

Adjacency spectrum of power graphs of $\mathbb{Z}_m \times \mathbb{Z}_n$

Research Article

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Abstract: The power graph $\mathcal{P}(G)$ of a group G is defined as the simple graph with vertex set G , and two distinct vertices x and y are joined by an edge if and only if either $x = y^k$ or $y = x^k$, $k \in \mathbb{N}$. In this paper, first, we obtain the characteristic polynomial of $\mathcal{P}(\mathbb{Z}_m \times \mathbb{Z}_n)$ for any two positive integers m and n . Then, we determine the full spectrum of $\mathcal{P}(\mathbb{Z}_m \times \mathbb{Z}_n)$ for some particular values of m and n .

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1. Introduction

All graphs considered in this article are simple and undirected. A graph H is typically denoted as $H = (V(H), E(H))$, where $V(H)$ is the vertex set and $E(H)$ is the edge set of H . For $u, v \in V(H)$, by the notation $u \sim v$, we mean u and v are adjacent. For $S_1, S_2 \subseteq V(H)$, $S_1 \cap S_2 = \emptyset$, $S_1 \sim S_2$ means all the vertices in S_1 are adjacent with all the vertices in S_2 . For $u \in V(H)$, $N(u)$ denotes the set of all neighbours of u . An r -regular graph is the one in which every vertex is of degree r . The complete graph K_n on n vertices, is an $(n-1)$ -regular graph. The characteristic polynomial of a square matrix N is defined by $\det(xI - N)$, which we represent as $\psi(N, x)$. For any simple graph H with n vertices, the adjacency matrix $A(H) = (a_{ij})$ is defined as an $n \times n$ matrix, where $a_{ij} = 1$ if the i^{th} vertex is adjacent to the j^{th} vertex, and zero otherwise. The spectrum of H is the multiset of all eigenvalues of $A(H)$ and is denoted by $\{\lambda_1^{\{m_1\}}, \lambda_2^{\{m_2\}}, \dots, \lambda_r^{\{m_r\}}\}$, where $\lambda_1, \lambda_2, \dots, \lambda_r$ are distinct eigenvalues of

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$A(H)$ with multiplicities $m_1, m_2, \dots, m_r$, respectively. The *Kronecker product* of two matrices F and N of order $p \times q$ and $r \times s$, respectively, denoted by $F \otimes N$, is the block matrix $[f_{ij}N]$, where $F = (f_{ij})_{p \times q}$.

A partition $V_1 \cup V_2 \cup \dots \cup V_m$ of $V(H)$ of a graph H is known as an *equitable partition* if, for all i, j , $1 \leq i, j \leq m$, and all $u, v \in V_i$, $|N(u) \cap V_j| = |N(v) \cap V_j|$. For $v \in V_i$, we take $q_{ij} = |N(v) \cap V_j|$, $1 \leq i, j \leq m$, and the matrix $Q = (q_{ij})_{m \times m}$. The matrix Q is called the *quotient matrix* of the partition $V_1 \cup V_2 \cup \dots \cup V_m$. The *join* $H_1 \vee H_2$ of graphs H_1 and H_2 is the graph obtained from the disjoint union of H_1 and H_2 by connecting every vertex in H_1 to every vertex in H_2 by an edge. For an n -vertex graph H with vertex set $V(H) = 1, 2, \dots, n$ and a family of vertex-disjoint graphs $H_1, H_2, \dots, H_n$, the *H-generalized join* of $H_1, H_2, \dots, H_n$ [3], denoted as $\tilde{G} = \bigvee_H \{H_1, H_2, \dots, H_n\}$, is a graph with vertex set $V(\tilde{G}) = \bigcup_{i=1}^n V(H_i)$ and edge set

$$E(\tilde{G}) = \left(\bigcup_{i=1}^n E(H_i) \right) \cup \left(\bigcup_{ij \in E(H)} \{uv : u \in V(H_i), v \in V(H_j)\} \right)$$

If n_i is the order of H_i , $1 \leq i \leq n$, and $v_i \in V(H_i)$, then $\deg_{\tilde{G}}(v_i) = \deg_{H_i}(v_i) + \sum_{ij \in E(H)} n_j$. If $H = K_2$, then the *H-generalized join* coincides with the join of two graphs.

Theorem 1.1 ([14]). *Let H_i be an r_i -regular graph, $1 \leq i \leq m$, and H be an m -vertex graph. Then $V(H_1) \cup V(H_2) \cup \dots \cup V(H_m)$ is an equitable partition of $H[H_1, H_2, \dots, H_m]$, and the characteristic polynomial of $H[H_1, H_2, \dots, H_m]$ is $\psi(H[H_1, H_2, \dots, H_m], x) = \psi(Q, x) \prod_{i=1}^m \psi(H_i, x)/(x - r_i)$, where Q is the quotient matrix.*

Chakrabarty et al. [4] defined ingeniously the *power graph* $\mathcal{P}(G)$ of a group G , where the vertex set of $\mathcal{P}(G)$ is G , and two distinct vertices are adjacent when one is the integral power of the other. The direct product of two groups G_1 and G_2 , is the group $G_1 \times G_2 = (a, b) : a \in G_1, b \in G_2$, where the operation is component-wise. Mehranian et al. [10] studied the adjacency spectrum of power graphs of cyclic groups, elementary abelian groups of prime power order, the Mathieu group M_{11} , and the dihedral group of order twice a prime power. For distinct primes p, q , and r , Ghorbani et al. [7] obtained the characteristic polynomial for power graphs of groups with order p^3 and pqr . Chattopadhyay et al. [5] determined the characteristic polynomial of power graphs of the dihedral group and the generalized quaternion group. Bhuniya et al. [2] explored some relationships of $\mathcal{P}(G_1 \times G_2)$ with $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$. Fast forward to 2021, Jafari and Chattopadhyay [8] contributed to the spectrum of proper power graphs of the direct product of finite element prime order groups (*EPO-groups*). Singh [15] determined the Laplacian spectra of power graphs of certain prime power abelian groups. Banerjee [1] studied the structural and spectral properties of the reduced power graph of finite groups. In 2024, Romdhini et al. [13] obtained the spectrum of the power graph of the generalized quaternion group. Very recently, Rajkumar et al. [12] determined the adjacency spectrum of the reduced power graph of finite groups. For more interesting findings on the power graph, one may refer to [9].

In this article, we undertake an analysis of the spectrum of power graphs within the direct product of cyclic groups. In Section 2, we focus on the structural properties of $\mathcal{P}(\mathbb{Z}_m \times \mathbb{Z}_n)$. In Section 3, we provide the characteristic polynomial of $\mathcal{P}(\mathbb{Z}_m \times \mathbb{Z}_n)$ when $m \mid n$. In addition to this, we elaborate the characteristic polynomial of $\mathcal{P}(\mathbb{Z}_m \times \mathbb{Z}_n)$ for $m = p, pq$, and p^2 . In Section 4, we find the full spectrum of $\mathcal{P}(\mathbb{Z}_p \times \mathbb{Z}_{pq})$ and $\mathcal{P}(\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2})$. Finally, in Section 5, we obtain the characteristic polynomial of $\mathcal{P}(\mathbb{Z}_m \times \mathbb{Z}_n)$ when $m \nmid n$, and then provide explicitly the characteristic polynomial of $\mathcal{P}(\mathbb{Z}_{p^2} \times \mathbb{Z}_n)$ for $p^2 \nmid n$ but $p \mid n$.

2. Properties of $\mathcal{P}(G_1 \times G_2)$

In this section, we present some properties of $\mathcal{P}(G_1 \times G_2)$, where G_1 and G_2 are any two finite groups.

Notation 2.1. We denote J , I , M and O as the matrix comprising of all ones, the identity matrix, permutation matrix and the zero matrix of appropriate orders, respectively. $E_{r \times s}$ denotes a matrix of order $r \times s$ which has all one in the last row and zero otherwise. $L_{r \times s}$ denotes a matrix of order $r \times s$ in which $(r, s)^{\text{th}}$ element is 1 and zero otherwise. For finite groups G_1, G_2 and $(a, b) \in G_1 \times G_2$, $\langle(a, b)\rangle$ denotes the subgroup of $G_1 \times G_2$ generated by (a, b) , and $|\langle(a, b)\rangle|$ denotes the order of $\langle(a, b)\rangle$. $T_{\langle(a, b)\rangle}$ denotes the set of all generators of $\langle(a, b)\rangle$. We denote all the divisors of n as $d_1, d_2, \dots, d_k$, for some integer k . For any integer m , $\phi(m)$ denotes Euler's totient function.

We define a relation ρ on $\mathcal{P}(G_1 \times G_2)$ as $(a_1, b_1) \rho (a_2, b_2)$ if and only if $\langle(a_1, b_1)\rangle = \langle(a_2, b_2)\rangle$. Clearly, ρ is an equivalence relation. The disjoint equivalence classes of ρ are exactly $T_{\langle(a_1, b_1)\rangle}, T_{\langle(a_2, b_2)\rangle}, \dots, T_{\langle(a_l, b_l)\rangle}$, for some positive integer l and $(a_1, b_1), (a_2, b_2), \dots, (a_l, b_l) \in V(\mathcal{P}(G_1 \times G_2))$. Then we have the following partition of $V(\mathcal{P}(G_1 \times G_2))$

$$V(\mathcal{P}(G_1 \times G_2)) = T_{\langle(a_1, b_1)\rangle} \cup T_{\langle(a_2, b_2)\rangle} \cup \dots, T_{\langle(a_l, b_l)\rangle} \quad (1)$$

Throughout the paper, we represent $A(\mathcal{P}(G))$ as a block matrix whose rows and columns are indexed by $T_{\langle(a_1, b_1)\rangle}, T_{\langle(a_2, b_2)\rangle}, \dots, T_{\langle(a_l, b_l)\rangle}$.

Lemma 2.1 (Corollary 2.9, [11]). *Let S_1, S_2 be distinct cyclic subgroups of group G with generator sets T_{S_1} and T_{S_2} , respectively. Then an element of T_{S_1} is adjacent to an element of T_{S_2} in $\mathcal{P}(G)$ if and only if all the elements of T_{S_1} are adjacent to all the elements of T_{S_2} .*

Lemma 2.2. *For $(a, b) \in G_1 \times G_2$, $T_{\langle(a, b)\rangle}$ induces a complete subgraph of $\mathcal{P}(G_1 \times G_2)$ of order $\phi(|\langle(a, b)\rangle|)$.*

Proof. As $T_{\langle(a, b)\rangle}$ is the generator set of the cyclic subgroup $\langle(a, b)\rangle$, $|T_{\langle(a, b)\rangle}| = \phi(|\langle(a, b)\rangle|)$. By the definition of the power graph, the result is straightforward. $\square$

Lemma 2.3. *The partition $T_{\langle(a_1, b_1)\rangle} \cup T_{\langle(a_2, b_2)\rangle} \cup \dots \cup T_{\langle(a_l, b_l)\rangle}$ (with respect to ρ) is an equitable partition of the vertex set of $\mathcal{P}(G_1 \times G_2)$ with $Q = (q_{ij})_{l \times l}$, where*

$$q_{ij} = \begin{cases} \phi(|\langle(a_j, b_j)\rangle|) & \text{if } i \neq j \text{ and } T_{\langle(a_i, b_i)\rangle} \sim T_{\langle(a_j, b_j)\rangle}, \\ \phi(|\langle(a_i, b_i)\rangle|) - 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Proof. From Lemma 2.1 and Lemma 2.2, the adjacency matrix of $\mathcal{P}(G_1 \times G_2)$ can be represented in terms of the following block matrix:

$$\begin{matrix} & T_{\langle(a_1, b_1)\rangle} & \cdots & T_{\langle(a_j, b_j)\rangle} & \cdots & T_{\langle(a_l, b_l)\rangle} \\ \begin{matrix} T_{\langle(a_1, b_1)\rangle} \\ \vdots \\ T_{\langle(a_i, b_i)\rangle} \\ \vdots \\ T_{\langle(a_l, b_l)\rangle} \end{matrix} & \left[\begin{array}{cccccc} & & & & \vdots & \\ & & & & \vdots & \\ & & & & \vdots & \\ \cdots & & \cdots & B(a_i, a_j, b_i, b_j) & \cdots & \cdots \\ & & & \vdots & & \\ & & & \vdots & & \end{array} \right] \end{matrix}$$

where

$$B(a_i, a_j, b_i, b_j) = \begin{cases} J_{\phi(|\langle(a_i, b_i)\rangle|) \times \phi(|\langle(a_j, b_j)\rangle|)} & \text{if } i \neq j \text{ and } T_{\langle(a_i, b_i)\rangle} \sim T_{\langle(a_j, b_j)\rangle}, \\ (J - I)_{\phi(|\langle(a_i, b_i)\rangle|) \times \phi(|\langle(a_i, b_i)\rangle|)} & \text{if } i = j, \\ O_{\phi(|\langle(a_i, b_i)\rangle|) \times \phi(|\langle(a_j, b_j)\rangle|)} & \text{otherwise.} \end{cases}$$

We obtain that the row sum of the matrix $B(a_i, a_j, b_i, b_j)$ is q_{ij} as given in Equation (2). Thus, the given partition is equitable. $\square$

Lemma 2.4. In $\mathcal{P}(\mathbb{Z}_m \times \mathbb{Z}_n)$, a vertex of $T_{\langle(a_i, b_i)\rangle}$ is adjacent to a vertex of $T_{\langle(a_j, b_j)\rangle}$ if and only if either $\langle(a_i, b_i)\rangle \subset \langle(a_j, b_j)\rangle$ or $\langle(a_j, b_j)\rangle \subset \langle(a_i, b_i)\rangle$ in $\mathbb{Z}_m \times \mathbb{Z}_n$.

Proof. Suppose a vertex of $T_{\langle(a_i, b_i)\rangle}$ is adjacent to a vertex of $T_{\langle(a_j, b_j)\rangle}$. By Lemma 2.1, all vertices of $T_{\langle(a_i, b_i)\rangle}$ are adjacent to all vertices of $T_{\langle(a_j, b_j)\rangle}$. Without loss of generality, by the definition of the power graph there exists an integer t such that $(a_i, b_i)^t = (a_j, b_j)$. This implies that $(a_j, b_j) \in \langle(a_i, b_i)\rangle$. Then we get $\langle(a_j, b_j)\rangle \subset \langle(a_i, b_i)\rangle$.

Conversely, without loss of generality, let $\langle(a_j, b_j)\rangle \subset \langle(a_i, b_i)\rangle$. Then $(a_j, b_j) \in \langle(a_i, b_i)\rangle$. That means, there exists an integer t such that $(a_i, b_i)^t = (a_j, b_j)$. This implies that (a_i, b_i) is adjacent to (a_j, b_j) in $\mathcal{P}(\mathbb{Z}_m \times \mathbb{Z}_n)$. $\square$

Lemma 2.5. Consider a graph H with $V(H) = \{1, 2, \dots, l\}$ and $i \sim j$ in H if and only if either $\langle(a_i, b_i)\rangle \subset \langle(a_j, b_j)\rangle$ or $\langle(a_j, b_j)\rangle \subset \langle(a_i, b_i)\rangle$ in $\mathbb{Z}_m \times \mathbb{Z}_n$. Then,

$$\mathcal{P}(\mathbb{Z}_m \times \mathbb{Z}_n) = H[K_{\phi(|\langle(a_1, b_1)\rangle|)}, K_{\phi(|\langle(a_2, b_2)\rangle|)}, \dots, K_{\phi(|\langle(a_l, b_l)\rangle|)}].$$

Proof. The result follows from Lemmas 2.4, 2.2, and 2.1. $\square$

Lemma 2.6. [6] The spectrum of the complete graph K_n is $\{n-1, (-1)^{\{n-1\}}\}$.

3. Characteristic polynomial of $\mathcal{P}(\mathbb{Z}_m \times \mathbb{Z}_n)$ with $m \mid n$

In this section, since we are taking $m \mid n$, the order of cyclic subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ are divisors of n . In the rest of the paper, we take $S = \{d_1, d_2, \dots, d_k\}$ as the set of all divisors of n .

Theorem 3.1. Let $G = \mathbb{Z}_m \times \mathbb{Z}_n$ with $m \mid n$. The characteristic polynomial of $A(\mathcal{P}(G))$ is

$$\psi(A(\mathcal{P}(G)), x) = (1+x)^\alpha \psi(Q, x),$$

where $\alpha = \sum_{i=1}^k n_i(\phi(d_i) - 1)$, n_i is the number of cyclic subgroups of G of order d_i and $Q = (q_{ij})_{l \times l}$ with

$$q_{ij} = \begin{cases} \phi(|\langle(a_j, b_j)\rangle|) & \text{if } i \neq j \text{ and } T_{\langle(a_i, b_i)\rangle} \sim T_{\langle(a_j, b_j)\rangle}, \\ \phi(|\langle(a_i, b_i)\rangle|) - 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As $m \mid n$, all the elements of the group G have possible orders $d_1, d_2, \dots, d_k$. This implies that all the cyclic subgroups of G are of orders $d_1, d_2, \dots, d_k$. Let $n_1, n_2, \dots, n_k$ be positive integers with $\sum_{i=1}^k n_i = l$. Without loss of generality, suppose $\langle(a_1, b_1)\rangle, \langle(a_2, b_2)\rangle, \dots, \langle(a_{n_1}, b_{n_1})\rangle$ are the cyclic subgroups of G of order d_1 , and for $1 \leq t \leq k-1$, $\langle(a_{\sum_{i=1}^t n_i+1}, b_{\sum_{i=1}^t n_i+1})\rangle, \langle(a_{\sum_{i=1}^t n_i+2}, b_{\sum_{i=1}^t n_i+2})\rangle, \dots, \langle(a_{\sum_{i=1}^{t+1} n_i}, b_{\sum_{i=1}^{t+1} n_i})\rangle$ are the cyclic subgroups of G of order d_{t+1} .

We index the rows and columns of $A(\mathcal{P}(G))$ in the order of the generator sets of the subgroups listed above. By Lemma 2.1 and Lemma 2.2, the block matrix of $\mathcal{P}(G)$ corresponding to $T_{\langle(a_i, b_i)\rangle} \times T_{\langle(a_j, b_j)\rangle}$ is $B(T_{\langle(a_i, b_i)\rangle}, T_{\langle(a_j, b_j)\rangle})$, where

$$B(T_{\langle(a_i, b_i)\rangle}, T_{\langle(a_j, b_j)\rangle}) = \begin{cases} J_{\phi(|\langle(a_i, b_i)\rangle|) \times \phi(|\langle(a_j, b_j)\rangle|)} & \text{if } i \neq j \text{ and } T_{\langle(a_i, b_i)\rangle} \sim T_{\langle(a_j, b_j)\rangle}, \\ (J - I)_{\phi(|\langle(a_i, b_i)\rangle|) \times \phi(|\langle(a_i, b_i)\rangle|)} & \text{if } i = j, \\ O_{\phi(|\langle(a_i, b_i)\rangle|) \times \phi(|\langle(a_j, b_j)\rangle|)} & \text{otherwise.} \end{cases}$$

By Lemma 2.3, Lemma 2.5, Lemma 2.6, and Theorem 1.1, we get the required result. $\square$

We apply Theorem 3.1 in the next few results, namely Theorems 3.3, 3.5, and 3.7, to study spectrum of $\mathcal{P}(\mathbb{Z}_m \times \mathbb{Z}_n)$ where $m = p$, $m = pq$, and $m = p^2$ respectively, for distinct primes p and q . If $p \nmid n$ then $\mathbb{Z}_p \times \mathbb{Z}_n \cong \mathbb{Z}_{pn}$. The spectrum of $\mathbb{Z}_n$ is discussed in [10]. So in Theorem 3.3 below, we consider the case $p \mid n$. Before that, we present the following lemma.

Lemma 3.2. *Let prime p divides n . Without loss of generality, suppose the first r elements in S are divisible by p^2 , then the next t elements in S are divisible by p but not by p^2 , and the rest of the elements in S are relatively prime to p . Then all possible cyclic subgroups of $\mathbb{Z}_p \times \mathbb{Z}_n$ are as given below: $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle((p-1), \frac{n}{d_i})\rangle$ for $1 \leq i \leq r$; $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle((p-1), \frac{n}{d_i})\rangle, \langle(1, \frac{np}{d_i})\rangle$ for $r+1 \leq i \leq r+t$; and $\langle(0, \frac{n}{d_i})\rangle$ for $r+t+1 \leq i \leq k$.*

Proof. There may not be any d_i which is divisible by p^2 , and so r may be zero. Since $p \mid n$, there exists at least one d_i such that $p \mid d_i$ but $p^2 \nmid d_i$. So $t \geq 1$. And we have at least one d_i which is relatively prime to n . We have following cases depending on values of r and t .

Case:1 If $r \geq 1$, then $d_1, d_2, \dots, d_r$ are divisible by p^2 . So, $\gcd(p, d_i) = p$ and $\gcd(p, \frac{d_i}{p}) = p$ for $1 \leq i \leq r$. The number of cyclic subgroups of G of order d_i is $\frac{(\phi(1)\phi(d_i) + \phi(p)\phi(\frac{d_i}{p}))}{\phi(d_i)} = p$. These p subgroups are $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle((p-1), \frac{n}{d_i})\rangle$ for $1 \leq i \leq r$.

Case:2 Here $d_{r+1}, d_{r+2}, \dots, d_{r+t}$ are divisible by p but not by p^2 . So, $\gcd(p, d_i) = p$ and $\gcd(p, \frac{d_i}{p}) = 1$ for $r+1 \leq i \leq r+t$. The number of cyclic subgroups of G of order d_i is $\frac{(\phi(1)\phi(d_i) + \phi(p)\phi(\frac{d_i}{p}))}{\phi(d_i)} = (p+1)$. These $(p+1)$ subgroups are $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle((p-1), \frac{n}{d_i})\rangle, \langle(1, \frac{np}{d_i})\rangle$ for $r+1 \leq i \leq r+t$.

Case:3 Here $d_{r+t+1}, d_{r+t+2}, \dots, d_k$ are relatively prime to p . So, $\gcd(p, d_i) = 1$ for $r+t+1 \leq i \leq k$ then the number of cyclic subgroup of G of order d_i is $\frac{(\phi(1)\phi(d_i))}{\phi(d_i)} = 1$. This cyclic subgroup is $\langle(0, \frac{n}{d_i})\rangle$ for $r+t+1 \leq i \leq k$. $\square$

By Lemma 3.2, there are total $pr + (p+1)t + (k-r-t) = l(\text{say})$ cyclic subgroups of $\mathbb{Z}_p \times \mathbb{Z}_n$. We list all these in the order as follows : the cyclic subgroups of Case 1 (of Lemma 3.2) are $S_1 = \langle(0, \frac{n}{d_1})\rangle, S_2 = \langle(1, \frac{n}{d_1})\rangle, \dots, S_{pr} = \langle((p-1), \frac{n}{d_r})\rangle$, then the cyclic subgroup of Case 2 are $S_{pr+1} = \langle(0, \frac{n}{d_{r+1}})\rangle, S_{pr+2} = \langle(1, \frac{n}{d_{r+1}})\rangle, \dots, S_{pr+p(t+1)} = \langle(1, \frac{np}{d_{r+t}})\rangle$ and finally the cyclic subgroups of Case 3 are $S_{pr+p(t+1)+1} = \langle(0, \frac{n}{d_{r+t+1}})\rangle, \dots, S_l = \langle(0, \frac{n}{d_k})\rangle$.

Theorem 3.3. *Let r and t be as given in Lemma 3.2 and $G = \mathbb{Z}_p \times \mathbb{Z}_n$ with $p \mid n$. Then,*

$$\psi(A(\mathcal{P}(G)), x) = (1+x)^\alpha \psi(Q, x),$$

where $\alpha = p \sum_{i=1}^r (\phi(d_i) - 1) + (p+1) \sum_{i=r+1}^{r+t} (\phi(d_i) - 1) + \sum_{i=r+t+1}^k (\phi(d_i) - 1)$ and $Q = (q_{ij})_{l \times l}$ with

$$q_{ij} = \begin{cases} \phi(|S_j|) & \text{if } i \neq j \text{ and } T_{S_i} \sim T_{S_j}, \\ \phi(|S_i|) - 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have $V(\mathcal{P}(\mathbb{Z}_p \times \mathbb{Z}_n)) = T_{S_1} \cup T_{S_2} \cup \dots \cup T_{S_l}$. By Lemma 3.2, number of cyclic subgroups of order d_i is p for $1 \leq i \leq r$; $p+1$ for $r+1 \leq i \leq r+t$; 1 for $r+t+1 \leq i \leq k$. Then, by Lemma 2.5 and Theorem 3.1, we get the required result. $\square$

Remark 3.1. Taking $n = p^2$ yields Corollary 3.2 of [7] as a particular case of Theorem 3.3.

If $p \nmid n$ or $q \nmid n$ then $\mathbb{Z}_{pq} \times \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_q \times \mathbb{Z}_{np}$ or $\mathbb{Z}_p \times \mathbb{Z}_{nq}$ respectively, and this case is discussed in Theorem 3.3. So in the next theorem, we consider $pq \mid n$.

Lemma 3.4. *For distinct primes p and q , let $pq \mid n$. Without loss of generality, in S , suppose the first r_1 elements are relatively prime to pq , next r_2 elements are divisible by p but not by p^2 and q , next r_3 (may be zero) elements are divisible by p^2 but not by q , next r_4 elements are divisible by*

q but not by q^2 and p , next r_5 (may be zero) elements are divisible by q^2 but not by p , next r_6 elements are divisible by pq but not by p^2q and pq^2 , next r_7 (may be zero) elements are divisible by p^2q but not by pq^2 , next r_8 (may be zero) elements are divisible by pq^2 but not by p^2q , and the remaining elements (may be zero) are multiple of p^2q^2 . Then all possible cyclic subgroup of $\mathbb{Z}_{pq} \times \mathbb{Z}_n$ are as given below: $\langle(0, \frac{n}{d_i})\rangle$ for $1 \leq i \leq r_1$; $\langle(0, \frac{n}{d_i})\rangle, \langle(q, \frac{n}{d_i})\rangle, \langle(2q, \frac{n}{d_i})\rangle, \dots, \langle(q(p-1), \frac{n}{d_i})\rangle, \langle(q, \frac{np}{d_i})\rangle$ for $r_1 + 1 \leq i \leq \sum_{j=1}^2 r_j \leq \sum_{j=1}^3 r_j$; $\langle(0, \frac{n}{d_i})\rangle, \langle(q, \frac{n}{d_i})\rangle, \langle(2q, \frac{n}{d_i})\rangle, \dots, \langle(q(p-1), \frac{n}{d_i})\rangle$ for $\sum_{j=1}^2 r_j + 1 \leq i \leq \sum_{j=1}^3 r_j$; $\langle(0, \frac{n}{d_i})\rangle, \langle(p, \frac{n}{d_i})\rangle, \langle(2p, \frac{n}{d_i})\rangle, \dots, \langle(p(q-1), \frac{n}{d_i})\rangle, \langle(p, \frac{np}{d_i})\rangle$ for $\sum_{j=1}^3 r_j + 1 \leq i \leq \sum_{j=1}^4 r_j \leq \sum_{j=1}^5 r_j$; $\langle(0, \frac{n}{d_i})\rangle, \langle(p, \frac{n}{d_i})\rangle, \langle(2p, \frac{n}{d_i})\rangle, \dots, \langle(p(q-1), \frac{n}{d_i})\rangle$ for $\sum_{j=1}^4 r_j + 1 \leq i \leq \sum_{j=1}^5 r_j$; $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle(pq-1, \frac{n}{d_i})\rangle, \langle(1, \frac{np}{d_i})\rangle, \dots, \langle(1, \frac{np(q-1)}{d_i})\rangle, \langle(q, \frac{np}{d_i})\rangle, \langle(1, \frac{nq}{d_i})\rangle, \langle(1, \frac{n2q}{d_i})\rangle, \dots, \langle(1, \frac{n(p-1)q}{d_i})\rangle, \langle(p, \frac{nq}{d_i})\rangle, \langle(1, \frac{npq}{d_i})\rangle$ for $\sum_{j=1}^5 r_j + 1 \leq i \leq \sum_{j=1}^6 r_j \leq k$; $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle(pq-1, \frac{n}{d_i})\rangle, \langle(p, \frac{nq}{d_i})\rangle, \langle(1, \frac{nq}{d_i})\rangle, \langle(2, \frac{nq}{d_i})\rangle, \dots, \langle(p-1, \frac{nq}{d_i})\rangle$ for $\sum_{j=1}^6 r_j < i \leq \sum_{j=1}^7 r_j \leq k$; $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle(pq-1, \frac{n}{d_i})\rangle, \langle(1, \frac{np}{d_i})\rangle, \langle(1, \frac{n2p}{d_i})\rangle, \dots, \langle(1, \frac{np(q-1)}{d_i})\rangle, \langle(q, \frac{np}{d_i})\rangle$ for $\sum_{j=1}^7 r_j < i \leq \sum_{j=1}^8 r_j \leq k$; $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle(pq-1, \frac{n}{d_i})\rangle$ for $\sum_{j=1}^8 r_j < i \leq k$.

Proof. As $pq \mid n$, the order of an element of $\mathbb{Z}_{pq} \times \mathbb{Z}_n$ is a divisor of n . Then, we have the following cases depending on the values of $r_1, r_2, r_3, r_4, r_5, r_6, r_7$ and r_8 .

Case:1 We have at least one d_i which is relatively prime to pq . Let $\gcd(pq, d_i) = 1$ for $1 \leq i \leq r_1$. Then the number of cyclic subgroup of G of order d_i is $\frac{\phi(1)\phi(d_i)}{\phi(d_i)} = 1$. This cyclic subgroup is $\langle(0, \frac{n}{d_i})\rangle$ for $1 \leq i \leq r_1$.

Case:2 Since $p \mid n$, there exists at least one d_i such that $p \mid d_i$ but $pq \nmid d_i$. Let $\gcd(pq, d_i) = p$ for $r_1 + 1 \leq i \leq \sum_{j=1}^3 r_j \leq k$. Then, the following two subcases arise depending upon $p^2 \mid d_i$ or $p^2 \nmid d_i$:

Subcase:2(a) If $\gcd(pq, \frac{d_i}{p}) = 1$ for $r_1 + 1 \leq i \leq \sum_{j=1}^2 r_j \leq \sum_{j=1}^3 r_j$, then the number of cyclic subgroups of G of order d_i is $\frac{\phi(1)\phi(d_i) + \phi(p)\phi(d_i) + \phi(p)\phi(\frac{d_i}{p})}{\phi(d_i)} = (p+1)$. These $(p+1)$ cyclic subgroups are $\langle(0, \frac{n}{d_i})\rangle, \langle(q, \frac{n}{d_i})\rangle, \langle(2q, \frac{n}{d_i})\rangle, \dots, \langle(q(p-1), \frac{n}{d_i})\rangle, \langle(q, \frac{np}{d_i})\rangle$ for $r_1 + 1 \leq i \leq \sum_{j=1}^2 r_j \leq \sum_{j=1}^3 r_j$.

Subcase:2(b) If $\sum_{j=1}^2 r_j < \sum_{j=1}^3 r_j$, then there exists at least one d_i such that $p^2 \mid d_i$. Let $\gcd(pq, \frac{d_i}{p}) = p$ for $\sum_{j=1}^2 r_j + 1 \leq i \leq \sum_{j=1}^3 r_j$. Then, the number of cyclic subgroups of G of order d_i is $\frac{\phi(1)\phi(d_i) + \phi(p)\phi(d_i)}{\phi(d_i)} = p$. These p cyclic subgroups are $\langle(0, \frac{n}{d_i})\rangle, \langle(q, \frac{n}{d_i})\rangle, \langle(2q, \frac{n}{d_i})\rangle, \dots, \langle(q(p-1), \frac{n}{d_i})\rangle$ for $\sum_{j=1}^2 r_j + 1 \leq i \leq \sum_{j=1}^3 r_j$.

Case:3 Since $q \mid n$, there exists at least one d_i such that $q \mid d_i$ but not by pq . Let $\gcd(pq, d_i) = q$ for $\sum_{j=1}^3 r_j + 1 \leq i \leq \sum_{j=1}^5 r_j$. Then, the following two subcases arise depending upon $q^2 \mid d_i$ or $q^2 \nmid d_i$:

Subcase:3(a) If $\gcd(pq, \frac{d_i}{q}) = 1$ for $\sum_{j=1}^3 r_j + 1 \leq i \leq \sum_{j=1}^4 r_j \leq \sum_{j=1}^5 r_j$, then the number of cyclic subgroups of G of order d_i is $\frac{\phi(1)\phi(d_i) + \phi(q)\phi(d_i) + \phi(q)\phi(\frac{d_i}{q})}{\phi(d_i)} = (q+1)$. These $(q+1)$ cyclic subgroups are $\langle(0, \frac{n}{d_i})\rangle, \langle(p, \frac{n}{d_i})\rangle, \langle(2p, \frac{n}{d_i})\rangle, \dots, \langle(p(q-1), \frac{n}{d_i})\rangle, \langle(p, \frac{nq}{d_i})\rangle$ for $\sum_{j=1}^3 r_j + 1 \leq i \leq \sum_{j=1}^4 r_j \leq \sum_{j=1}^5 r_j$.

Subcase:3(b) If $\sum_{j=1}^4 r_j < \sum_{j=1}^5 r_j$, then there exists at least one d_i such that $q^2 \mid d_i$. Let $\gcd(pq, \frac{d_i}{q}) = q$ for $\sum_{j=1}^4 r_j + 1 \leq i \leq \sum_{j=1}^5 r_j$. Then, the number of cyclic subgroups of G of order d_i is $\frac{\phi(1)\phi(d_i) + \phi(q)\phi(d_i)}{\phi(d_i)} = q$. These q cyclic subgroups are $\langle(0, \frac{n}{d_i})\rangle, \langle(p, \frac{n}{d_i})\rangle, \langle(2p, \frac{n}{d_i})\rangle, \dots, \langle(p(q-1), \frac{n}{d_i})\rangle$ for $\sum_{j=1}^4 r_j + 1 \leq i \leq \sum_{j=1}^5 r_j$.

Case:4 Since $pq \mid n$, there exists at least one d_i such that $pq \mid d_i$. Let $\gcd(pq, d_i) = pq$ for $\sum_{j=1}^5 r_j + 1 \leq i \leq k$. Then, there are four subcases arise:

Subcase:4(a) Since $pq \mid d_i$, there exists at least one d_i such that $\gcd(pq, \frac{d_i}{pq}) = 1$ i.e. $d_i = pq$. Let $\gcd(pq, \frac{d_i}{pq}) = 1$ for $\sum_{j=1}^5 r_j + 1 \leq i \leq \sum_{j=1}^6 r_j \leq k$. Then, the number of cyclic subgroups of G of order d_i is $\frac{\phi(1)\phi(d_i) + \phi(p)\phi(d_i) + \phi(q)\phi(d_i) + \phi(pq)\phi(d_i) + \phi(pq)\phi(\frac{d_i}{p}) + \phi(pq)\phi(\frac{d_i}{q}) + \phi(pq)\phi(\frac{d_i}{pq}) + \phi(p)\phi(\frac{d_i}{p}) + \phi(q)\phi(\frac{d_i}{q})}{\phi(d_i)} = pq + p + q + 1$. These $(pq + p + q + 1)$ cyclic subgroups is $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle(pq-1, \frac{n}{d_i})\rangle, \langle(1, \frac{np}{d_i})\rangle, \langle(1, \frac{n2p}{d_i})\rangle, \dots, \langle(1, \frac{np(q-1)}{d_i})\rangle, \langle(q, \frac{np}{d_i})\rangle, \langle(1, \frac{nq}{d_i})\rangle, \langle(1, \frac{n2q}{d_i})\rangle, \dots, \langle(1, \frac{n(p-1)q}{d_i})\rangle, \langle(p, \frac{nq}{d_i})\rangle, \langle(1, \frac{npq}{d_i})\rangle$ for

$$\sum_{j=1}^5 r_j + 1 \leq i \leq \sum_{j=1}^6 r_j \leq k.$$

If $\sum_{j=1}^6 r_j < k$, then these three subcases may occur:

Subcase:4(b) If $\gcd(pq, \frac{d_i}{pq}) = p$ for $\sum_{j=1}^6 r_j < i \leq \sum_{j=1}^7 r_j \leq k$, then the number of cyclic subgroups of G of order d_i is $\frac{\phi(1)\phi(d_i)+\phi(p)\phi(d_i)+\phi(q)\phi(d_i)+\phi(pq)\phi(d_i)+\phi(q)\phi(\frac{d_i}{q})+\phi(pq)\phi(\frac{d_i}{p})}{\phi(d_i)} = pq + p$. These $(pq + p)$ cyclic subgroups are $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle(pq - 1, \frac{n}{d_i})\rangle, \langle(p, \frac{nq}{d_i})\rangle, \langle(1, \frac{nq}{d_i})\rangle, \langle(2, \frac{nq}{d_i})\rangle, \dots, \langle(p - 1, \frac{nq}{d_i})\rangle$ for $\sum_{j=1}^6 r_j < i \leq \sum_{j=1}^7 r_j \leq k$.

Subcase:4(c) If $\gcd(pq, \frac{d_i}{pq}) = q$ for $\sum_{j=1}^7 r_j + 1 < i \leq \sum_{j=1}^8 r_j \leq k$, then the number of cyclic subgroups of G of order d_i is $\frac{\phi(1)\phi(d_i)+\phi(p)\phi(d_i)+\phi(q)\phi(d_i)+\phi(pq)\phi(d_i)+\phi(p)\phi(\frac{d_i}{p})+\phi(pq)\phi(\frac{d_i}{p})}{\phi(d_i)} = pq + q$. These $(pq + q)$ cyclic subgroups are $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle(pq - 1, \frac{n}{d_i})\rangle, \langle(1, \frac{np}{d_i})\rangle, \langle(1, \frac{n2p}{d_i})\rangle, \dots, \langle(1, \frac{np(q-1)}{d_i})\rangle, \langle(q, \frac{np}{d_i})\rangle$ for $\sum_{j=1}^7 r_j < i \leq \sum_{j=1}^8 r_j \leq k$.

Subcase:4(d) If $\gcd(pq, \frac{d_i}{pq}) = pq$ for $\sum_{j=1}^8 r_j < i \leq k$, then the number of cyclic subgroups of G of order d_i is $\frac{\phi(1)\phi(d_i)+\phi(p)\phi(d_i)+\phi(q)\phi(d_i)+\phi(pq)\phi(d_i)}{\phi(d_i)} = pq$. These pq cyclic subgroups are $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle(pq - 1, \frac{n}{d_i})\rangle$ for $\sum_{j=1}^8 r_j < i \leq k$. $\square$

By Lemma 3.4, there are total $r_1 + (p + 1)r_2 + pr_3 + (q + 1)r_4 + qr_5 + (pq + p + q + 1)r_6 + (pq + p)r_7 + (pq + q)r_8 + pq(k - \sum_{j=1}^8 r_j) = l(\text{say})$ cyclic subgroups of $\mathbb{Z}_{pq} \times \mathbb{Z}_n$. We list all these in the order of their occurrence in the statement of Lemma 3.4, that is $S_1 = \langle(0, \frac{n}{d_1})\rangle, S_2 = \langle(0, \frac{n}{d_2})\rangle, \dots, S_l = \langle(pq - 1, \frac{n}{d_k})\rangle$.

Theorem 3.5. Let $r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8$ be as given in Lemma 3.4 and $G = \mathbb{Z}_{pq} \times \mathbb{Z}_n$ with $pq \mid n$. Then,

$$\psi(A(\mathcal{P}(G)), x) = (1 + x)^\alpha \psi(Q, x),$$

where $\alpha = \sum_{i=1}^{t_1} (\phi(d_i) - 1) + (p + 1) \sum_{i=t_1+1}^{t_2} (\phi(d_i) - 1) + p \sum_{i=t_2+1}^{t_3} (\phi(d_i) - 1) + (q + 1) \sum_{i=t_3+1}^{t_4} (\phi(d_i) - 1) + q \sum_{i=t_4+1}^{t_5} (\phi(d_i) - 1) + (pq + p + q + 1) \sum_{i=t_5+1}^{t_6} (\phi(d_i) - 1) + (pq + p) \sum_{i=t_6+1}^{t_7} (\phi(d_i) - 1) + (pq + q) \sum_{i=t_7+1}^{t_8} (\phi(d_i) - 1) + pq \sum_{i=t_8+1}^k (\phi(d_i) - 1)$, and $t_1 = r_1, t_2 = \sum_{j=1}^2 r_j, t_3 = \sum_{j=1}^3 r_j, t_4 = \sum_{j=1}^4 r_j, t_5 = \sum_{j=1}^5 r_j, t_6 = \sum_{j=1}^6 r_j, t_7 = \sum_{j=1}^7 r_j, t_8 = \sum_{j=1}^8 r_j$. and $Q = (q_{ij})_{l \times l}$ with

$$q_{ij} = \begin{cases} \phi(|S_j|) & \text{if } i \neq j \text{ and } T_{S_i} \sim T_{S_j}, \\ \phi(|S_i|) - 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have $V(\mathcal{P}(\mathbb{Z}_{pq} \times \mathbb{Z}_n)) = T_{S_1} \cup T_{S_2} \cup \dots \cup T_{S_l}$. By Lemma 3.4, number of cyclic subgroups of order d_i is 1 for $1 \leq i \leq t_1$; $(p + 1)$ for $t_1 + 1 \leq i \leq t_2$; p for $t_2 + 1 \leq i \leq t_3$; $(q + 1)$ for $t_3 + 1 \leq i \leq t_4$; q for $t_4 + 1 \leq i \leq t_5$; $(pq + p + q + 1)$ for $t_5 + 1 \leq i \leq t_6$; $pq + p$ for $t_6 + 1 \leq i \leq t_7$; $pq + q$ for $t_7 + 1 \leq i \leq t_8$; pq for $t_8 + 1 \leq i \leq k$. Then, by Lemma 2.5 and Theorem 3.1, we get the required result. $\square$

Remark 3.2. Theorem 3.5 can be extended to the case $\mathbb{Z}_m \times \mathbb{Z}_n$ when m is product of distinct primes. Let $m = p_1 p_2 \dots p_r$. Then, the divisors of m are in order $1, p_i, p_i p_j (1 \leq i < j \leq r), p_i p_j p_l (1 \leq i < j < l \leq r), \dots$, and $p_1 p_2 \dots p_r$. We name these divisors as $\xi_1, \xi_2, \dots, \xi_{2^r}$. Analogous to Lemma 3.6, we will have a lemma where there will be 2^r cases, one corresponding to each $\xi_i, 1 \leq i \leq 2^r$. If $\xi_i = p_{i_1} p_{i_2} \dots p_{i_t}$ then the case corresponding to ξ_i will have 2^t subcases. Number of cyclic subgroups in a subcase corresponding to ξ_i can be determined by the number of possible arrangement of divisors of ξ_i and d_j (some divisor of n) such that their least common multiple is d_j . Finally we will have a theorem analogous to Theorem 3.5.

If $p \nmid n$ then $\mathbb{Z}_{p^2} \times \mathbb{Z}_n \cong \mathbb{Z}_{p^2 n} = \mathbb{Z}_{n_1}$ for some $n_1 \in \mathbb{N}$, and this case is discussed in [10]. Thus, we consider the case $p^2 \mid n$.

Lemma 3.6. For prime p , let $p^2 \mid n$. Without loss of generality, in S , suppose the first r_1 elements are divisible by p^3 , the next r_2 elements are divisible by p^2 but not by p^3 , the next r_3 elements are divisible by p but not by p^2 , and the rest of the elements are relatively prime to p^2 . All possible cyclic subgroups of $\mathbb{Z}_{p^2} \times \mathbb{Z}_n$ are as given below: $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle(p^2 - 1, \frac{n}{d_i})\rangle$ for $1 \leq i \leq r_1$; $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle(p^2 - 1, \frac{n}{d_i})\rangle, \langle(1, \frac{np}{d_i})\rangle, \langle(2, \frac{np}{d_i})\rangle, \dots, \langle(p - 1, \frac{np}{d_i})\rangle, \langle(1, \frac{np^2}{d_i})\rangle$ for $r_1 + 1 \leq i \leq r_1 + r_2$; $\langle(0, \frac{n}{d_i})\rangle, \langle(p, \frac{n}{d_i})\rangle, \langle(2p, \frac{n}{d_i})\rangle, \dots, \langle(p(p - 1), \frac{n}{d_i})\rangle, \langle(p, \frac{np}{d_i})\rangle$ for $r_1 + r_2 + 1 \leq i \leq r_1 + r_2 + r_3$; $\langle(0, \frac{n}{d_i})\rangle$ for $r_1 + r_2 + r_3 + 1 \leq i \leq k$.

Proof. As $p^2 \mid n$, the order of all the elements of the group G are the divisors of n . There may not be any d_i which is divisible by p^3 , and so r_1 may be zero. Since $p^2 \mid n$, there exists at least one d_i such that $p^2 \mid d_i$ i.e. $r_2 \geq 1$. Since $p \mid n$, there exists at least one d_i such that $p \mid d_i$ i.e. $r_3 \geq 1$. And we have at least one d_i which is relatively prime to n . We have following cases depending on values of r_1, r_2 and r_3 .

Case:1 If $r_1 \geq 1$, then $d_1, d_2, \dots, d_{r_1}$ are divisible by p^3 . So, $\gcd(p^2, d_i) = p^2$ and $\gcd(p^2, \frac{d_i}{p}) = p^2$ for $1 \leq i \leq r_1$. The number of cyclic subgroups of G of order d_i is $\frac{\phi(1)\phi(d_i) + \phi(p)\phi(d_i) + \phi(p^2)\phi(d_i)}{\phi(d_i)} = p^2$. These p^2 subgroups are $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle(p^2 - 1, \frac{n}{d_i})\rangle$ for $1 \leq i \leq r_1$.

Case:2 Here $d_{r_1+1}, d_{r_1+2}, \dots, d_{r_1+r_2}$ are divisible by p^2 but not by p^3 . So, $\gcd(p^2, d_i) = p^2$ and $\gcd(p^2, \frac{d_i}{p}) = p$ for $r_1 + 1 \leq i \leq r_1 + r_2$. The number of cyclic subgroups of G of order d_i is $\frac{\phi(1)\phi(d_i) + \phi(p)\phi(d_i) + \phi(p^2)\phi(d_i)}{\phi(d_i)} + \frac{\phi(p^2)\phi(\frac{d_i}{p}) + \phi(p^2)\phi(\frac{d_i}{p^2})}{\phi(d_i)} = (p^2 + p)$. These $(p^2 + p)$ subgroups are $\langle(0, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i})\rangle, \langle(2, \frac{n}{d_i})\rangle, \dots, \langle(p^2 - 1, \frac{n}{d_i})\rangle, \langle(1, \frac{np}{d_i})\rangle, \langle(2, \frac{np}{d_i})\rangle, \dots, \langle(p - 1, \frac{np}{d_i})\rangle, \langle(1, \frac{np^2}{d_i})\rangle$ for $r_1 + 1 \leq i \leq r_1 + r_2$.

Case:3 Here $d_{r_1+r_2+1}, d_{r_1+r_2+2}, \dots, d_{r_1+r_2+r_3}$ are divisible by p but not by p^2 . So, $\gcd(p^2, d_i) = p$ for $r_1 + r_2 + 1 \leq i \leq r_1 + r_2 + r_3$. The number of cyclic subgroups of G of order d_i is $\frac{\phi(1)\phi(d_i) + \phi(p)\phi(d_i) + \phi(p)\phi(\frac{d_i}{p})}{\phi(d_i)} = (p + 1)$. These $(p + 1)$ cyclic subgroups are $\langle(0, \frac{n}{d_i})\rangle, \langle(p, \frac{n}{d_i})\rangle, \langle(2p, \frac{n}{d_i})\rangle, \dots, \langle(p(p - 1), \frac{n}{d_i})\rangle, \langle(p, \frac{np}{d_i})\rangle$ for $r_1 + r_2 + 1 \leq i \leq r_1 + r_2 + r_3$.

Case:4 Here $d_{r_1+r_2+r_3+1}, d_{r_1+r_2+r_3+2}, \dots, d_k$ are relatively prime to p . So, $\gcd(p^2, d_i) = 1$ for $r_1 + r_2 + r_3 + 1 \leq i \leq k$. The number of cyclic subgroup of G of order d_i is $\frac{\phi(1)\phi(d_i)}{\phi(d_i)} = 1$. This cyclic subgroup is $\langle(0, \frac{n}{d_i})\rangle$ for $r_1 + r_2 + r_3 + 1 \leq i \leq k$. $\square$

By Lemma 3.6, there are total $p^2 r_1 + (p^2 + p)r_2 + (p + 1)r_3 + (k - r_1 - r_2 - r_3) = l$ (say) cyclic subgroups of $\mathbb{Z}_p \times \mathbb{Z}_n$. We list all these in the order of their occurrence in the statement of Lemma 3.6, that is $S_1 = \langle(0, \frac{n}{d_1})\rangle, S_2 = \langle(1, \frac{n}{d_1})\rangle, \dots, S_l = \langle(0, \frac{n}{d_k})\rangle$.

Theorem 3.7. Let r_1, r_2 and r_3 be as given in Lemma 3.6 and $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_n$ with $p^2 \mid n$. Then,

$$\psi(A(G), x) = (1 + x)^\alpha \psi(Q, x),$$

where $\alpha = p \sum_{i=1}^{r_1} (\phi(d_i) - 1) + (p + 1) \sum_{i=r_1+1}^{r_1+r_2} (\phi(d_i) - 1) + p^2 \sum_{i=r_1+r_2+1}^{r_1+r_2+r_3} (\phi(d_i) - 1) + (p^2 + p) \sum_{i=r_1+r_2+r_3+1}^k (\phi(d_i) - 1)$, $Q = (q_{ij})_{l \times l}$ with

$$q_{ij} = \begin{cases} \phi(|S_j|) & \text{if } i \neq j \text{ and } T_{S_i} \sim T_{S_j}, \\ \phi(|S_i|) - 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have $V(\mathcal{P}(\mathbb{Z}_{p^2} \times \mathbb{Z}_n)) = T_{S_1} \cup T_{S_2} \cup \dots \cup T_{S_l}$. By Lemma 3.6, number of cyclic subgroups of order d_i is p for $1 \leq i \leq r_1$; $(p + 1)$ for $r_1 + 1 \leq i \leq r_1 + r_2$; p^2 for $r_1 + r_2 + 1 \leq i \leq r_1 + r_2 + r_3$; $(p^2 + p)$ for $r_1 + r_2 + r_3 + 1 \leq i \leq k$. By Lemma 2.5 and Theorem 3.1, we get the required result. $\square$

4. Full spectrum of some $\mathcal{P}(\mathbb{Z}_m \times \mathbb{Z}_n)$

Theorem 4.1. Let $G = \mathbb{Z}_p \times \mathbb{Z}_{pq}$, where p, q are distinct prime numbers. Then, the spectrum of $\mathcal{P}(G)$ consists of:

1. -1 with multiplicity $p^2q - p - 4$.
2. $(p-1)q - 1$ with multiplicity p .
3. And the roots of the polynomial

$$-x^4 + (pq - 4)x^3 + (p^2q^2 - p^2q - pq^2 + 3pq + p^2 + q - 7)x^2 + (-p^3(q-1)^2 + 3p^2q^2 - 5p^2q - pq^2 + 4p^2 - q^2 + pq + p + 5q - 8)x - p^4(q-1)^2 + 3p^2q^2 - 6p^2q - pq^2 + pq + 4p^2 - q^2 + 4q - 4.$$

Proof. For $G = \mathbb{Z}_p \times \mathbb{Z}_{pq}$, an element of G is of the order pq, q, p or 1 . We can easily deduce that G has exactly $\frac{\phi(pq)(\phi(p)+2)}{\phi(pq)} = (p+1)$ cyclic subgroups of order pq , $\frac{\phi(p)(\phi(p)+2)}{\phi(p)} = (p+1)$ cyclic subgroups of order p , one cyclic subgroup each of order q and 1 . Then by Case 2 of the proof of Lemma 3.4 the cyclic subgroups of order pq are $\langle(0,1)\rangle, \langle(1,1)\rangle, \langle(2,1)\rangle, \dots, \langle(p-1,1)\rangle, \langle(1,p)\rangle$ and the cyclic subgroups of order p are $\langle(0,q)\rangle, \langle(1,q)\rangle, \dots, \langle(p-1,q)\rangle, \langle(1,0)\rangle$. And by Case 3 of the proof of Lemma 3.4 the cyclic subgroup of order q is $\langle(0,p)\rangle$ and the subgroup of order one is $\langle(0,0)\rangle$. Let $\Omega_{pq} = \{T_{\langle(0,1)\rangle}, T_{\langle(1,1)\rangle}, \dots, T_{\langle(p-1,1)\rangle}, T_{\langle(1,p)\rangle}\}$, $\Omega_p = \{T_{\langle(0,q)\rangle}, T_{\langle(1,q)\rangle}, T_{\langle(2,q)\rangle}, \dots, T_{\langle(p-1,q)\rangle}, T_{\langle(1,0)\rangle}\}$, $\Omega_q = T_{\langle(0,p)\rangle}$, and $\Omega_1 = \{T_{\langle(0,0)\rangle}\}$. By Lemma 2.2, each element of the set $\Omega_{pq} \cup \Omega_p \cup \Omega_q \cup \Omega_1$ induces a complete subgraph in $\mathcal{P}(G)$. No two distinct elements of Ω_{pq} are adjacent because otherwise the corresponding subgroups will coincide. Similarly, no two distinct elements of Ω_p are adjacent. Every element of Ω_{pq} is adjacent to exactly one (distinct) element of Ω_p and one element of Ω_q , because every cyclic group has a unique subgroup of particular order. No element of Ω_p is adjacent to an element of Ω_q because an element of order p cannot be generated by an element of order q and vice-versa. Every element of the set $\Omega_{pq} \cup \Omega_p \cup \Omega_q$ is adjacent to the element of Ω_1 . It is to be noted that the rows and columns of $A(\mathcal{P}(G))$ are indexed in order by the elements of $\Omega_{pq} \cup \Omega_p \cup \Omega_q \cup \Omega_1$. So, $A(\mathcal{P}(G))$ is given by

$$\begin{bmatrix} I_{(p+1) \times (p+1)} \otimes (J-I)_{\phi(pq) \times \phi(pq)} & M_{(p+1) \times (p+1)}^T \otimes J_{\phi(pq) \times \phi(p)} & J_{(p+1) \times 1} \otimes J_{\phi(pq) \times \phi(q)} & J_{(p+1) \times 1} \otimes J_{\phi(pq) \times 1} \\ M_{(p+1) \times (p+1)} \otimes J_{\phi(p) \times \phi(pq)} & I_{(p+1) \times (p+1)} \otimes (J-I)_{\phi(p) \times \phi(p)} & O_{(p+1) \times 1} \otimes O_{\phi(p) \times \phi(q)} & J_{(p+1) \times 1} \otimes J_{\phi(p) \times 1} \\ J_{1 \times (p+1)} \otimes J_{\phi(q) \times \phi(pq)} & O_{1 \times (p+1)} \otimes O_{\phi(q) \times \phi(p)} & I_{1 \times 1} \otimes (J-I)_{\phi(q) \times \phi(q)} & J_{1 \times 1} \otimes J_{\phi(q) \times 1} \\ J_{1 \times (p+1)} \otimes J_{1 \times \phi(pq)} & J_{1 \times (p+1)} \otimes J_{1 \times \phi(p)} & J_{1 \times 1} \otimes J_{1 \times \phi(q)} & O_{1 \times 1} \end{bmatrix}$$

where M is a permutation matrix.

By Lemma 2.5 and Theorem 3.3, we have

$$\psi(A(\mathcal{P}(G)), x) = \psi(Q, x)(1+x)^{[(p+1)(\phi(pq)+\phi(p)-2)+\phi(q)-1]}, \quad (3)$$

where

$$\psi(Q, x) = \begin{vmatrix} (\phi(pq) - 1 - x)I_{(p+1) \times (p+1)} & \phi(p)M_{(p+1) \times (p+1)}^T & \phi(q)J_{(p+1) \times 1} & J_{(p+1) \times 1} \\ \phi(pq)M_{(p+1) \times (p+1)} & (\phi(p) - 1 - x)I_{(p+1) \times (p+1)} & O_{(p+1) \times 1} & J_{(p+1) \times 1} \\ \phi(pq)J_{1 \times (p+1)} & O_{1 \times (p+1)} & (\phi(q) - 1 - x) & 1 \\ \phi(pq)J_{1 \times (p+1)} & \phi(p)J_{1 \times (p+1)} & \phi(q) & -x \end{vmatrix}.$$

After suitable interchange of rows and columns simultaneously (no effect on sign of determinant), we can convert M to an identity matrix and this will not effect other blocks. Then,

$$\psi(Q, x) = \begin{vmatrix} (\phi(pq) - 1 - x)I_{(p+1) \times (p+1)} & \phi(p)I_{(p+1) \times (p+1)} & \phi(q)J_{(p+1) \times 1} & J_{(p+1) \times 1} \\ \phi(pq)I_{(p+1) \times (p+1)} & (\phi(p) - 1 - x)I_{(p+1) \times (p+1)} & O_{(p+1) \times 1} & J_{(p+1) \times 1} \\ \phi(pq)J_{1 \times (p+1)} & O_{1 \times (p+1)} & (\phi(q) - 1 - x) & 1 \\ \phi(pq)J_{1 \times (p+1)} & \phi(p)J_{1 \times (p+1)} & \phi(q) & -x \end{vmatrix}$$

Taking common $\phi(pq)$ from the first $(p+1)$ columns, $\phi(p)$ from the next $(p+1)$ columns and $\phi(q)$

from the last column and taking $a = \frac{(\phi(pq)-1-x)}{\phi(pq)}$, $b = \frac{(\phi(p)-1-x)}{\phi(p)}$, $c = \frac{(\phi(q)-1-x)}{\phi(q)}$, and $d = -x$, we have

$$\psi(Q, x) = \phi(pq)^{(p+1)} \phi(p)^{(p+1)} \phi(q) \begin{vmatrix} aI_{(p+1) \times (p+1)} & I_{(p+1) \times (p+1)} & J_{(p+1) \times 1} & J_{(p+1) \times 1} \\ I_{(p+1) \times (p+1)} & bI_{(p+1) \times (p+1)} & O_{(p+1) \times 1} & J_{(p+1) \times 1} \\ J_{1 \times (p+1)} & O_{1 \times (p+1)} & c & 1 \\ J_{1 \times (p+1)} & J_{1 \times (p+1)} & 1 & d \end{vmatrix}.$$

Applying the row operations $R_i \rightarrow R_i - aR_{i+p+1}$ for $1 \leq i \leq p+1$, $R_{2p+3} \rightarrow R_{2p+3} - (R_{p+2} + R_{p+3} + \dots + R_{2p+2})$ and $R_{2p+4} \rightarrow R_{2p+4} - (R_{p+2} + R_{p+3} + \dots + R_{2p+2})$, we get

$$\psi(Q, x) = \phi(pq)^{(p+1)} \phi(p)^{(p+1)} \phi(q) \begin{vmatrix} O_{(p+1) \times (p+1)} & (1-ab)I_{(p+1) \times (p+1)} & J_{(p+1) \times 1} & (1-a)J_{(p+1) \times 1} \\ I_{(p+1) \times (p+1)} & bI_{(p+1) \times (p+1)} & O_{(p+1) \times 1} & J_{(p+1) \times 1} \\ O_{1 \times (p+1)} & -bJ_{1 \times (p+1)} & c & 1 - (p+1) \\ O_{1 \times (p+1)} & (1-b)J_{1 \times (p+1)} & 1 & d - (p+1) \end{vmatrix}.$$

Now we apply the row operation on the last row $R_{2p+4} \rightarrow R_{2p+4} - R_{2p+3}$. Then, we expand the determinant along first $(p+1)$ columns successively and get

$$\psi(Q, x) = \phi(pq)^{(p+1)} \phi(p)^{(p+1)} \phi(q) \begin{vmatrix} (1-ab)I_{(p+1) \times (p+1)} & J_{(p+1) \times 1} & (1-a)J_{(p+1) \times 1} \\ -bJ_{1 \times (p+1)} & c & -p \\ J_{1 \times p+1} & 1-c & d-1 \end{vmatrix}.$$

We write the above equation in the form:

$$\psi(Q, x) = \phi(pq)^{(p+1)} \phi(p)^{(p+1)} \phi(q) \begin{vmatrix} (1-ab)I_{p \times p} & O_{p \times 1} & J_{p \times 1} & (1-a)J_{p \times 1} \\ O_{1 \times p} & (1-ab) & 1 & (1-a) \\ -bJ_{1 \times p} & -b & c & -p \\ J_{1 \times p} & 1 & 1-c & d-1 \end{vmatrix}.$$

Applying the column operations $C_i \rightarrow C_i - C_{p+1}$ for $1 \leq i \leq p$, we get

$$\psi(Q, x) = \phi(pq)^{(p+1)} \phi(p)^{(p+1)} \phi(q) \begin{vmatrix} (1-ab)I_{p \times p} & O_{p \times 1} & J_{p \times 1} & (1-a)J_{p \times 1} \\ -(1-ab)J_{1 \times p} & (1-ab) & 1 & 1-a \\ O_{1 \times p} & -b & c & -p \\ O_{1 \times p} & 1 & 1-c & d-1 \end{vmatrix}.$$

We apply the row operation $R_{p+1} \rightarrow R_1 + R_2 + \dots + R_p + R_{p+1}$ in the above determinant. Then,

$$\psi(Q, x) = \phi(pq)^{(p+1)} \phi(p)^{(p+1)} \phi(q) \begin{vmatrix} (1-ab)I_{p \times p} & O_{p \times 1} & J_{p \times 1} & (1-a)J_{p \times 1} \\ O_{1 \times p} & (1-ab) & (p+1) & (1-a)(p+1) \\ O_{1 \times p} & -b & c & -p \\ O_{1 \times p} & 1 & 1-c & d-1 \end{vmatrix}.$$

Applying the row operations $R_{p+1} \rightarrow R_{p+1} - (1-ab)R_{p+3}$, $R_{p+2} \rightarrow R_{p+2} + bR_{p+3}$ and by taking common $(1-ab)$ from first p columns, we get

$$\psi(Q, x) = \phi(pq)^{(p+1)} \phi(p)^{(p+1)} \phi(q) (1-ab)^p \begin{vmatrix} I_{p \times p} & O_{p \times 1} & J_{p \times 1} & (1-a)J_{p \times 1} \\ O_{1 \times p} & 0 & p+c+ab(1-c) & (1-a)(p+1) - (1-ab)(d-1) \\ O_{1 \times p} & 0 & c+b(1-c) & -p+b(d-1) \\ O_{1 \times p} & 1 & 1-c & d-1 \end{vmatrix}.$$

Expanding the above determinant along the first $(p+1)$ columns and then applying the row operation $R_1 \rightarrow R_1 - aR_2$ to the resulting determinant, we have

$$\psi(Q, x) = \phi(pq)^{(p+1)} \phi(p)^{(p+1)} \phi(q) (1-ab)^p \det(C),$$

where

$$\det(C) = \begin{vmatrix} p+c-ac & p+(1-a)+(1-d) \\ b+c-bc & -p-b(1-d) \end{vmatrix}.$$

Thus, $\psi(Q, x) = \phi(pq)^{(p+1)} \phi(p)^{(p+1)} \phi(q)(1-ab)^p [(p+c-ac)(-p-b(1-d)) - (b+c-bc)(p+(1-a)+(1-d))]$. The factor $(1-ab)^p = \left(\frac{(1+x)(\phi(pq)+\phi(p)-1-x)}{\phi(pq)\phi(p)}\right)^p$ gives the eigenvalues -1 and $\phi(pq) + \phi(p) - 1$ with both multiplicities p . And $\phi(pq)\phi(p)\phi(q)\det(C)$ is the quartic polynomial given in the hypothesis and from this factor we get four eigenvalues. Finally by Equation (3), we get multiplicity of -1 is $p^2q - p - 4$. $\square$

Example 4.2. We illustrate Theorem 4.1 for $p = 3$ and $q = 2$. So we get the partition of $V(\mathcal{P}(\mathbb{Z}_3 \times \mathbb{Z}_6))$ as

$$V(\mathcal{P}(\mathbb{Z}_3 \times \mathbb{Z}_6)) = T_{(0,1)} \cup T_{(1,1)} \cup T_{(2,1)} \cup T_{(1,3)} \cup T_{(0,2)} \cup T_{(1,2)} \cup T_{(2,2)} \cup T_{(1,0)} \cup T_{(0,3)} \cup T_{(0,0)}.$$

By indexing the rows and columns in the order of the above partition, we get

$$A(\mathcal{P}(\mathbb{Z}_3 \times \mathbb{Z}_6)) = \begin{bmatrix} I_4 \otimes (J - I)_{2 \times 2} & M_4^T \otimes J_{2 \times 2} & J_{4 \times 1} \otimes J_{2 \times 1} & J_{4 \times 1} \otimes J_{2 \times 1} \\ M_4 \otimes J_{2 \times 2} & I_4 \otimes (J - I)_{2 \times 2} & O_{4 \times 1} \otimes O_{2 \times 1} & J_{4 \times 1} \otimes J_{2 \times 1} \\ J_{1 \times 4} \otimes J_{1 \times 2} & O_{1 \times 4} \otimes O_{1 \times 2} & I_1 \otimes (J - I)_{1 \times 1} & J_{1 \times 1} \otimes J_{1 \times 1} \\ J_{1 \times 4} \otimes J_{1 \times 2} & J_{1 \times 4} \otimes J_{1 \times 2} & J_{1 \times 1} \otimes J_{1 \times 1} & O_{1 \times 1} \end{bmatrix},$$

where M is a permutation matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then by Theorem 3.3, we get $\psi(A(\mathcal{P}(\mathbb{Z}_3 \times \mathbb{Z}_6)), x) = \psi(Q, x) (1+x)^8$, where $\psi(Q, x)$ is given by

$$\begin{bmatrix} (1-x)I_4 & 2M_4^T & J_{4 \times 1} & J_{4 \times 1} \\ 2M_4 & (1-x)I_4 & O_{4 \times 1} & J_{4 \times 1} \\ 2J_{1 \times 4} & O_{1 \times 4} & -x & 1 \\ 2J_{1 \times 4} & 2J_{1 \times 4} & 1 & -x \end{bmatrix}$$

After interchange the rows R_6 and R_7 , and columns C_6 and C_7 simultaneously, we get

$$\psi(Q, x) = \begin{vmatrix} (1-x)I_4 & 2I_4 & J_{4 \times 1} & J_{4 \times 1} \\ 2I_4 & (1-x)I_4 & O_{4 \times 1} & J_{4 \times 1} \\ 2J_{1 \times 4} & O_{1 \times 4} & -x & 1 \\ 2J_{1 \times 4} & 2J_{1 \times 4} & 1 & -x \end{vmatrix} = (2)^4 (2)^4 \begin{vmatrix} \frac{(1-x)}{2}I_4 & I_4 & J_{4 \times 1} & J_{4 \times 1} \\ I_4 & \frac{(1-x)}{2}I_4 & O_{4 \times 1} & J_{4 \times 1} \\ J_{1 \times 4} & O_{1 \times 4} & -x & 1 \\ J_{1 \times 4} & J_{1 \times 4} & 1 & -x \end{vmatrix}$$

We put $a = \frac{(1-x)}{2}$, $b = -x$, apply the row operations $R_i \rightarrow R_i - aR_{i+4}$ for $1 \leq i \leq 4$, $R_{10} \rightarrow R_{10} - R_9$, and get

$$\psi(Q, x) = (2)^8 \begin{vmatrix} O_4 & (1-a^2)I_4 & J_{4 \times 1} & (1-a)J_{4 \times 1} \\ I_4 & aI_4 & O_{4 \times 1} & J_{4 \times 1} \\ J_{1 \times 4} & O_{1 \times 4} & b & 1 \\ O_{1 \times 4} & J_{1 \times 4} & 1-b & b-1 \end{vmatrix}.$$

Now applying the row operation $R_9 \rightarrow R_9 - (R_5 + R_6 + R_7 + R_8)$, and expanding along the first four

columns, we get

$$|\psi(Q, x)| = (2)^8 \begin{vmatrix} (1-a^2)I_4 & J_{4 \times 1} & (1-a)J_{4 \times 1} \\ -aJ_{1 \times 4} & b & -3 \\ J_{1 \times 4} & 1-b & b-1 \end{vmatrix}.$$

First, applying the column operations $C_i \rightarrow C_i - C_4$, $1 \leq i \leq 3$, then row operation $R_4 \rightarrow R_4 + R_3 + R_2 + R_1$, and expanding along first three columns, we have

$$\psi(Q, x) = 2^8(1-a^2)^3 \begin{vmatrix} (1-a^2) & 4 & 4(1-a) \\ -a & b & -3 \\ 1 & 1-b & b-1 \end{vmatrix}.$$

After putting the values of a and b , we get $|Q - xI| = (1+x)^3(3-x)^3(x+3)(x-1)(x^2-4x-17)$. So, the adjacency spectrum of $\mathcal{P}(\mathbb{Z}_3 \times \mathbb{Z}_6)$ is

$$\begin{pmatrix} -3 & -1 & 2-\sqrt{21} & 1 & 3 & 2+\sqrt{21} \\ 1 & 11 & 1 & 1 & 3 & 1 \end{pmatrix}.$$

Theorem 4.3. Let $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$, where p is a prime number. The spectrum of $\mathcal{P}(G)$ consists of:

1. -1 with multiplicity $p^4 - p^2 - 2p - 2$.
2. $p^2 - p - 1$ with multiplicity $p^2 - 1$.
3. $\frac{1}{2}(p^2 \pm (p-1)\sqrt{5p^2 - 2p + 1} - 3)$ with multiplicity p .
4. And the roots of the polynomial $(x^3 + (3-p^2)x^2 + (-2p^4 + 3p^3 - 4p^2 + p + 3)x - p^5 + p^4 - 2p^2 + p + 1)$.

Proof. For $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$, an element of the group G is of order 1, p or p^2 . We get that G has exactly $\frac{\phi(p^2)(2+2\phi(p)+\phi(p^2))}{\phi(p^2)} = p(p+1)$ cyclic subgroups of order p^2 , $\frac{(2\phi(1)+\phi(p))\phi(p)}{\phi(p)} = p+1$ cyclic subgroups of order p , and one subgroup of order 1.

By Case 2 of the proof of Lemma 3.6, the cyclic subgroups of order p^2 are $\langle(0, 1)\rangle, \langle(1, 1)\rangle, \langle(2, 1)\rangle, \dots, \langle(p^2-1, 1)\rangle, \langle(1, p)\rangle, \langle(2, p)\rangle, \dots, \langle(p-1, p)\rangle, \langle(1, 0)\rangle$. By Case 3 of the proof of Lemma 3.6, the cyclic subgroups of order p are $\langle(0, p)\rangle, \langle(p, p)\rangle, \langle(2p, p)\rangle, \dots, \langle((p-1)p, p)\rangle, \langle(p, 0)\rangle$ and by Case 4 of the same lemma, the subgroup of order 1 is $\langle(0, 0)\rangle$. Let $\Omega_{p^2} = T_{\langle(0,1)\rangle}, T_{\langle(1,1)\rangle}, T_{\langle(2,1)\rangle}, \dots, T_{\langle(p^2-1,1)\rangle}, T_{\langle(1,p)\rangle}, T_{\langle(2,p)\rangle}, \dots, T_{\langle(p-1,p)\rangle}, T_{\langle(1,0)\rangle}$, $\Omega_p = T_{\langle(0,p)\rangle}, T_{\langle(p,p)\rangle}, T_{\langle(2p,p)\rangle}, \dots, T_{\langle(p(p-1),p)\rangle}, T_{\langle(p,0)\rangle}$ and $\Omega_1 = T_{\langle(0,0)\rangle}$. No two distinct elements of Ω_{p^2} are adjacent because otherwise the corresponding subgroups will coincide. Similarly, no two distinct elements of Ω_p are adjacent. Each element of Ω_{p^2} is adjacent to exactly one (distinct) element of Ω_p because every cyclic group has a unique subgroup of particular order. And every element from the set $\Omega_{p^2} \cup \Omega_p$ is adjacent to the element of Ω_1 . It is to be noted that the rows and columns of $A(\mathcal{P}(G))$ are indexed in order by the elements of $\Omega_1 \cup \Omega_{p^2} \cup \Omega_p$. So, $A(\mathcal{P}(G))$ is given by

$$A(\mathcal{P}(G)) = \begin{bmatrix} 0 & J_{1 \times (p^2+p)} \otimes J_{1 \times \phi(p^2)} & J_{1 \times (p+1)} \otimes J_{1 \times \phi(p)} \\ J_{(p^2+p) \times 1} \otimes J_{\phi(p^2) \times 1} & I_{(p^2+p) \times (p^2+p)} \otimes (J - I)_{\phi(p^2) \times \phi(p^2)} & J_{p \times 1} \otimes (M_{(p+1) \times (p+1)} \otimes J_{\phi(p^2) \times \phi(p)}) \\ J_{(p+1) \times 1} \otimes J_{\phi(p) \times 1} & J_{1 \times p} \otimes (M_{(p+1) \times (p+1)} \otimes J_{\phi(p) \times \phi(p^2)})^T & I_{(p+1) \times (p+1)} \otimes (J - I)_{\phi(p) \times \phi(p)} \end{bmatrix}$$

where M is a permutation matrix.

Then by Lemma 2.5 and Theorem 3.1, we get

$$\psi(A(\mathcal{P}(G)), x) = (1+x)^{(\phi(p^2)-1)(p^2+p)+(\phi(p)-1)(p+1)} \psi(Q, x),$$

where

$$\psi(Q, x) = \begin{vmatrix} -x & \phi(p^2)J_{1 \times (p^2+p)} & \phi(p)J_{1 \times (p+1)} \\ J_{(p^2+p) \times 1} & (\phi(p^2) - 1 - x)I_{(p^2+p) \times (p^2+p)} & J_{p \times 1} \otimes (\phi(p)M_{(p+1) \times (p+1)}) \\ J_{(p+1) \times 1} & J_{1 \times p} \otimes (\phi(p^2)M_{(p+1) \times (p+1)}^T) & (\phi(p) - 1 - x)I_{(p+1) \times (p+1)} \end{vmatrix}.$$

After suitable interchange of rows and columns simultaneously (no effect on sign of determinant), we can convert M to an identity matrix and this will not effect other blocks. Taking common 1 from first column, $\phi(p^2)$ from next $(p^2 + p)$ columns and $\phi(p)$ from last $(p + 1)$ columns, we get

$$\psi(Q, x) = \phi(p^2)^{p^2+p} \phi(p)^{p+1} \begin{vmatrix} -x & J_{1 \times (p^2+p)} & J_{1 \times (p+1)} \\ J_{(p^2+p) \times 1} & \frac{(\phi(p^2)-1-x)}{\phi(p^2)} I_{(p^2+p) \times (p^2+p)} & J_{p \times 1} \otimes (I_{(p+1) \times (p+1)}) \\ J_{(p+1) \times 1} & J_{1 \times p} \otimes (I_{(p+1) \times (p+1)}) & \frac{(\phi(p)-1-x)}{\phi(p)} I_{(p+1) \times (p+1)} \end{vmatrix}.$$

By taking $\frac{(\phi(p^2)-1-x)}{\phi(p^2)} = a$, $\frac{(\phi(p)-1-x)}{\phi(p)} = b$, and $-x = c$, we have

$$\psi(Q, x) = \phi(p^2)^{p^2+p} \phi(p)^{p+1} \begin{vmatrix} c & J_{1 \times (p^2+p)} & J_{1 \times (p+1)} \\ J_{(p^2+p) \times 1} & aI_{(p^2+p) \times (p^2+p)} & J_{p \times 1} \otimes (I_{(p+1) \times (p+1)}) \\ J_{(p+1) \times 1} & J_{1 \times p} \otimes (I_{(p+1) \times (p+1)}) & bI_{(p+1) \times (p+1)} \end{vmatrix}.$$

By applying the row operations on blocks $R_i \rightarrow R_i - R_{(p+1)}$ for $2 \leq i \leq p$, we get

$$\psi(Q, x) = \phi(p^2)^{p^2+p} \phi(p)^{p+1} \begin{vmatrix} c & J_{1 \times (p+1)} & J_{1 \times (p+1)} & \cdots & J_{1 \times (p+1)} & J_{1 \times (p+1)} \\ O_{(p+1) \times (p+1)} & aI_{(p+1) \times (p+1)} & O_{(p+1) \times (p+1)} & \cdots & -aI_{(p+1) \times (p+1)} & O_{(p+1) \times (p+1)} \\ O_{(p+1) \times (p+1)} & O_{(p+1) \times (p+1)} & aI_{(p+1) \times (p+1)} & \cdots & -aI_{(p+1) \times (p+1)} & O_{(p+1) \times (p+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J_{(p+1) \times 1} & O_{(p+1) \times (p+1)} & O_{(p+1) \times (p+1)} & \cdots & aI_{(p+1) \times (p+1)} & I_{(p+1) \times (p+1)} \\ J_{(p+1) \times 1} & I_{(p+1) \times (p+1)} & I_{(p+1) \times (p+1)} & \cdots & I_{(p+1) \times (p+1)} & bI_{(p+1) \times (p+1)} \end{vmatrix}.$$

Applying the column operation on $(p + 1)^{th}$ block, $C_{p+1} \rightarrow C_2 + C_3 + \dots + C_{p+1}$, we get

$$\psi(Q, x) = \phi(p^2)^{p^2+p} \phi(p)^{p+1} \begin{vmatrix} c & J_{1 \times (p+1)} & J_{1 \times (p+1)} & \cdots & pJ_{1 \times (p+1)} & J_{1 \times (p+1)} \\ O & aI & O & \cdots & O & O \\ O & O & aI & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J_{(p+1) \times 1} & O & O & \cdots & aI & I \\ J_{(p+1) \times 1} & I & I & \cdots & pI & bI \end{vmatrix}.$$

Expanding along the rows $R_2, R_3, \dots, R_{p^2}$, we get

$$\psi(Q, x) = \phi(p^2)^{p^2+p} \phi(p)^{p+1} a^{(p+1)(p-1)} \begin{vmatrix} c & pJ_{1 \times (p+1)} & J_{1 \times (p+1)} \\ J_{(p+1) \times 1} & aI_{(p+1) \times (p+1)} & I_{(p+1) \times (p+1)} \\ J_{(p+1) \times 1} & pI_{(p+1) \times (p+1)} & bI_{(p+1) \times (p+1)} \end{vmatrix}.$$

By applying the row operations $R_1 \rightarrow R_1 - (R_2 + R_3 + \dots + R_{p+2})$, $R_{p+2+i} \rightarrow R_{p+2+i} - bR_{i+1}$ for $1 \leq i \leq p + 1$, we get

$$\psi(Q, x) = \phi(p^2)^{p^2+p} \phi(p)^{p+1} a^{(p^2-1)} \begin{vmatrix} c - (p + 1) & (p - a)J_{(p+1) \times 1} & O_{1 \times (p+1)} \\ J_{(p+1) \times 1} & aI_{(p+1) \times (p+1)} & I_{(p+1) \times (p+1)} \\ (1 - b)J_{(p+1) \times 1} & (p - ab)I_{(p+1) \times (p+1)} & O_{(p+1) \times (p+1)} \end{vmatrix}.$$

Expanding along the last $p + 1$ columns, we get

$$\psi(Q, x) = \phi(p^2)^{p^2+p} \phi(p)^{p+1} a^{(p^2-1)} \begin{vmatrix} c - (p+1) & (p-a)J_{(p+1) \times 1} \\ (1-b)J_{(p+1) \times 1} & (p-ab)I_{(p+1) \times (p+1)} \end{vmatrix}.$$

Applying the row operations $R_i \rightarrow R_i - R_2$ for $3 \leq i \leq p+2$ and then the column operation $C_2 \rightarrow C_2 + C_3 + \dots + C_{p+2}$, we get

$$\psi(Q, x) = \phi(p^2)^{p^2+p} \phi(p)^{p+1} a^{(p^2-1)} \begin{vmatrix} c - (p+1) & (p+1)(p-a) & (p-a) & \cdots & (p-a) \\ (1-b) & p-ab & 0 & \cdots & 0 \\ 0 & 0 & p-ab & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p-ab \end{vmatrix}.$$

This implies that

$$\psi(Q, x) = \phi(p^2)^{p^2+p} \phi(p)^{p+1} a^{(p^2-1)} (p-ab)^p [(p-ab)(c - (p+1)) - (p+1)(1-b)(p-a)] \quad (4)$$

Here, $(p-ab)$ is a quadratic polynomial and $[(p-ab)(c - (p+1)) - (p+1)(1-b)(p-a)]$ is a cubic polynomial. By putting values of a , b and c in Equation (4), then $\psi(Q, x)$ is

$$\phi(p^2)^{p^2+p} \phi(p)^{p+1} (\phi(p^2) - 1 - x)^{(p^2-1)} (-x^2 + (p^2 - 3)x + p^4 - 3p^3 + 4p^2 - p - 2)^p [x^3 + (3 - p^2)x^2 + (-2p^4 + 3p^3 - 4p^2 + p + 3)x - p^5 + p^4 - 2p^2 + p + 1]. \quad \square$$

5. Characteristic polynomial of $\mathcal{P}(\mathbb{Z}_m \times \mathbb{Z}_n)$, $m \nmid n$

If $m \nmid n$ and $\gcd(m, n) = 1$, then $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$. So here we study the spectrum of $\mathcal{P}(\mathbb{Z}_m \times \mathbb{Z}_n)$ when $m \nmid n$ and $\gcd(m, n) \neq 1$. Recall that we have taken the set of all divisors of n as $S = \{d_1, d_2, \dots, d_k\}$. Let $S' = \{\xi_1, \xi_2, \dots, \xi_t, \xi_{t+1}, \dots, \xi_s\}$ be the set of divisors of m . Since $\gcd(m, n) \neq 1$, without loss of generality let $S \cap S' = \{d_{r+1}, d_{r+2}, \dots, d_{k=r+h}\} = \{\xi_{t+1}, \xi_{t+2}, \dots, \xi_{s=t+h}\}$ for some positive integers r , t and h . Let $T = \{d_i \xi_j : \gcd(d_i, \xi_j) = 1, 1 \leq i \leq k, 1 \leq j \leq s\}$, and $T' = T / \{S \cup S'\}$.

Theorem 5.1. Let $G = \mathbb{Z}_m \times \mathbb{Z}_n$ with $m \nmid n$ and $\gcd(m, n) \neq 1$. Then,

$$\psi(A(\mathcal{P}(G)), x) = (1+x)^\alpha \psi(Q, x),$$

where $\alpha = \sum_{i=1}^k n_i(\phi(d_i) - 1) + \sum_{j=1}^t m_j(\phi(\xi_j) - 1) + \sum_{a \in T'} \eta_a(\phi(a) - 1)$, n_i, m_j, η_a are respectively the number of cyclic subgroups of G of order d_i, ξ_j, a , and $Q = (q_{ij})_{l \times l}$ with

$$q_{ij} = \begin{cases} \phi(|\langle (a_j, b_j) \rangle|) & \text{if } i \neq j \text{ and } T_{\langle (a_i, b_i) \rangle} \sim T_{\langle (a_j, b_j) \rangle}, \\ \phi(|\langle (a_i, b_i) \rangle|) - 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We obtain that $S \cup S' \cup T'$ is the set of all possible orders of the cyclic subgroups of G . Then we proceed exactly in the similar way as in the proof of Theorem 3.1 and get the required result. $\square$

Lemma 5.2. Let $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_n$ with $p \mid n$ and $p^2 \nmid n$. Let the first c elements in S be relatively prime to p^2 , and the next $k-c$ elements in S be divisible by p . Then, the cyclic subgroups of G of different orders are as given below: the cyclic subgroups of order d_i are $\langle (0, \frac{n}{d_i}) \rangle$ for $1 \leq i \leq c$; $\langle (0, \frac{n}{d_i}) \rangle, \langle (p, \frac{n}{d_i}) \rangle, \langle (2p, \frac{n}{d_i}) \rangle, \dots, \langle (p(p-1), \frac{n}{d_i}) \rangle, \langle (p, \frac{np}{d_i}) \rangle$ for $c+1 \leq i \leq k$; and the cyclic subgroups of order $d_i p^2$ are $\langle (1, \frac{n}{d_i}) \rangle, \langle (1, \frac{n}{d_i p}) \rangle, \langle (2, \frac{n}{d_i p}) \rangle, \dots, \langle (p-1, \frac{n}{d_i p}) \rangle$ for $1 \leq i \leq c$.

Proof. We have at least one d_i which is relatively prime to p^2 . So $c \geq 1$. Since $p|n$, there exists at least one d_i such that $p|d_i$. So $k - c \geq 1$. Then, the order of the elements of G are $d_1, d_2, \dots, d_k, d_1p^2, d_2p^2, \dots, d_cp^2$. Now the cyclic subgroups of G of different orders are obtained as below.

Case:1 For each i , $1 \leq i \leq c$, the number of cyclic subgroup of G of order d_i is $\frac{(\phi(1)\phi(d_i))}{\phi(d_i)} = 1$. This cyclic subgroup is $\langle(0, \frac{n}{d_i})\rangle$ for $1 \leq i \leq c$.

Case:2 For each i , $c + 1 \leq i \leq k$, the number of cyclic subgroups of group G of order d_i is $\frac{[\phi(1)\phi(d_i) + \phi(p)\phi(d_i) + \phi(p)\phi(\frac{d_i}{p})]}{\phi(d_i)} = (p + 1)$. These $(p + 1)$ cyclic subgroups are $\langle(0, \frac{n}{d_i})\rangle, \langle(p, \frac{n}{d_i})\rangle, \langle(2p, \frac{n}{d_i})\rangle, \dots, \langle(p(p-1), \frac{n}{d_i})\rangle, \langle(p, \frac{np}{d_i})\rangle$ for $c + 1 \leq i \leq k$.

Case:3 For each i , $1 \leq i \leq c$, the number of cyclic subgroups of group G of order $d_i p^2$ is $\frac{\phi(p^2)\phi(d_i) + \phi(p^2)\phi(d_i p)}{\phi(d_i)} = p$. These p cyclic subgroups are $\langle(1, \frac{n}{d_i})\rangle, \langle(1, \frac{n}{d_i p})\rangle, \langle(2, \frac{n}{d_i p})\rangle, \dots, \langle(p-1, \frac{n}{d_i p})\rangle$ for $1 \leq i \leq c$. $\square$

By Lemma 5.2, there are total $c + (p + 1)(k - c) + pc = l(\text{say})$ cyclic subgroups of $\mathbb{Z}_{p^2} \times \mathbb{Z}_n$. We list all these in the order of their occurrence in the statement of Lemma 5.2, that is $S_1 = \langle(0, \frac{n}{d_1})\rangle, S_2 = \langle(0, \frac{n}{d_2})\rangle, \dots, S_l = \langle(p-1, \frac{n}{d_{kp}})\rangle$. Without loss of generality, let $d_c = 1$. $S' = \{p^2, p, 1\}$ and $T' = \{d_1 p^2, d_2 p^2, \dots, d_{c-1} p^2\}$.

Theorem 5.3. Let $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_n$ with $p | n$ and $p^2 \nmid n$. Let the first c elements in S be relatively prime to p^2 , and the next $k - c$ elements in S be divisible by p . Then,

$$\psi(A(\mathcal{P}(G)), x) = (1 + x)^\alpha \psi(Q, x),$$

where $\alpha = \sum_{i=1}^c (\phi(d_i) - 1) + (p + 1) \sum_{i=c+1}^k (\phi(d_i) - 1) + p \sum_{i=1}^c (\phi(d_i p^2) - 1)$ and $Q = (q_{ij})_{l \times l}$ with

$$q_{ij} = \begin{cases} \phi(|S_j|) & \text{if } i \neq j \text{ and } T_{S_i} \sim T_{S_j}, \\ \phi(|S_i|) - 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have $V(\mathcal{P}(G)) = T_{S_1} \cup T_{S_2} \cup \dots \cup T_{S_l}$. By Lemma 5.2, number of cyclic subgroups of order d_i is 1 for $1 \leq i \leq c$; $(p + 1)$ for $c + 1 \leq i \leq k$. The number of cyclic subgroups of order p^2 is p . And the number of cyclic subgroups of order $d_i p^2$ is p for $1 \leq i \leq c - 1$. Then, by Lemma 2.5 and Theorem 5.1, we get the required result. $\square$

Theorem 5.4. Let $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_{pq}$, where p, q are distinct prime numbers. Then, the spectrum of $\mathcal{P}(G)$ consists of:

1. -1 with multiplicity $p^3 q - 2p - 6$.
2. $p^2 q - pq - 1$ with multiplicity $p - 1$.
3. $pq - q - 1$ with multiplicity $p - 1$.
4. And the remaining 8 eigenvalues are the roots of the determinant

$$\begin{vmatrix} 1 - ab & 0 & -b & 1 - b & -b & 1 - b \\ 1 & 0 & c & 1 & 1 & 1 \\ 0 & 1 - cd & 0 & 0 & -d & 1 - d \\ 1 - a & 0 & 1 - p & d - p & -p & 1 - p \\ 1 & 1 & 1 & 0 & e & 1 \\ 1 - a & 1 - c & 1 - p & 1 - p & 1 - 2p & f - 2p \end{vmatrix} \quad (5)$$

where $a = \frac{\phi(p^2 q) - 1 - x}{\phi(p^2 q)}$, $b = \frac{\phi(p^2) - 1 - x}{\phi(p^2)}$, $c = \frac{\phi(pq) - 1 - x}{\phi(pq)}$, $d = \frac{\phi(p) - 1 - x}{\phi(p)}$, $e = \frac{\phi(q) - 1 - x}{\phi(q)}$ and $f = -x$.

Proof. The possible orders of the elements of G are $1, q, p, pq, p^2$ and p^2q . By Lemma 5.2, the cyclic subgroups of G of different orders are as follow: order p^2q are $\langle(1, 1)\rangle, \langle(2, 1)\rangle, \dots, \langle(p-1, 1)\rangle, \langle(1, p)\rangle$; order p^2 are $\langle(1, q)\rangle, \langle(2, q)\rangle, \dots, \langle(p-1, q)\rangle, \langle(1, 0)\rangle$; order pq are $\langle(0, 1)\rangle, \langle(p, 1)\rangle, \langle(2p, 1)\rangle, \dots, \langle((p-1)p, 1)\rangle, \langle(p, p)\rangle$; order p are $\langle(0, q)\rangle, \langle(p, q)\rangle, \langle(2p, q)\rangle, \dots, \langle((p-1)p, q)\rangle, \langle(p, 0)\rangle$; order q is $\langle(0, p)\rangle$, and the order 1 is $\langle(0, 0)\rangle$. Let $\Omega_{p^2q} = \{T_{\langle(1,1)\rangle}, T_{\langle(2,1)\rangle}, \dots, T_{\langle((p-1),1)\rangle}, T_{\langle(1,p)\rangle}\}$; $\Omega_{p^2} = \{T_{\langle(1,q)\rangle}, T_{\langle(2,q)\rangle}, \dots, T_{\langle((p-1),q)\rangle}, T_{\langle(1,0)\rangle}\}$; $\Omega_{pq} = \{T_{\langle(0,1)\rangle}, T_{\langle(p,1)\rangle}, T_{\langle(2p,1)\rangle}, \dots, T_{\langle((p-1)p,1)\rangle}, T_{\langle(p,p)\rangle}\}$; $\Omega_p = \{T_{\langle(0,q)\rangle}, T_{\langle(p,q)\rangle}, T_{\langle(2p,q)\rangle}, \dots, T_{\langle((p-1)p,q)\rangle}, T_{\langle(p,0)\rangle}\}$; $\Omega_q = \{T_{\langle(0,p)\rangle}\}$ and $\Omega_1 = \{T_{\langle(0,0)\rangle}\}$.

No two distinct elements of Ω_{p^2q} are adjacent because otherwise, the corresponding subgroups would be the same. Similarly, no two distinct elements of $\Omega_{p^2}, \Omega_{pq}$ or Ω_p are adjacent. Each element of Ω_{p^2q} is adjacent to exactly one distinct element of Ω_{p^2} because every cyclic group has a unique subgroup of a particular order. Every element of Ω_{p^2q} is adjacent to $T_{\langle(p,p)\rangle} \in \Omega_{pq}$, $T_{\langle(p,0)\rangle} \in \Omega_p$ and $T_{\langle(0,p)\rangle} \in \Omega_q$. No elements of Ω_{p^2} are adjacent to elements of Ω_{pq} or Ω_q because $p^2 \nmid pq$ and $p^2 \nmid q$ respectively. Each element of Ω_{p^2} is adjacent to $T_{\langle(p,0)\rangle} \in \Omega_p$. Each element of Ω_{pq} is adjacent to exactly one distinct element of Ω_p and to $T_{\langle(0,p)\rangle} \in \Omega_q$. No element of Ω_p is adjacent to an element of Ω_q . Every element of $\Omega_{p^2q} \cup \Omega_{p^2} \cup \Omega_{pq} \cup \Omega_p \cup \Omega_q$ is adjacent to Ω_1 . It is to be noted that the rows and columns of $A(\mathcal{P}(G))$ are indexed in order by the elements of $\Omega_{p^2q} \cup \Omega_{p^2} \cup \Omega_{pq} \cup \Omega_p \cup \Omega_q \cup \Omega_1$. So, $A(\mathcal{P}(G))$ is given by

$$\begin{bmatrix} I_{p \times p} \otimes (J-I)_{\phi(p^2q) \times \phi(p^2q)} & M_{p \times p}^T \otimes J_{\phi(p^2q) \times \phi(p^2)} & E_{(p+1) \times p}^T \otimes J_{\phi(p^2q) \times \phi(pq)} & E_{(p+1) \times p}^T \otimes J_{\phi(p^2q) \times \phi(p)} \\ M_{p \times p} \otimes J_{\phi(p^2) \times \phi(p^2q)} & I_{p \times p} \otimes (J-I)_{\phi(p^2) \times \phi(p^2)} & O_{p\phi(p^2) \times (p+1)\phi(pq)} & E_{(p+1) \times p}^T \otimes J_{\phi(p^2) \times \phi(p)} \\ E_{(p+1) \times p} \otimes J_{\phi(pq) \times \phi(p^2q)} & O_{(p+1)\phi(pq) \times p\phi(p^2)} & I_{(p+1) \times (p+1)} \otimes (J-I)_{\phi(pq) \times \phi(pq)} & N_{(p+1) \times (p+1)}^T \otimes J_{\phi(pq) \times \phi(p)} \\ E_{(p+1) \times p} \otimes J_{\phi(p) \times \phi(p^2q)} & E_{(p+1) \times p} \otimes J_{\phi(p) \times \phi(p^2)} & N_{(p+1) \times (p+1)} \otimes J_{\phi(p) \times \phi(pq)} & I_{(p+1) \times (p+1)} \otimes (J-I)_{\phi(p) \times \phi(p)} \\ J_{1 \times p} \otimes J_{\phi(q) \times \phi(p^2q)} & O_{1 \times p\phi(p^2)} & J_{1 \times (p+1)} \otimes J_{\phi(q) \times \phi(pq)} & O_{1 \times (p+1)\phi(p)} \\ J_{1 \times p\phi(p^2q)} & J_{1 \times p\phi(p^2)} & J_{1 \times (p+1)\phi(pq)} & J_{1 \times p\phi(p)} \end{bmatrix} \cdot \begin{bmatrix} J_{p \times 1} \otimes J_{\phi(p^2q) \times \phi(q)} & J_{p \times 1} \otimes J_{\phi(p^2q) \times 1} \\ O_{p\phi(p^2) \times 1} & J_{p \times 1} \otimes J_{\phi(p^2) \times 1} \\ J_{(p+1) \times 1} \otimes J_{\phi(pq) \times \phi(q)} & J_{(p+1) \times 1} \otimes J_{\phi(pq) \times 1} \\ O_{p\phi(p) \times \phi(q)} & J_{p \times 1} \otimes J_{\phi(p) \times 1} \\ (J-I)_{\phi(q) \times \phi(q)} & J_{\phi(q) \times 1} \\ J_{1 \times \phi(q)} & 0 \end{bmatrix}$$

where M and N are permutation matrices.

By Lemma 2.5 and Theorem 5.1, we have

$$\psi(A(\mathcal{P}(G), x) = \psi(Q, x)(1+x)^{[p(\phi(p^2q)+\phi(p^2)-2)+(p+1)(\phi(pq)+\phi(p)-2)+\phi(q)-1]} \quad (6)$$

where $\psi(Q, x)$ is

$$\begin{vmatrix} (\phi(p^2q) - 1 - x)I_{p \times p} & \phi(p^2)M_{p \times p}^T & \phi(pq)E_{(p+1) \times p}^T & \phi(p)E_{(p+1) \times p}^T & \phi(q)J_{p \times 1} & J_{p \times 1} \\ \phi(p^2q)M_{p \times p} & (\phi(p^2) - 1 - x)I_{p \times p} & O_{p \times (p+1)} & \phi(p)E_{(p+1) \times p}^T & O_{p \times 1} & J_{p \times 1} \\ \phi(p^2q)E_{(p+1) \times p} & O_{(p+1) \times p} & (\phi(pq) - 1 - x)I_{(p+1) \times (p+1)} & \phi(p)N_{(p+1) \times (p+1)}^T & \phi(q)J_{(p+1) \times 1} & J_{(p+1) \times 1} \\ \phi(p^2q)E_{(p+1) \times p} & \phi(p^2)E_{(p+1) \times p} & \phi(pq)N_{(p+1) \times (p+1)} & (\phi(p) - 1 - x)I_{(p+1) \times (p+1)} & O_{1 \times (p+1)} & J_{(p+1) \times 1} \\ \phi(p^2q)J_{1 \times p} & O_{1 \times p} & \phi(pq)J_{1 \times (p+1)} & O_{1 \times p+1} & \phi(q) - 1 - x & 1 \\ \phi(p^2q)J_{1 \times p} & \phi(p^2)J_{1 \times p} & \phi(pq)J_{1 \times (p+1)} & \phi(p)J_{1 \times (p+1)} & 1 & -x \end{vmatrix}.$$

After suitable interchange of rows and columns simultaneously (no effect on sign of determinant), we can convert M, N to identity matrices and this will not effect other blocks. Then,

$$\psi(Q, x) = \begin{vmatrix} (\phi(p^2q) - 1 - x)I_{p \times p} & \phi(p^2)I_{p \times p} & \phi(pq)E_{(p+1) \times p}^T & \phi(p)E_{(p+1) \times p}^T & \phi(q)J_{p \times 1} & J_{p \times 1} \\ \phi(p^2q)I_{p \times p} & (\phi(p^2) - 1 - x)I_{p \times p} & O_{p \times (p+1)} & \phi(p)E_{(p+1) \times p}^T & O_{p \times 1} & J_{p \times 1} \\ \phi(p^2q)E_{(p+1) \times p} & O_{(p+1) \times p} & (\phi(pq) - 1 - x)I_{(p+1) \times (p+1)} & \phi(p)I_{(p+1) \times (p+1)} & \phi(q)J_{(p+1) \times 1} & J_{(p+1) \times 1} \\ \phi(p^2q)E_{(p+1) \times p} & \phi(p^2)E_{(p+1) \times p} & \phi(pq)I_{(p+1) \times (p+1)} & (\phi(p) - 1 - x)I_{(p+1) \times (p+1)} & O_{1, (p+1)} & J_{(p+1) \times 1} \\ \phi(p^2q)J_{1 \times p} & O_{1 \times p} & \phi(pq)J_{1 \times (p+1)} & O_{1 \times (p+1)} & \phi(q) - 1 - x & 1 \\ \phi(p^2q)J_{1 \times p} & \phi(p^2)J_{1 \times p} & \phi(pq)J_{1 \times (p+1)} & \phi(p)J_{1 \times (p+1)} & 1 & -x \end{vmatrix}.$$

Taking common $\phi(p^2q), \phi(p^2), \phi(pq), \phi(p)$ and $\phi(q)$ from the first p columns, the next p columns, the next $(p+1)$ columns, the next $(p+1)$ columns and the last second column respectively. Let $a = \frac{\phi(p^2q)-1-x}{\phi(p^2q)}$, $b = \frac{\phi(p^2)-1-x}{\phi(p^2)}$, $c = \frac{\phi(pq)-1-x}{\phi(pq)}$, $d = \frac{\phi(p)-1-x}{\phi(p)}$, $e = \frac{\phi(q)-1-x}{\phi(q)}$ and $f = -x$. Then, we get

$$\psi(Q, x) = \phi(p^2q)^p \phi(p^2)^p \phi(pq)^{p+1} \phi(p)^{p+1} \phi(q) \begin{vmatrix} aI_{p \times p} & I_{p \times p} & E_{(p+1) \times p}^T & E_{(p+1) \times p}^T & J_{p \times 1} & J_{p \times 1} \\ I_{p \times p} & bI_{p \times p} & O_{p \times (p+1)} & O_{p \times (p+1)} & O_{p \times 1} & J_{p \times 1} \\ E_{(p+1) \times p} & O_{(p+1) \times p} & cI_{(p+1) \times (p+1)} & I_{(p+1) \times (p+1)} & J_{(p+1) \times 1} & J_{(p+1) \times 1} \\ E_{(p+1) \times p} & E_{(p+1) \times p} & I_{(p+1) \times (p+1)} & dI_{(p+1) \times (p+1)} & O_{(p+1) \times 1} & J_{(p+1) \times 1} \\ J_{1 \times p} & O_{1 \times p} & J_{1 \times (p+1)} & O_{1 \times (p+1)} & e & 1 \\ J_{1 \times p} & J_{1 \times p} & J_{1 \times (p+1)} & J_{1 \times (p+1)} & 1 & f \end{vmatrix}.$$

Suppose $w = \phi(p^2q)^p \phi(p^2)^p \phi(pq)^{p+1} \phi(p)^{p+1} \phi(q)$. Apply the row operations $R_{p+i} \rightarrow R_{p+i} - bR_i$ for $1 \leq i \leq p$, $R_{4p+2} \rightarrow R_{4p+2} - (R_1 + R_2 + \dots + R_p)$ and $R_{4p+4} \rightarrow R_{4p+4} - (R_1 + R_2 + \dots + R_p)$. We get

$$\psi(Q, x) = w \begin{vmatrix} aI_{p \times p} & I_{p \times p} & E_{(p+1) \times p}^T & E_{(p+1) \times p}^T & J_{p \times 1} & J_{p \times 1} \\ (1-ab)I_{p \times p} & O_{p \times p} & -bE_{(p+1) \times p}^T & (1-b)E_{(p+1) \times p}^T & -bJ_{p \times 1} & (1-b)J_{p \times 1} \\ E_{(p+1) \times p} & O_{(p+1) \times p} & cI_{(p+1) \times (p+1)} & I_{(p+1) \times (p+1)} & J_{(p+1) \times 1} & J_{(p+1) \times 1} \\ (1-a)E_{(p+1) \times p} & O_{(p+1) \times p} & (I - pL)_{(p+1) \times (p+1)} & (dI - pL)_{(p+1) \times (p+1)} & -pL_{(p+1) \times 1} & (J - pL)_{(p+1) \times 1} \\ J_{1 \times p} & O_{1 \times p} & J_{1 \times (p+1)} & O_{1 \times (p+1)} & e & 1 \\ (1-a)J_{1 \times p} & O_{1 \times p} & (J - pL)_{1 \times (p+1)} & (J - pL)_{1 \times (p+1)} & 1 - p & f - p \end{vmatrix}.$$

Expanding the determinant along the columns $C_{p+1}, C_{p+2}, \dots, C_{2p}$, we get

$$\psi(Q, x) = w \begin{vmatrix} (1-ab)I_{p \times p} & -bE_{(p+1) \times p}^T & (1-b)E_{(p+1) \times p}^T & -bJ_{p \times 1} & (1-b)J_{p \times 1} \\ E_{(p+1) \times p} & cI_{(p+1) \times (p+1)} & I_{(p+1) \times (p+1)} & L_{(p+1) \times 1} & L_{(p+1) \times 1} \\ (1-a)E_{(p+1) \times p} & (I - pL)_{(p+1) \times (p+1)} & (dI - pL)_{(p+1) \times (p+1)} & -pL_{(p+1) \times 1} & (J - pL)_{(p+1) \times 1} \\ J_{1 \times p} & J_{1 \times (p+1)} & O_{1 \times (p+1)} & e & 1 \\ (1-a)J_{1 \times p} & (J - pL)_{1 \times (p+1)} & (J - pL)_{1 \times (p+1)} & 1 - p & f - p \end{vmatrix}$$

Applying the column operations $C_{p+i} \rightarrow C_{p+i} - cC_{2p+1+i}$ for $1 \leq i \leq p$, $C_{3p+3} \rightarrow C_{3p+3} - (C_{2p+2} + C_{2p+3} + \dots + C_{3p+1})$ and $C_{3p+4} \rightarrow C_{3p+4} - (C_{2p+2} + C_{2p+3} + \dots + C_{3p+1})$, the determinant is

$$w \begin{vmatrix} (1-ab)I_{p \times p} & -bE_{(p+1) \times p}^T & (1-b)E_{(p+1) \times p}^T & -bJ_{p \times 1} & (1-b)J_{p \times 1} \\ E_{(p+1) \times p} & cL_{(p+1) \times (p+1)} & I_{(p+1) \times (p+1)} & L_{(p+1) \times 1} & L_{(p+1) \times 1} \\ (1-a)E_{(p+1) \times p} & ((1-cd)I - (p-cd)L)_{(p+1) \times (p+1)} & (dI - pL)_{(p+1) \times (p+1)} & (-dJ + (d-p)L)_{(p+1) \times 1} & ((1-d)J + (d-p)L)_{(p+1) \times 1} \\ J_{1 \times p} & J_{1 \times (p+1)} & O_{1 \times (p+1)} & e & 1 \\ (1-a)J_{1 \times p} & ((1-c)J - (p-c)L)_{1 \times (p+1)} & (J - pL)_{1 \times (p+1)} & 1 - 2p & f - 2p \end{vmatrix}.$$

Expanding along the rows $R_{p+1}, R_{p+2}, \dots, R_{2p}$, $\psi(Q, x)$ is

$$w \begin{vmatrix} (1-ab)I_{p \times p} & -bE_{(p+1) \times p}^T & (1-b)J_{p \times 1} & -bJ_{p \times 1} & (1-b)J_{p \times 1} \\ J_{1 \times p} & cL_{1 \times (p+1)} & 1 & 1 & 1 \\ (1-a)E_{(p+1) \times p} & ((1-cd)I - (p-cd)L)_{(p+1) \times (p+1)} & (d-p)L_{(p+1) \times 1} & (-dJ + (d-p)L)_{(p+1) \times 1} & ((1-d)J + (d-p)L)_{(p+1) \times 1} \\ J_{1 \times p} & J_{1 \times (p+1)} & 0 & e & 1 \\ (1-a)J_{1 \times p} & ((1-c)J - (p-c)L)_{1 \times (p+1)} & (1-p) & 1 - 2p & f - 2p \end{vmatrix}.$$

Apply the column operations $C_i \rightarrow C_i - C_p$ for $1 \leq i \leq p-1$ and $C_j \rightarrow C_j - C_{2p}$ for $p+1 \leq j \leq 2p-1$,

we get

$$\psi(Q, x) = w \begin{vmatrix} (1-ab)(I-E+L)_{p \times p} & O_{p \times p} & -bJ_{p \times 1} & (1-b)J_{p \times 1} & -bJ_{p \times 1} & (1-b)J_{p \times 1} \\ L_{1 \times p} & O_{1 \times p} & c & 1 & 1 & 1 \\ O_{p \times p} & (1-cd)(I-E+L)_{p \times p} & O_{p \times 1} & -dJ_{p \times 1} & (1-d)J_{p \times 1} & \\ (1-a)L_{1 \times p} & O_{1 \times p} & (1-p) & d-p & -p & 1-p \\ L_{1 \times p} & L_{1 \times p} & 1 & 0 & e & 1 \\ (1-a)L_{1 \times p} & L_{1 \times p} & 1-p & 1-p & 1-2p & f-2p \end{vmatrix}.$$

Taking common $(1-ab)$ from the first $(p-1)$ columns and $(1-cd)$ from $C_{p+1}, C_{p+2}, \dots, C_{2p-1}$, then $\psi(Q, x)$ is

$$w(1-ab)^{p-1}(1-cd)^{p-1} \begin{vmatrix} (I-E+(1-ab)L)_{p \times p} & O_{p \times p} & -bJ_{p \times 1} & (1-b)J_{p \times 1} & -bJ_{p \times 1} & (1-b)J_{p \times 1} \\ L_{1 \times p} & O_{1 \times p} & c & 1 & 1 & 1 \\ O_{p \times p} & (I-E+(1-cd)L)_{p \times p} & O_{p \times 1} & O_{p \times 1} & -dJ_{p \times 1} & (1-d)J_{p \times 1} \\ (1-a)L_{1 \times p} & O_{1 \times p} & 1-p & d-p & -p & 1-p \\ L_{1 \times p} & L_{1 \times p} & 1 & 0 & e & 1 \\ (1-a)L_{1 \times p} & (1-c)L_{1 \times p} & 1-p & 1-p & 1-2p & f-2p \end{vmatrix}.$$

Apply the column operations $C_p \rightarrow C_p - (C_1 + C_2 + \dots + C_{p-1})$, $C_{2p+1} \rightarrow C_{2p+1} - (C_{p+2} + C_{p+3} + \dots + C_{2p})$, $\psi(Q, x)$ is

$$w(1-ab)^{p-1}(1-cd)^{p-1} \begin{vmatrix} (I-abL)_{p \times p} & O_{p \times p} & -bJ_{p \times 1} & (1-b)J_{p \times 1} & -bJ_{p \times 1} & (1-b)J_{p \times 1} \\ L_{1 \times p} & O_{1 \times p} & c & 1 & 1 & 1 \\ O_{p \times p} & (I-cdL)_{p \times p} & O_{p \times 1} & O_{p \times 1} & -dJ_{p \times 1} & (1-d)J_{p \times 1} \\ (1-a)L_{1 \times p} & O_{1 \times p} & 1-p & d-p & -p & 1-p \\ L_{1 \times p} & L_{1 \times p} & 1 & 0 & e & 1 \\ (1-a)L_{1 \times p} & (1-c)L_{1 \times p} & 1-p & 1-p & 1-2p & f-2p \end{vmatrix}.$$

Expand along the columns $C_1, C_2, \dots, C_{p-1}$ and $C_{p+1}, C_{p+2}, \dots, C_{2p-1}$, we get

$$\psi(Q, x) = w(1-ab)^{p-1}(1-cd)^{p-1} \begin{vmatrix} 1-ab & 0 & -b & 1-b & -b & 1-b \\ 1 & 0 & c & 1 & 1 & 1 \\ 0 & 1-cd & 0 & 0 & -d & 1-d \\ 1-a & 0 & 1-p & d-p & -p & 1-p \\ 1 & 1 & 1 & 0 & e & 1 \\ 1-a & 1-c & 1-p & 1-p & 1-2p & f-2p \end{vmatrix}.$$

The factor $(1-ab)^{p-1} = \left(\frac{(1+x)(p^2q-pq-1-x)}{\phi(p^2q)\phi(p^2)}\right)^{p-1}$ gives the eigenvalues -1 and $(p^2q-pq-1-x)$ with both multiplicities $p-1$, and $(1-cd)^{p-1} = \left(\frac{(1+x)(pq-q-1-x)}{\phi(p^2)\phi(p)}\right)^{p-1}$ gives the eigenvalues -1 and $(pq-q-1-x)$ with both multiplicities $p-1$. By putting value of w , $\psi(Q, x)$ is

$$\phi(p^2q)\phi(p^2)\phi(q)\phi(pq)^2\phi(p)^2(1+x)^{2p-2}(p^2q-pq-1-x)^{p-1}(pq-q-1-x)^{p-1}\det(C),$$

where $\det(C)$ as given in Equation (5). Finally by Equation (6), we get multiplicity of -1 is $p^3q-2p-6$. $\square$

6. Conclusion

We determined the characteristic polynomial of the power graph $\mathcal{P}(\mathbb{Z}_m \times \mathbb{Z}_n)$ under the conditions $m|n$ and $m \nmid n$. We also investigated the complete spectrum for several particular cases, which provided deeper insight into the algebraic and spectral properties of these graphs. Future work may focus on extending these spectral investigations to more general finite groups and exploring further graph invariants associated with power graphs.

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References

- [1] S. Banerjee, [On structural and spectral properties of reduced power graph of finite groups](#), Asian-Eur. J. Math. 16(9) (2023) 2350170.
- [2] A. K. Bhuniya and S. K. Mukherjee, [On the power graph of the direct product of two groups](#), Electron. Notes Discrete Math. 63 (2017) 197–202.
- [3] D. M. Cardoso, M. A. A. de Freitas, E. A. Martins, and M. Robbiano, [Spectra of graphs obtained by a generalization of the join graph operation](#), Discrete Math. 313(5) (2013) 733–741.
- [4] I. Chakrabarty, S. Ghosh, and M. K. Sen, [Undirected power graphs of semigroups](#), Semigroup Forum 78(3) (2009) 410–426.
- [5] S. Chattopadhyay, P. Panigrahi, and F. Atik, [Spectral radius of power graphs on certain finite groups](#), Indag. Math. 29(2) (2018) 730–737.
- [6] D. Cvetković, P. Rowlinson, and S. Simić, [An Introduction to the Theory of Graph Spectra](#), vol. 75 in London Mathematical Society Student Texts, Cambridge University Press, Cambridge (2009).
- [7] M. Ghorbani and F. Abbasi-Barfaraz, [On the characteristic polynomial of power graphs](#), Filomat 32(12) (2018) 4375–4387.
- [8] S. H. Jafari and S. Chattopadhyay, [Spectrum of proper power graphs of the direct product of certain finite groups](#), Linear Multilinear Algebra 70(20) (2022) 5460–5481.
- [9] A. Kumar, L. Selvaganesh, P. J. Cameron, and T. T. Chelvam, [Recent developments on the power graph of finite groups—a survey](#), AKCE Int. J. Graphs Comb. 18(2) (2021) 65–94.
- [10] Z. Mehranian, A. Gholami, and A. R. Ashrafi, [The spectra of power graphs of certain finite groups](#), Linear Multilinear Algebra 65(5) (2017) 1003–1010.
- [11] R. P. Panda and K. V. Krishna, [On connectedness of power graphs of finite groups](#), J. Algebra Appl. 17(10) (2018) 1850184.
- [12] R. Rajkumar and T. Anitha, [Adjacency spectrum of reduced power graphs of finite groups](#), Algebra Colloq. 31(4) (2024) 661–674.
- [13] M. U. Romdhini, A. Nawawi, F. Al-Sharqi, and A. Al-Quran, [Spectral power graph of generalized quaternion groups](#), Asia Pac. J. Math. 11(36) (2024) 1–9.
- [14] A. J. Schwenk, [Computing the Characteristic Polynomial of a Graph](#), in: R. A. Bari and F. Harary, eds., Graphs and Combinatorics, vol. 406 in Lecture Notes in Mathematics, Springer, Berlin, Heidelberg (1974) 153–172.
- [15] S. N. Singh, [Laplacian spectra of power graphs of certain prime-power Abelian groups](#), Asian-Eur. J. Math. 15(2) (2022) 2250026.