to W_3 .

Case 1:

Let $k = 2$. We define a mapping $\alpha : V(n\text{-LP}_1) \rightarrow \{1, 2\}$ as follows: $\alpha(u_i) = \alpha(v_i) = \alpha(h_s) = 1$, $\alpha(r_i) = \alpha(t_i) = \alpha(y_i) = \alpha(x_i) = 2$ and

$$\alpha(w_i) = \begin{cases} 1, & \text{if } w_i \in W_2 \\ 2, & \text{otherwise} \end{cases}$$

Here, the vertices u_i, v_i, h_s , and $w_i \in W_2$ are nonadjacent to each other. Thus, every vertex assigned color 1 has all its neighbors colored 2. Moreover, the vertices r_i, t_i, y_i, x_i , and $w_i \notin W_2$ are adjacent to at least one vertex assigned color 2 and at least one vertex assigned color 1. Therefore, the graph is 2-role colorable with role graph R_2 .

Case 2:

Consider the case $k = 3$. We define a mapping $\alpha : V(n\text{-LP}_1) \rightarrow \{1, 2, 3\}$ as follows: $\alpha(h_s) = 1$,

$$\alpha(w_i) = \alpha(x_i) = \alpha(y_i) = \begin{cases} 2, & \text{if } w_i \in W'_1 \text{ or } w_i \in W_2 \\ 3, & \text{otherwise} \end{cases}$$

$$\alpha(u_i) = \alpha(v_i) = \begin{cases} 1, & \text{if } w_i \in W_1 \\ 2, & \text{if } w_i \in W_3 \\ 3, & \text{if } w_i \in W_2 \end{cases}$$

$$\alpha(r_i) = \alpha(t_i) = \begin{cases} 1, & \text{if } w_i \in W_2 \cup W_3 \\ 2, & \text{if } w_i \in W_1 \setminus W'_1 \\ 3, & \text{if } w_i \in W'_1 \end{cases}$$

Now we prove that α is a valid 3-role coloring with role graph $R_{3,3}$, as shown in Fig. 10. It suffices to show that each color class forms an independent dominating set. For color class 1, the vertices $\{u_i, v_i, h_s\}$ are assigned color 1 when $w_i \in W_1$, and the vertices $\{r_i, t_i\}$ are assigned color 1 when $w_i \in W_2 \cup W_3$. All vertices in color class 1 dominate all other vertices in the graph $n\text{-LP}_1$. For color class 2, consider the vertices $\{w_i, x_i, y_i\}$ when $w_i \in W'_1$ or $w_i \in W_2$, and the vertices $\{r_i, t_i\}$ when $w_i \in W_1 \setminus W'_1$. These vertices dominate all vertices in the corresponding gadget H_i . When $w_i \in W_3$, the vertices $\{u_i, v_i\}$ are in color class 2; in this case, w_i is the only vertex in H_i not dominated by u_i and v_i , and it is dominated by some $w_j \in W_2$. For color class 3, consider the vertices $\{w_i, x_i, y_i\}$ when $w_i \in W_1 \setminus W'_1$ or $w_i \in W_3$, and the vertices $\{r_i, t_i\}$ when $w_i \in W'_1$. These vertices dominate all vertices in the corresponding gadget H_i . When $w_i \in W_2$, the vertices $\{u_i, v_i\}$ are in color class 3; here, w_i is the only vertex in H_i not dominated by u_i and v_i , and it is dominated by some $w_j \in W_3$. Also, the three color classes are pairwise disjoint. Hence, each color class forms an independent dominating set, and therefore α is a valid 3-role coloring. $\square$

Observation 1. If $n\text{-LP}_2$ is an infinite family of twisted Loupekin snark, then $n\text{-LP}_2$ is k -role colorable for $k \leq 3$.

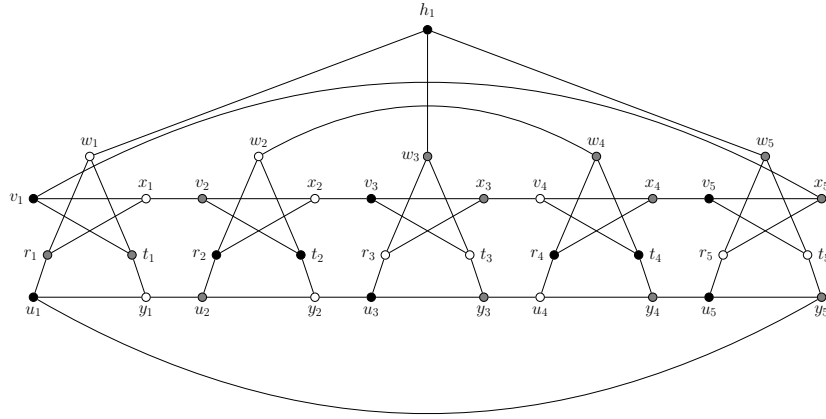


Figure 10. A 3-role coloring of $5-LP_1$ snark

2.4. Generalized Blanuša snark

Watkins [19, 20] generalized the construction of first and second Blanuša snarks, defining two infinite families of generalized Blanuša snarks. Let $B^1 = \{B_1^1, B_2^1, B_3^1, \dots\}$ be the first family of generalized Blanuša snarks and let $B^2 = \{B_1^2, B_2^2, B_3^2, \dots\}$ be the second family of generalized Blanuša snarks. The first member of B^1 is the first Blanuša snark, while the first member of B^2 is the second Blanuša snark. In order to construct the first family, we consider the drawing of the Petersen Graph shown in the left of Fig. 11(a). In this figure, the leftmost drawing shows the first operand, and the next drawing shows the second operand. In order to construct B_1^1 , remove edges u_1v_1 and u_2v_2 from the first operand, remove vertices z_1 and z_2 from the second operand, and add edges $u_1y_1, v_1x_1, u_2y_2, v_2x_2$. Then we obtain the graph B_1^1 . Each member B_i^1 of B^1 , with $i > 1$, is obtained by the dot product $B_{i-1}^1 \cdot P$, with B_{i-1}^1 playing the role of P in the construction of B_1^1 . Then assign the labels for B_i^1 according to Fig. 11(b). The construction of family B^2 is similar to the construction of family B^1 [9].

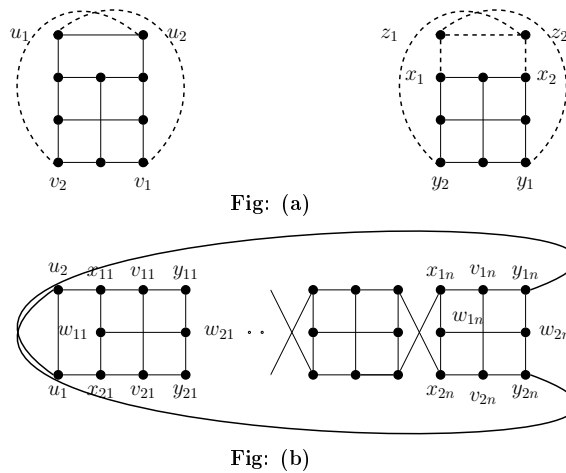


Figure 11. (a) Basic gadget of first Blanuša construction, (b) Generalized Blanuša B_i^1 of the first family B^1

Fig. 12 shows an illustration of the second Blanuša snark B^2 .

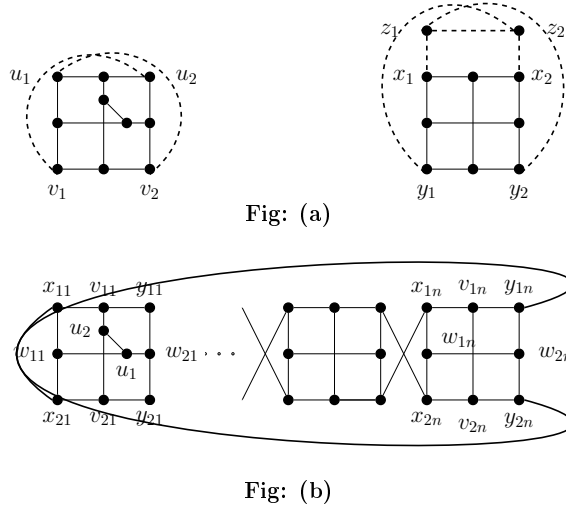


Figure 12. (a) Basic gadget of second Blanuša construction, (b) Generalized Blanuša B_i^2 of the second family B^2

Theorem 2.4. Let B_i^1 be the generalized Blanuša snark of the first family B^1 of order $8i + 10$. Then B_i^1 is k -role colorable for $k \leq 3$.

Proof. Let B_i^1 be the generalized Blanuša snark obtained by the construction using the basic gadget. Let B_1^1 be the first snark in the family B_i^1 .

Case 1:

Let $k = 2$. Define a mapping $\alpha : V(B_i^1) \rightarrow \{1, 2\}$ by, $\alpha(u_1) = 1$, $\alpha(u_2) = 2$,

$$\alpha(x_{1i}) = \alpha(v_{1i}) = \alpha(v_{2i}) = \alpha(y_{2i}) = \begin{cases} 1, & \text{if } i \text{ is even} \\ 2, & \text{if } i \text{ is odd} \end{cases}$$

$$\alpha(y_{1i}) = \alpha(w_{1i}) = \alpha(w_{2i}) = \alpha(x_{2i}) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$$

To verify the adjacency condition, first consider the vertices u_1 and u_2 . Their neighborhoods are $N(u_1) = \{x_{21}, y_{1n}, u_2\}$, $N(u_2) = \{x_{11}, y_{2n}, u_1\}$. Since $\alpha(x_{21}) = \alpha(u_1) = 1$ and $\alpha(x_{11}) = \alpha(u_2) = 2$, we have $\alpha(N(u_1)) = \alpha(N(u_2)) = \{1, 2\}$. For the vertices x_{1i} , we have $N(x_{11}) = \{u_2, v_{11}, w_{11}\}$, $N(x_{1i}) = \{y_{2(i-1)}, v_{1i}, w_{1i}\}$ for $i \neq 1$. For the vertices y_{1i} , $N(y_{1i}) = \{v_{1i}, w_{2i}, x_{2(i+1)}\}$ for $i \neq n$, $N(y_{1n}) = \{v_{1n}, w_{2n}, u_1\}$. Using $\alpha(v_{1i}) = \alpha(x_{1i}) \neq \alpha(w_{1i})$ and $\alpha(w_{2i}) = \alpha(y_{1i}) \neq \alpha(v_{1i})$, it follows that $\alpha(N(x_{1i})) = \alpha(N(y_{1i})) = \{1, 2\}$. The same argument applies to the vertices x_{2i} and y_{2i} . Finally, for the vertices v_{1i} and w_{1i} , $N(v_{1i}) = \{x_{1i}, y_{1i}, v_{2i}\}$, $N(w_{1i}) = \{x_{1i}, x_{2i}, w_{2i}\}$. Since $\alpha(v_{1i}) = \alpha(v_{2i}) \neq \alpha(y_{1i})$ and $\alpha(w_{1i}) = \alpha(w_{2i}) \neq \alpha(x_{1i})$, we obtain $\alpha(N(v_{1i})) = \alpha(N(w_{1i})) = \{1, 2\}$. The same holds for v_{2i} and w_{2i} . Hence, it is a valid 2-role coloring of B_i^1 with role graph R_3 .

Case 2:

Let $k = 3$. Define $\alpha : V(B_i^1) \rightarrow \{1, 2, 3\}$ by $\alpha(u_1) = \alpha(u_2) = \alpha(v_{1i}) = \alpha(v_{2i}) = \alpha(y_{1i}) = \alpha(y_{2i}) = 1$, $\alpha(x_{1i}) = \alpha(x_{2i}) = \alpha(w_{2i}) = 2$, and $\alpha(w_{1i}) = 3$. We show that α is a valid 3-role coloring. For the vertices u_1 and u_2 , we have $N(u_1) = \{u_2, x_{21}, y_{1n}\}$, $N(u_2) = \{u_1, x_{11}, y_{2n}\}$. Since $\alpha(u_2) = \alpha(y_{1n}) = 1$

and $\alpha(x_{21}) = 2$, it follows that $\alpha(N(u_1)) = \{1, 2\}$. Similarly, $\alpha(u_1) = \alpha(y_{2n}) = 1$ and $\alpha(x_{11}) = 2$, giving $\alpha(N(u_2)) = \{1, 2\}$. Next, the vertices x_{1i} , x_{2i} , and w_{2i} each have a common neighbor w_{1i} of color 3 and all other neighbors of color 1. Hence, $\alpha(N(x_{1i})) = \alpha(N(x_{2i})) = \alpha(N(w_{2i})) = \{1, 3\}$. For w_{1i} , we have $N(w_{1i}) = \{x_{1i}, x_{2i}, w_{2i}\}$, and all its neighbors have color 2. Thus, $\alpha(N(w_{1i})) = \{2\}$. Finally, the vertices v_{1i} , v_{2i} , y_{1i} , and y_{2i} each have two neighbors of color 1 and one neighbor of color 2. Hence, $\alpha(N(v_{1i})) = \alpha(N(v_{2i})) = \alpha(N(y_{1i})) = \alpha(N(y_{2i})) = \{1, 2\}$. Therefore, α is a valid 3-role coloring of B_i^1 with role graph $R_{3,1}$, as illustrated in Fig. 13. $\square$

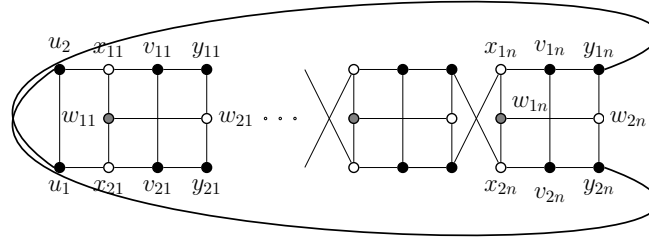


Figure 13. 3-role coloring of generalized Blanuša B_i^1 of the first family B^1

Theorem 2.5. Let B_i^2 be the generalized Blanuša snark of the second family B^2 of order $8i + 10$. Then B_i^2 is k -role colorable for $k \leq 3$.

Proof. Let B_i^2 be the generalized Blanuša snark obtained by the construction using the basic blocks.

Case 1:

Let $k = 2$. Define a mapping $\alpha : V(B_i^2) \rightarrow \{1, 2\}$ by $\alpha(u_1) = 2$, $\alpha(u_2) = 1$, $\alpha(x_{1i}) = \alpha(v_{1i}) = \alpha(v_{2i}) = \alpha(y_{2i}) = 1$, and $\alpha(y_{1i}) = \alpha(w_{1i}) = \alpha(w_{2i}) = \alpha(x_{2i}) = 2$. We claim that α is a valid 2-role coloring of B_i^2 . First, consider the vertices u_1 and u_2 . Their neighborhoods are $N(u_1) = \{w_{11}, w_{21}, u_2\}$ and $N(u_2) = \{v_{11}, v_{21}, u_1\}$. Since $\alpha(w_{11}) = \alpha(w_{21}) = 2$ and $\alpha(u_2) = 1$, and $\alpha(v_{11}) = \alpha(v_{21}) = 1$ and $\alpha(u_1) = 2$, we obtain $\alpha(N(u_1)) = \alpha(N(u_2)) = \{1, 2\}$. Next, consider the vertices x_{1i} and y_{1i} . Their neighborhoods are $N(x_{11}) = \{v_{11}, w_{11}, y_{2n}\}$, $N(x_{1i}) = \{v_{1i}, w_{1i}, y_{2(i-1)}\}$ for $i \neq 1$, and $N(y_{1i}) = \{v_{1i}, w_{2i}, x_{2(i+1)}\}$ for $i \neq n$, while $N(y_{1n}) = \{v_{1n}, w_{2n}, x_{21}\}$. Since $\alpha(v_{1i}) = 1$ and $\alpha(w_{1i}) = \alpha(w_{2i}) = 2$, it follows that $\alpha(N(x_{1i})) = \alpha(N(y_{1i})) = \{1, 2\}$. The same argument applies to the vertices x_{2i} and y_{2i} . Finally, consider the vertices v_{1i} and w_{1i} . Their neighborhoods are $N(v_{11}) = \{x_{11}, y_{11}, u_2\}$, $N(v_{1i}) = \{x_{1i}, y_{1i}, v_{2i}\}$ for $i \neq 1$, $N(w_{11}) = \{x_{11}, x_{21}, u_1\}$, and $N(w_{1i}) = \{x_{1i}, x_{2i}, w_{2i}\}$ for $i \neq 1$. Since $\alpha(x_{1i}) = 1$ and $\alpha(x_{2i}) = \alpha(y_{2i}) = 2$, we obtain $\alpha(N(v_{1i})) = \alpha(N(w_{1i})) = \{1, 2\}$. The same reasoning applies to the vertices v_{2i} and w_{2i} . Therefore, α is a valid 2-role coloring of B_i^2 with role graph R_3 .

Case 2:

Let $k = 3$. Define a mapping $\alpha : V(B_i^2) \rightarrow \{1, 2, 3\}$ as follows: $\alpha(u_2) = \alpha(w_{21}) = \alpha(v_{1i}) = \alpha(y_{1i}) = \alpha(v_{2i}) = \alpha(y_{2i}) = 1$, $\alpha(u_1) = \alpha(x_{1i}) = \alpha(x_{2i}) = 2$, $\alpha(w_{2i}) = 2$ for $i \neq 1$, and $\alpha(w_{1i}) = 3$ for all i . We show that α is a valid 3-role coloring of B_i^2 . Consider the vertex u_1 with $N(u_1) = \{w_{11}, w_{21}, u_2\}$. Since $\alpha(w_{11}) = 3$ and $\alpha(w_{21}) = \alpha(u_2) = 1$, we obtain $\alpha(N(u_1)) = \{1, 3\}$. For the vertex u_2 with $N(u_2) = \{v_{11}, v_{21}, u_1\}$, we have $\alpha(v_{11}) = \alpha(v_{21}) = 1$ and $\alpha(u_1) = 2$, so $\alpha(N(u_2)) = \{1, 2\}$. Next, consider the vertices x_{1i} , x_{2i} , and w_{2i} for $i \neq 1$. Each has a common neighbor w_{1i} of color 3 and all other neighbors of color 1. Thus, $\alpha(N(x_{1i})) = \alpha(N(x_{2i})) = \alpha(N(w_{2i})) = \{1, 3\}$ for $i \neq 1$. Now consider the vertex w_{21} with $N(w_{21}) = \{y_{11}, y_{21}, u_1\}$. Since $\alpha(u_1) = 2$ and $\alpha(y_{11}) = \alpha(y_{21}) = 1$, we have $\alpha(N(w_{21})) = \{1, 2\}$. For the vertices w_{1i} , all neighbors have color 2, and hence $\alpha(N(w_{1i})) = \{2\}$. Finally, consider the vertices v_{1i} , y_{1i} , v_{2i} , and y_{2i} , all of which have color 1. Each such vertex has two neighbors of color 1 and exactly one neighbor of color 2. Therefore, $\alpha(N(v_{1i})) = \alpha(N(v_{2i})) = \alpha(N(y_{1i})) = \alpha(N(y_{2i})) = \{1, 2\}$. Hence, α is a valid 3-role coloring of B_i^2 with role graph $R_{3,1}$, as illustrated in Fig. 14. $\square$

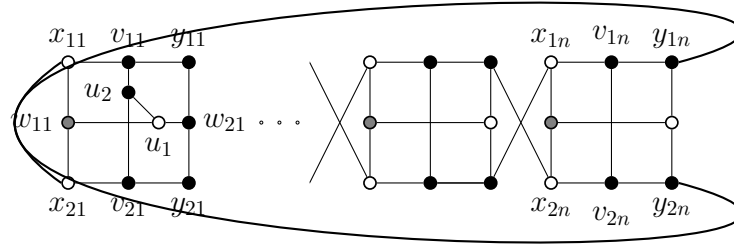


Figure 14. 3-role coloring of generalized Blanuša B_i^2 of the second family B^2

3. k -role coloring of some specific snark graphs

3.1. Double star snark

The double star snark $DS(n, r)$ with $n = 5$ and $r = 2$ has 30 vertices and 45 edges. Its construction is as follows: Let $\{u_i, v_i, w_i, x_i, y_i, z_i\}$ be the vertices of double star snark $DS(5, 2)$, where $1 \leq i \leq 5$. The link edges of $DS(5, 2)$ form two $2n$ -cycles on the vertex sets $\{x_1, y_2, x_3, y_4, x_5, y_1, x_2, y_3, x_4, y_5\}$ and $\{w_1, z_3, w_5, z_2, w_4, z_1, w_3, z_5, w_2, z_4\}$.

Theorem 3.1. *Let $DS(5, 2)$ be the double star snark on 30 vertices and 45 edges. Then $DS(5, 2)$ is k -role colorable for $k \leq 3$.*

Proof. Let $DS(5, 2)$ be the double star snark. Let $\{u_i, v_i, x_i, y_i, w_i, z_i\}$ be the vertices of a double star, where $1 \leq i \leq 5$.

Case 1:

Let $k = 2$. Define a mapping $\alpha : V(DS(5, 2)) \rightarrow \{1, 2\}$ as follows: $\alpha(u_i) = \alpha(v_i) = 1$ and $\alpha(x_i) = \alpha(y_i) = \alpha(w_i) = \alpha(z_i) = 2$. Then, $\alpha(N(u_i)) = \alpha(N(v_i)) = \{1, 2\}$, and $\alpha(N(x_i)) = \alpha(N(y_i)) = \alpha(N(w_i)) = \alpha(N(z_i)) = \{1, 2\}$. This gives a 2-role coloring of $DS(5, 2)$ with role graph R_3 .

Case 2:

Let $k = 3$. Define a mapping $\alpha : V(DS(5, 2)) \rightarrow \{1, 2, 3\}$ as follows: $\alpha(u_i) = \alpha(v_i) = 1$, $\alpha(x_i) = \alpha(w_i) = 2$, and $\alpha(y_i) = \alpha(z_i) = 3$, for $i = 1, \dots, 5$. The neighborhoods of the vertices are given by $N(u_i) = \{v_i, x_i, y_i\}$ and $N(v_i) = \{u_i, w_i, z_i\}$, so that $\alpha(N(u_i)) = \alpha(N(v_i)) = \{1, 2, 3\}$. For vertices of color 2, we have $N(x_i) = \{u_i, y_{i+1}, y_{i-1}\}$ and $N(w_i) = \{v_i, z_{i+2}, z_{i+3}\}$, with indices modulo 5. Therefore, $\alpha(N(x_i)) = \alpha(N(w_i)) = \{1, 3\}$. For vertices of color 3, the neighborhoods are $N(y_i) = \{u_i, x_{i+1}, x_{i-1}\}$ and $N(z_i) = \{v_i, w_{i+2}, w_{i+3}\}$, with indices modulo 5. Therefore, $\alpha(N(y_i)) = \alpha(N(z_i)) = \{1, 2\}$. Thus, α defines a valid 3-role coloring of $DS(5, 2)$ with role graph $R_{3,4}$, as illustrated in Fig. 15. $\square$

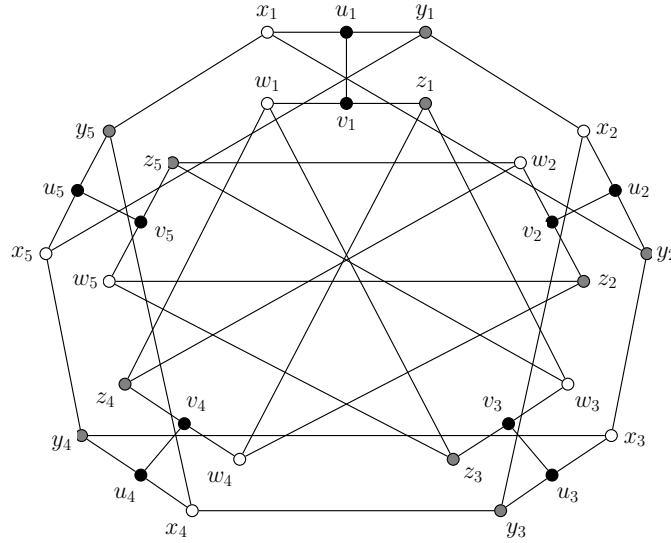


Figure 15. 3-role coloring of double star snark

3.2. Watkins snark

The Watkins snark is a cubic graph discovered by John J. Watkins in 1989. It has 50 vertices and 75 edges. Let W_{50} denote this snark, which contains a 25-cycle formed by the vertices $\{a_1, a_2, \dots, a_{25}\}$ and five vertex-disjoint 5-cycles composed of the vertices $\{b_1, \dots, b_{25}\}$. Each 5-cycle C_j is defined as follows: $V(C_1) = \{b_1, b_{19}, b_2, b_{20}, b_3\}$, $V(C_2) = \{b_{11}, b_4, b_{12}, b_5, b_{13}\}$, $V(C_3) = \{b_{14}, b_{22}, b_{15}, b_{23}, b_{21}\}$, $V(C_4) = \{b_{24}, b_7, b_{25}, b_8, b_6\}$, $V(C_5) = \{b_9, b_{17}, b_{10}, b_{18}, b_{16}\}$.

Theorem 3.2. Let W_{50} be the Watkins snark on 50 vertices and 75 edges. Then W_{50} is k -role colorable for $k \leq 3$.

Proof. Let $\{a_1, a_2, \dots, a_{25}\}$ be the vertices of 25-cycle and C_j denotes the 5-vertex disjoint cycles in W_{50} .

Case 1:

Let $k = 2$. Define a mapping $\alpha : V(W_{50}) \rightarrow \{1, 2\}$ such that, for all $a_i, b_i \in V(W_{50})$ with $1 \leq i \leq 25$, we have $\alpha(a_i) = 1$ and $\alpha(b_i) = 2$. Here, each vertex of color 1 is adjacent to two vertices of color 1 and one vertex of color 2. Similarly, each vertex of color 2 is adjacent to two vertices of color 2 and one vertex of color 1. Hence, $\alpha(N(a_i)) = \alpha(N(b_i)) = \{1, 2\}$. This assignment defines a valid 2-role coloring of W_{50} with role graph R_3 .

Case 2:

Let $k = 3$. Define $\alpha : V(W_{50}) \rightarrow \{1, 2, 3\}$ as follows,

$$\alpha(a_i) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod{5} \\ 2, & \text{if } i \equiv 0 \text{ or } 2 \pmod{5} \\ 3, & \text{if } i \equiv 3 \text{ or } 4 \pmod{5} \end{cases}$$

$$\alpha(b_i) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod{5} \\ 2, & \text{if } i \equiv 3 \text{ or } 4 \pmod{5} \\ 3, & \text{if } i \equiv 0 \text{ or } 2 \pmod{5} \end{cases}$$

We now verify the adjacency condition. Here, each vertex of color 1 is adjacent to two vertices of color 2 and one vertex of color 1. Each vertex of color 2 is adjacent to two vertices of color 3 and one vertex of

color 1. Also, each vertex of color 3 is adjacent to two vertices of color 2 and one vertex of color 3. Thus, α is a valid 3-role coloring with role graph $R_{3,2}$, illustrated in Fig. 16. $\square$

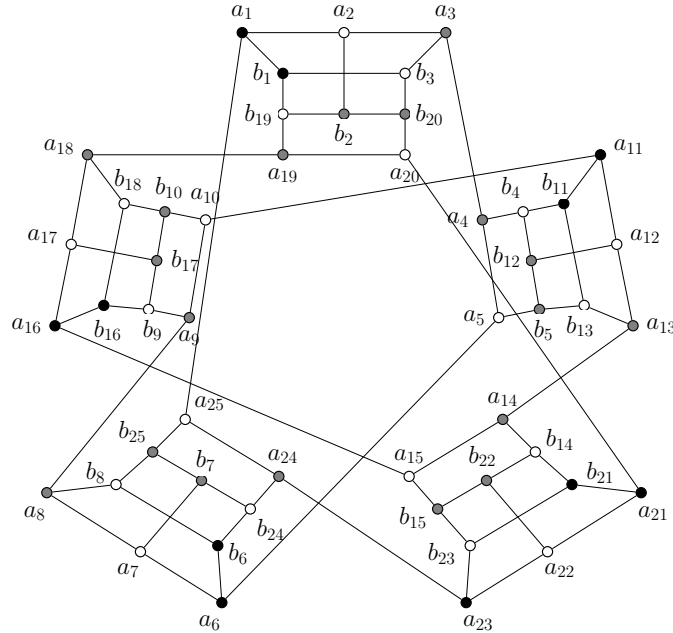


Figure 16. 3-role coloring of Watkins snark

3.3. Szekeres snark

The Szekeres snark S_{50} was discovered by George Szekeres [18] in 1973. It is a snark with 50 vertices and 75 edges. The construction of the Szekeres snark S_{50} consists of 5 gadgets H_i , with vertex set $V(S_{50}) = V_1 \cup V_2$, where $V_1 = \{a_1, a_2, \dots, a_{25}\}$ and $V_2 = \{b_1, b_2, \dots, b_{25}\}$. The vertices $\{a_1, a_2, \dots, a_{25}\}$ form a 25-cycle, and the additional linking edges $E_j = \{b_2b_{14}, b_4b_{17}, b_7b_{20}, b_9b_{22}, b_{12}b_{24}\}$ complete the construction of the Szekeres snark.

Theorem 3.3. *Let S_{50} be a Szekeres snark on 50 vertices and 75 edges. Then S_{50} is k -role colorable for $k \leq 3$.*

Proof. Let S_{50} be a Szekeres snark by considering 5-copies of H_i with vertices $V(S_{50}) = V_1 \cup V_2$, where $V_1 = \{a_1, a_2, \dots, a_{25}\}$ and $V_2 = \{b_1, b_2, \dots, b_{25}\}$.

Case 1:

Let $k = 2$. Define a mapping $\alpha : V(S) \rightarrow \{1, 2\}$ as follows: $\alpha(a_i) = 1$ and $\alpha(b_i) = 2$ for all $1 \leq i \leq 25$. Each vertex a_i or b_i of color 1 has neighbors of colors 1 and 2. Similarly, each vertex a_i or b_i of color 2 has neighbors of colors 1 and 2. Hence, α defines a valid 2-role coloring of S_{50} with role graph R_3 .

Case 2:

Let $k = 3$. We define a mapping $\alpha : V(S_{50}) \rightarrow \{1, 2, 3\}$ as follows:

$$\alpha(a_i) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod{5} \\ 2, & \text{if } i \equiv 0 \text{ or } 2 \pmod{5} \\ 3, & \text{if } i \equiv 3 \text{ or } 4 \pmod{5} \end{cases}$$

$$\alpha(b_i) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod{5} \\ 2, & \text{if } i \equiv 3 \text{ or } 4 \pmod{5} \\ 3, & \text{if } i \equiv 0 \text{ or } 2 \pmod{5} \end{cases}$$

We now verify that α is a valid 3-role coloring. For all vertices a_i and b_i with $\alpha(a_i) = \alpha(b_i) = 1$, each vertex has neighbors of colors 1 and 3. For all vertices a_i and b_i with $\alpha(a_i) = \alpha(b_i) = 2$, we have $\alpha(N(a_i)) = \alpha(N(b_i)) = \{1, 3\}$. And for all vertices a_i and b_i with $\alpha(a_i) = \alpha(b_i) = 3$, we have $\alpha(N(a_i)) = \alpha(N(b_i)) = \{2, 3\}$. Hence, α defines a valid 3-role coloring of S_{50} with role graph $R_{3,2}$, illustrated in Fig. 17. $\square$

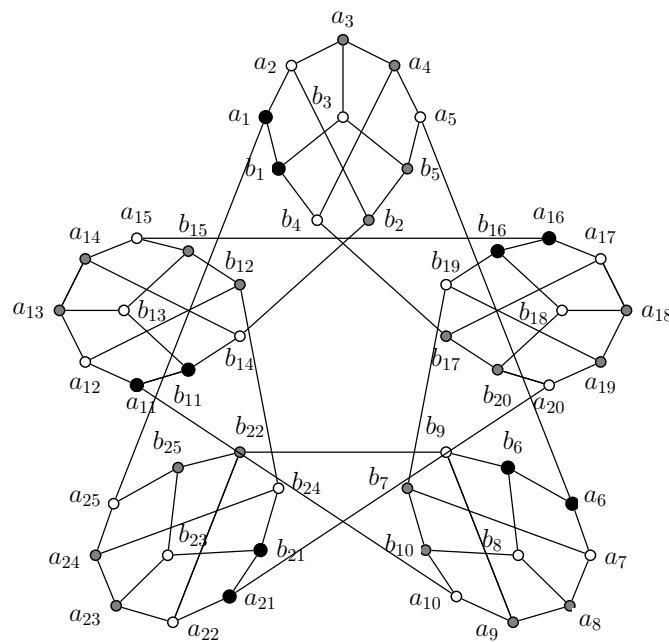


Figure 17. 3-role coloring of Szekeres snark

4. Conclusion and open problems

We present results on several snark families in this work, including the flower snarks, Goldberg snarks, Loupekin snarks, generalized Blanuša snarks, double star snark, Watkins snark, and Szekeres snark. Since snarks are 3-regular, we determined the 3-role coloring for these different snark families. This inspires us to ask the following open question: “Does every r -regular graph admit an r -role coloring when $r \geq 3$?” Furthermore, one can also investigate the k -role coloring of snarks for $k \geq 4$.

Declarations

Author contributions

All authors contributed to the manuscript and approved the final version.

Conflict of interest

The authors declare that they have no conflict of interest.

Data availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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Generative AI and AI-assisted technologies

The authors declare that no generative AI or AI-assisted technologies were used in the preparation of this manuscript, except for tools used solely for spelling, grammar, language editing, reference formatting, or similar routine editorial corrections.

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