



Projective systems and bounds on the length of codes of non-zero defect

Research Article

Tim Alderson^a • Zhipeng Zhang

Abstract: We derive bounds on the lengths of linear codes with fixed Singleton defect, working within the framework of projective systems. This geometric perspective allows us to unify and extend a range of existing results. We study the maximum length of a non-degenerate linear code when its dimension, field order, and Singleton defect are fixed, and refine this question by also prescribing the defect of the dual code. We further investigate the maximum dimension in which a code can attain the general length bound. Among our main results, we provide sufficient conditions under which a code and its dual have equal Singleton defect, addressing a gap in the existing literature. We show that codes of sufficient length must be projective, meet the Griesmer bound, and have an almost maximum distance separable dual. Our bounds subsume or improve several results in the literature. For field orders greater than three, we propose two conjectures on length-maximal codes of dimension at least five, supported by computational evidence.

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Tim Alderson (Corresponding Author); Mathematics and Statistics, University of New Brunswick, Saint John, New Brunswick, E2L 4L5, Canada, (tim@unb.ca).

Zhipeng Zhang; Mathematics and Statistics, University of New Brunswick, Saint John, New Brunswick, E2L 4L5, Canada, (zhipeng.zhang@unb.ca).

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1. Introduction and preliminaries

This section establishes the notation and background used throughout. We begin by recalling the definitions of linear codes and projective systems, then introduce the central parameters $m^s(k, q)$, $m_t^s(k, q)$, and $\kappa(s, q)$, and survey existing bounds in the literature. [Section 2](#) treats codes of small dimension, [Section 3](#) reviews bounds from the theory of classical arcs, and [Section 4](#) presents the main results. [Section 5](#) concludes with open questions and conjectures.

For $n \geq k$, a linear code C of length n , dimension k , and minimum distance d , denoted an $[n, k, d]_q$ code, is a k -dimensional subspace of $\mathbb{F}_q^n$ such that the minimum (Hamming) distance between any two elements (codewords) of C is d ; that is to say, there exist two codewords agreeing in $n - d$ coordinates and no two codewords agree in as many as $n - d + 1$ coordinates. A linear code $C \subset \mathbb{F}_q^n$ is degenerate if $C \subseteq \mathbb{F}_q^{n-1} \subset \mathbb{F}_q^n$, where $\mathbb{F}_q^{n-1}$ is the subspace of vectors with 0 at some fixed position, otherwise, we say that the code is non-degenerate. Unless otherwise stated, the codes considered here shall be non-degenerate. Given an $[n, k, d]_q$ code C , the dual code $C^\perp = \{y \in \mathbb{F}_q^n \mid y \cdot x = 0 \text{ for all } x \in C\}$ is an $[n, k^\perp, d^\perp]_q$ code where $k^\perp = n - k$.

Following Tsfasman and Vlăduț [32], we represent linear codes as *projective systems* — finite multisets of points in a projective space — thereby gaining a geometric vantage point from which to analyze code parameters. We now recall the necessary definitions.

The projective space of k dimensions over $\mathbb{F}_q$ will be denoted by $\text{PG}(k, q)$. A projective subspace of dimension m of $\text{PG}(k - 1, q)$ is called an m -flat; it is isomorphic to $\text{PG}(m, q)$ as an abstract projective space, though we retain the term flat to emphasize its embedding in the ambient space. We say that a set of m points of $\text{PG}(k - 1, q)$ are *independent* if they are not contained in an $(m - 2)$ -flat. More generally, the quotient of $\text{PG}(k, q)$ by a d -flat Ω is isomorphic to $\text{PG}(k - d - 1, q)$. The flats of the quotient space are precisely the flats of $\text{PG}(k, q)$ containing Ω , (see e.g. [3]).

An $[n, k, d]_q$ code C is determined by an associated generator matrix, G . Up to code equivalence, the columns of G can be considered as an n -multiset $\mathcal{G}$ of points in $\text{PG}(k - 1, q)$, at most $n - d$ per hyperplane (subspace of co-dimension 1). Any hyperplane meeting $\mathcal{G}$ in $n - d$ points is said to be a *secant* of $\mathcal{G}$. We may thus discuss codes in terms of equivalent objects known as *projective systems* (see e.g. [32]).

A projective $[n, k, d]_q$ system is a finite unordered family $\mathcal{G}$ of points in $\Sigma = \text{PG}(k - 1, q)$ that does not lie entirely in a hyperplane of Σ . Though generally, $\mathcal{G}$ may be a multiset, we shall employ a slight abuse of notation, writing $\mathcal{G} \subseteq \Sigma$, and $|\mathcal{G}|$ to denote the cardinality of $\mathcal{G}$ counting multiplicities. For a point P , we denote by $\mu(P)$ the multiplicity of P in $\mathcal{G}$. For $k = 1$, the hyperplane is the empty set. The parameters satisfy

$$n = |\mathcal{G}|, \quad k = \dim \Sigma + 1, \quad n - d = \max\{|\mathcal{G} \cap H| \mid H \text{ is a hyperplane of } \Sigma\}. \quad (1)$$

Similarly, the set of parameters $[n, k^\perp, d^\perp]_q$ of a dual code $\mathcal{G}^\perp$ can be characterized in terms of the projective system $\mathcal{G}$.

$$d^\perp = \min\{|\mathcal{Q}| : \mathcal{Q} \subset \mathcal{G}, |\mathcal{Q}| - \dim \text{lin} \langle \mathcal{Q} \rangle = 1\}, \quad (2)$$

where $\dim \text{lin} \langle \mathcal{Q} \rangle$ is the linear dimension of the span of $\mathcal{Q}$, one greater than its projective dimension. A submultiset $\mathcal{Q} \subseteq \mathcal{G}$ is *dependent* if $|\mathcal{Q}| - \dim \text{lin} \langle \mathcal{Q} \rangle \geq 1$. Every minimally dependent submultiset has this difference equal to 1, so (2) gives the minimum size of a dependent submultiset of $\mathcal{G}$. Note that degenerate codes are degenerate projective systems, in that they contain a point 0, corresponding to the 0-dimensional vector subspace $\langle 0 \rangle$. Though we generally avoid degenerate codes here, in such cases, the point 0 is said to have projective dimension -1 . Notice that according to (2), a code $\mathcal{G}$ is degenerate if and only if $d^\perp = 1$.

Our working definition of a (non-degenerate) linear code is as follows.

Definition 1.1. A linear $[n, k, d]_q$ code $\mathcal{G}$ is an n -(multi)set spanning $\Sigma = \text{PG}(k - 1, q)$ such that

$$n - d = \max\{|\mathcal{G} \cap H| \mid H \text{ is a hyperplane of } \Sigma\}.$$

That is to say, $\mathcal{G}$ is an $[n, k, d]_q$ projective system.

It follows immediately that an $[n, k, d]_q$ code $\mathcal{G}$ satisfies $n - d \geq k - 1$, which is the well known Singleton bound. The *Singleton defect* of an $[n, k, d]_q$ code is $S(\mathcal{G}) = n - k + 1 - d$. An $[n, k, d]_q$ code $\mathcal{G}$ with $S(\mathcal{G}) = 0$ is called a maximum distance separable (MDS) code. Codes of Singleton defect 1 are called Almost-MDS (AMDS) codes [10], and more generally, codes with $S(\mathcal{G}) = s$ are denoted A^s MDS codes. The dual code of an MDS code is MDS, but generally speaking the dual of an A^s MDS code need not be A^s MDS. If $S(\mathcal{G}) = S(\mathcal{G}^\perp) = 1$, $\mathcal{G}$ is said to be dually-AMDS (or Near MDS, or NMDS). This notation extends naturally to dually A^s MDS (or N^s MDS) codes.

A linear code $\mathcal{G}$ is *projective* if $\mathcal{G}$ is a set, that is to say if $P \in \mathcal{G}$ then $\mu(P) = 1$. A non-projective code is therefore equivalent to a code with one or more repeated coordinates. As we shall see, codes of sufficient length are necessarily projective.

Projective systems may be viewed as arcs or, sometimes as caps in $PG(k-1, q)$. An (n, r) -arc $\mathcal{K}$ in $\Pi = PG(k-1, q)$, $k \leq r+1$, is a collection of n points such that each hyperplane of Π is incident with at most r points in $\mathcal{K}$, and some hyperplane is incident with r points of $\mathcal{K}$. In the case where $\mathcal{K}$ is a multi-set, we shall use the term multi-arc. An $(n, k-1)$ -arc in $PG(k-1, q)$ is an n -arc. From the definitions, it is immediate that $(n, n-d)$ -arcs in $PG(k-1, q)$ and projective $[n, k, d]_q$ -codes are equivalent objects, as are $(n, n-d)$ -multi-arcs in $PG(k-1, q)$ and $[n, k, d]_q$ -codes. An (n, r) -arc in $PG(k-1, q)$ is said to be maximal if $n = (r - k + 2)(q + 1) + k - 2$. Such arcs are complete, in that they are not contained in an $(n+1, r)$ -arc. The (non)-existence question regarding maximal (n, r) -arcs in $PG(2, q)$ has largely been answered (see e.g. [6]). Here, for fixed q , and s and with $r = k + s - 1$, we investigate the parameter $\kappa(s, q)$, representing the maximum value k for which there exists a maximal (n, r) -arc in $PG(k-1, q)$.

An n -cap in $PG(k-1, q)$ is an n -set of points such that no three are collinear. It follows immediately that for $k \geq 3$, a projective $[n, k, d]_q$ code, $\mathcal{G}$ is an n -cap in $PG(k-1, q)$ if and only if $d^\perp \geq 4$ (equivalently, $S(\mathcal{G}^\perp) \leq k-3$).

A fundamental problem in coding theory is that of determining the maximum length of a code with k , q , and $S(\mathcal{G})$ fixed. To this end, we make the following definition.

Definition 1.2. We shall denote by $m_t^s(k, q)$ the maximum length of a (non-degenerate) $[n, k, d]_q$ A^s MDS code $\mathcal{G}$ such that $\mathcal{G}^\perp$ is an A^t MDS code. If t is not restricted, we shall simply write $m^s(k, q)$. In keeping with the standard notation, in the special cases $s = 0$, and $s = t = 1$ these parameters shall be denoted $m(k, q)$, and $m'(k, q)$ respectively.

Investigations into the bounds on $m_t^s(k, q)$ initially focused on codes of small defect. In particular, the case $s = 0$ has received much attention by way of the Main Conjecture on MDS codes (see Subsection 3.1). The focus of the current work is on codes of defect $s > 0$. For $s = 1$, De Boer [10] and Dodunekov and Landjev [17, 18] provide the following.

Proposition 1.3 ([10], Thm. 8; [17], Prop. 6.2). *If $q > 3$ and $k \geq 3$ then $m^1(k, q) \leq 2q + k - 2$.*

Recall the notation $m'(k, q) := m_1^1(k, q)$.

Theorem 1.4 ([18], Thm. 2.7). *Let $k \geq 2$.*

1. $m'(k, q) \leq 2q + k$;
2. $m'(2, q) = 2q + 2$;
3. $m'(k, q) \leq m'(k - \alpha, q) + \alpha$ for every integer $0 \leq \alpha \leq k - 2$;
4. $m'(k, q) = k + 1$ for $k > 2q$;
5. $m'(2q, q) = 2q + 2$ for $q > 3$.
6. $m'(2q - 1, q) = 2q + 1$ for $q > 3$.

Dodunekov and Landjev furthermore showed that an $(n, k, d)_q$ AMDS code with $n > q + k$ must in fact be NMDS. Written using our current notation, their result is as follows.

Theorem 1.5 ([17], Thm. 3.4). *For $k \geq 3$, if $m_t^1(k, q) > q + k$, then $t = 1$.*

In [31], Tong discusses the case $s = 2$, establishing the following bound in the binary case.

Proposition 1.6 ([31], Cor. 2). *If $k \geq 3$ then $m^2(k, 2) \leq k + 5$.*

Faldum and Willems [19] opened discussions to codes of arbitrary defect, providing bounds on $m^s(k, q)$. In particular, they provided the following result which subsumes [part 1 of Theorem 1.4](#), and [Theorem 1.5](#).

Lemma 1.7 ([19], Lem. 2).

1. *If $k \geq 2$, then $m^s(k, q) \leq (s + 1)(q + 1) + k - 2$.*
2. *If $k \geq 3$, and $m^s(k, q) = (s + 1)(q + 1) + k - 2$ then $s \leq q - 1$.*

If a code achieves the bound in [Lemma 1.7](#) then it is said to be *length-maximal*. Faldum and Willems go on to show a code of sufficient length must be dual to an AMDS code.

Theorem 1.8 ([19], Thm. 7). *If $s \geq 1$ and $k \geq 2$, and $m_t^s(k, q) > s(q + 1) + k - 1$, then $t \leq 1$.*

Corollary 1.9 ([19], Cor. 8). *Let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code with $k \geq 2$ and $n > s(q + 1) + k - 1$.*

1. *If $s = 1$, then $k \leq 2q$;*
2. *If $s > 1$, then $k \leq q$, and consequently ([Lemma 1.7](#)) $n \leq (q + 1)(s + 2) - 3$.*

In the sequel we shall establish bounds on $m_t^s(k, q)$ which subsume and generalize the existing bounds. We are also interested in the existence question of length-maximal codes. We conjecture that such codes do not exist for $k > 4$ when $q > 3$.

Let us briefly describe code shortening, a technique that shall be employed in the sequel.

1.1. Shortening codes via quotient geometries

Code shortening is a common technique employed in coding theory (see e.g. constructions Y1–Y4, p. 592 in [26]). We shall employ shortening in some of our proofs. Within the framework of projective systems, shortening can be performed by way of quotients, described as follows.

Suppose $\mathcal{G}$ is an $[n, k, d]_q$ A^s MDS code in $\Sigma = \text{PG}(k - 1, q)$, let Λ be an ℓ -flat of codimension $r \geq 2$, and let $\alpha = |\Lambda \cap \mathcal{G}|$ (including multiplicities). Let Σ^* be the quotient geometry taken at Λ , so that in particular, $\Sigma^* \cong \text{PG}(r - 1, q)$. Each point P of Σ^* is the image of an $(\ell + 1)$ -flat, Ω_P , where $\Lambda \subseteq \Omega_P$. If for each such point P , we assign the multiplicity $\mu(P) = |\Omega_P \cap \mathcal{G}| - \alpha$ then we obtain an $[n - \alpha, r, d^*]_q$ A^{s^*} MDS code $\mathcal{G}^*$, where $n - \alpha - d^* \leq n - d - \alpha$, giving $d^* \geq d$, and

$$s^* = n - \alpha - r + 1 - d^* = n - \alpha + \ell - k + 2 - d^* \leq n - \alpha + \ell - k + 2 - d = s - \alpha + \ell + 1. \quad (3)$$

Consequently, we have the following.

Proposition 1.10. *Let $\mathcal{G}$ and $\mathcal{G}^*$ be as above.*

1. *If $\alpha = |\Lambda \cap \mathcal{G}| \geq \ell + 1$, then $s^* \leq s$, and $n \leq m^{s^*}(r, q) + \alpha$.*
2. *$d^* = d$ if and only if Λ lies in a secant of $\mathcal{G}$. Moreover, $s^* = s + \ell + 1 - \alpha - (d^* - d)$. If $\alpha \geq \ell + 1$, then $s^* = s$ if and only if $d^* = d$ and $\alpha = \ell + 1$. In particular, for a point $\Lambda \in \mathcal{G}$, we have $s^* \leq s - \mu(\Lambda) + 1$, with equality if and only if Λ lies in a secant.*

Before delving into the main results of this paper, we explore bounds that follow from the basic theory, as well as those that follow from the established bounds on planar arcs.

2. Trivial codes and their bounds

We begin with 1-dimensional codes, where we temporarily widen the discussion to include degenerate codes. For any n, d, q one may construct the trivial (possibly degenerate) $[n, 1, d]_q$ A^sMDS code by taking as $\mathcal{G}$ a multiset from $\{0, 1\}$ ($=\text{PG}(0, q) \cup \{0\}$), where $s = \mu(0)$, and $d = \mu(1)$. By definition, $\mathcal{G}$ will be MDS and non-degenerate if $d = n$, and degenerate with $s > 0$ otherwise. The dual code $\mathcal{G}^\perp$ is an $[n, n-1, d^\perp]_q$ code, which is MDS ($d^\perp = 2$) if $d = n$, and is AMDS ($d^\perp = 1$) otherwise.

In the 2-dimensional case, for arbitrary n, d, q , and $s = n - d - 1$, one may clearly select a multiset of points from $\text{PG}(1, q)$ with the maximum multiplicity of any point being $n - d$. As such, one may construct an $[n, 2, d]_q$ -A^sMDS code if and only if $2 + s \leq n \leq (s+1)(q+1)$. If $\mathcal{G}$ is such a code with $\mathcal{G}^\perp$ an $[n, n-2, d^\perp]_q$ code, then (2) $d^\perp = 3$ if $s = 0$, and $d^\perp = 2$ otherwise. For $k > 2q$, any $[k+2, 2, d]_q$ -code necessarily satisfies $d^\perp = 2$ and is therefore dual to an $[k+2, k, 2]_q$ -AMDS code. Since an $[k+2, 2, d^\perp]_q$ code can only be AMDS if $k+2 \leq 2(q+1)$, part 4 of Theorem 1.4 follows immediately, as does each of the following.

Lemma 2.1.

1. If C is an $[n, 2, d]_q$ A^sMDS code, $s > 0$, then $C^\perp$ is AMDS.
2. $m(1, q)$ is unbounded, and $m^s(1, q) = 0$ for $s > 0$;
3. $m^1(k, q) \geq k+1$ for $k \geq 2$ and $m^1(k, q) \geq k+2$ for $k > 2q$.
4. $m'(k, q) \geq k+2$ for $2 \leq k \leq 2q$.
5. $m^s(2, q) = (s+1)(q+1)$

Remark 2.2. Note that part 2 of Theorem 1.4 is subsumed by Lemma 2.1, part 5.

Let us denote by e_i , $1 \leq i \leq k$, the projective point with homogeneous coordinates being the i 'th standard basis vector in $\mathbb{F}_q^k$. For fixed k, q, s , the code $\mathcal{G}$ comprising $\{e_1, e_2, e_3, \dots, e_k\}$, where $\mu(e_1) = s+1$, and $\mu(e_i) = 1$ for $2 \leq i \leq k$, is an $[k+s, k, 1]_q$ A^sMDS code, with $d = 1$, and $d^\perp = 2$. We thus have the following.

Lemma 2.3. For $k \geq 2$, $m^s(k, q) \geq k + s$.

Codes of interest here are those which are non-degenerate and “long”—in the sense of being (near) length-maximal. If $\mathcal{G}$ is an $[n, k, d]_q$ A^sMDS code with $d^\perp = 1$ then $\mathcal{G}$ is degenerate, and if $d = 1$, then $n = k + s$, so $\mathcal{G}$ is far from length-maximal. Thus, having discussed codes of dimension 2 or less, in the sequel we shall generally limit discussion to cases where $d, d^\perp > 1$ and $k > 2$.

3. Some established code bounds based on arcs

We now explore the relationship between (n, r) -arcs in projective planes and the bounds on the lengths of linear codes.

3.1. MDS codes

In 1952, Bush [13] established that if $k \geq q$ then $m(k, q) = k + 1$. For $k < q$ there is a long-standing conjecture regarding linear MDS codes: every linear $[n, k, n-k+1]_q$ MDS code with $1 < k < q$ satisfies $n \leq q+1$, except when q is even and $k = 3$ or $k = q-1$ in which cases $n \leq q+2$. This conjecture is called the main conjecture on linear MDS codes. Though the main conjecture has been shown to hold in many cases, the topic is far from moribund. Below we list some of the cases where the main conjecture holds, but for a more complete summary see [24] and [5].

Lemma 3.1. *Throughout, assume $2 \leq k < q$.*

1. $m(k, q) = q + 1$ for $k = 2, 4, 5$.
2. $m(3, q) = q + 1$ if q is odd, and $m(3, q) = q + 2$ if q is even.
3. For $k \geq 4$, $m(k, q) \leq q + k - 3$.
4. If $q = p^h$ and $4 \leq k \leq p$, then $m(k, q) = q + 1$.
5. If $q = p^h$, $h > 1$ and $k \leq 2p - 2$, then $m(k, q) = q + 1$.
6. If $q = p^{2h}$, p odd, and $k \leq \sqrt{q} - \sqrt{q}/p + 2$, then $m(k, q) = q + 1$.

Proof. [Part 1](#) is clear for $k = 2$. For $k = 4, 5$, the case q odd is due to Segre [28], whereas the case q even is due to Casse and Glynn [15], and Gulati and Kounias [21]. For [part 2](#) see [11] for q odd, and for q even see e.g. [23]. For [part 3](#), see [28] for $q > 4$ odd, [14] for $q > 4$ even. For [parts 4 to 6](#), see [2, 5, 7]. $\square$

3.2. Bounds on 3-dimensional codes

The results of Barlotti (1956) [8], when paired with the later developments of Ball et al. [4], give bounds on (n, r) -arcs in $\text{PG}(2, q)$ which provide the first three items in the following lemma. The fourth item is due to the construction of Denniston [16].

Lemma 3.2 ([8], [4], [16]).

1. $m^s(3, q) \leq (s + 1)(q + 1) + 1$.
2. If $0 < s < q - 2$, and $(s + 2, q) \neq (2^e, 2^h)$, then $m^s(3, q) \leq (s + 1)(q + 1) - 1$.
3. If $0 < s < q - 2$, $(s + 2) | q$, and $m^s(3, q) \geq (s + 1)(q + 1)$, then $m^s(3, q) = (s + 1)(q + 1) + 1$.
4. If $0 < s \leq q - 2$, and $(s + 2, q) = (2^e, 2^h)$, then $m^s(3, q) = (s + 1)(q + 1) + 1$.

Further, we have the following.

Lemma 3.3. $m^s(3, q) \leq (s + 1)q + 1$ in each of the following cases:

1. q is prime, and $s + 2 \leq (q + 3)/2$ ([1])
2. $\gcd(s + 2, q) = 1$ and $s < \sqrt{2q} - 1$ ([6])
3. q odd, $s + 2$ divides q , $s < \frac{1}{4}\sqrt{q} - 2$ ([33])
4. $\gcd(s + 2, q) = 1$ and there is a line disjoint from the corresponding projective system. ([9])

4. Main results

In this section, we present the primary findings, focusing on the bounds for the lengths of codes with non-zero Singleton defects. We first establish some elementary properties of $m^s(k, q)$.

Lemma 4.1. *Let $k \geq 2$. The following hold.*

1. $m^s(k, q) < m^{s+1}(k, q)$.
2. For $k \geq 3$, $m^s(k, q) \leq m^s(k - 1, q) + 1$.

3. For integers $0 \leq \alpha \leq k - 2$, $m^s(k, q) \leq m^s(k - \alpha, q) + \alpha$. In particular, for $k \geq 3$, $m^s(k, q) \leq m^s(3, q) + k - 3$.
4. If $s = k - 1 + b$, where $b = s_1 + s_2$, $s_1, s_2 \geq 0$, then $m^s(k, q) \geq m^{s_1}(k, q) + m^{s_2}(k, q)$.
5. $m^s(3, q) \geq \begin{cases} \frac{s+2}{2} \cdot m(3, q) & \text{if } s \text{ is even,} \\ \frac{s-1}{2} \cdot m(3, q) + m^1(3, q) & \text{if } s \text{ is odd.} \end{cases}$

Proof. Let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code, let $\mathcal{H}$ be a secant of $\mathcal{G}$, and let $P \in \mathcal{G} \cap \mathcal{H}$. If $n = m^s(k, q)$, then $\mathcal{G}' = \mathcal{G} \cup \{P\}$ is an $[n + 1, k, d]_q$ A^{s+1} MDS code (since $\mathcal{H}$ remains a secant), giving [part 1](#). For [part 2](#), suppose P has multiplicity m . Taking the quotient geometry by P , $\mathcal{G}^* = \mathcal{G} \setminus \{P\}$ corresponds to an $[n - m, k - 1, d]_q$ code in $\text{PG}(k - 2, q)$. By [part 1](#), $n - m \leq m^{s-m+1}(k - 1, q) \leq m^s(k - 1, q) - (m - 1)$. [Part 3](#) follows inductively from [part 2](#).

For [part 4](#), choose systems $\mathcal{G}_i$ of defect s_i and length $m^{s_i}(k, q)$, $i = 1, 2$, and apply a projectivity so that they share a secant H . Their multiset union has length $m^{s_1}(k, q) + m^{s_2}(k, q)$. Its maximum hyperplane intersection is $(k + s_1 - 1) + (k + s_2 - 1)$, attained at H , so its defect is $k - 1 + s_1 + s_2$.

[Part 5](#) is obtained by recursively applying [part 4](#). $\square$

Remark 4.2. For $0 \leq \alpha \leq k - 2$, [part 3](#) of [Theorem 1.4](#) is subsumed by [Lemma 4.1](#), [part 3](#).

Remark 4.3. For $k \geq 2$, deleting the $\alpha > 0$ all-zero coordinates of a degenerate $[n, k, d]_q$ A^s MDS code gives a non-degenerate $[n - \alpha, k, d]_q$ $A^{s-\alpha}$ MDS code. By [Lemma 4.1](#), [part 1](#), $n \leq m^{s-\alpha}(k, q) + \alpha \leq m^s(k, q)$. Thus, in searching for long codes, it suffices to consider non-degenerate codes.

The bounds in [Lemma 3.2](#) and [Lemma 3.3](#) may be extended to higher dimensions as follows.

Corollary 4.4. For $k \geq 3$, the following hold.

1. (cf. [Lemma 1.7 \(part 1\)](#)) $m^s(k, q) \leq (s + 1)(q + 1) + k - 2$.
2. If $0 < s < q - 2$, and $(s + 2, q) \neq (2^e, 2^h)$, then $m^s(k, q) \leq (s + 1)(q + 1) + k - 4$.
3. Under the conditions specified in [parts 1 to 3](#) of [Lemma 3.3](#), $m^s(k, q) \leq q(s + 1) + k - 2$.
4. If $\mathcal{G}$ is an $[n, k, d]_q$ A^s MDS code, $\gcd(s + 2, q) = 1$, and some hyperplane H satisfies $|H \cap \mathcal{G}| = k - 3$, then $n \leq q(s + 1) + k - 2$.

Remark 4.5. Note that [Proposition 1.3](#) is a particular case of [part 2](#) of [Corollary 4.4](#).

Proof. [Part 1](#), [part 2](#), and [part 3](#) follow immediately from the application of [part 2](#) of [Lemma 4.1](#) to [Lemma 3.2](#) and [Lemma 3.3](#).

For [part 4](#), if $k > 3$, quotient by a $(k - 4)$ -flat Ω with $H \cap \mathcal{G} \subseteq \Omega \subseteq H$; for $k = 3$, take $\mathcal{G}' = \mathcal{G}$. The resulting planar system $\mathcal{G}'$ has length $n' = n - k + 3$, defect $s' \leq s$, and a disjoint line. If $s' = s$, [Lemma 3.3](#), [part 4](#) gives $n' \leq q(s + 1) + 1$. If $s' < s$, the q affine lines in a parallel class each contain at most $s' + 2$ points, so $n' \leq q(s' + 2) \leq q(s + 1)$. Both cases give the result. $\square$

4.1. Projective codes

Recall that a code $\mathcal{G}$ is projective if $\mathcal{G}$ is a set, i.e. every point has multiplicity 1. We now establish conditions under which codes are necessarily projective, and derive bounds on their lengths.

Note that according to the equation [\(2\)](#), a (non-degenerate) code of dimension $k > 2$ is projective if and only if $d^\perp \neq 2$.

Theorem 4.6. Let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code, $k \geq 2$.

1. If $s = 0$ then $\mathcal{G}$ is projective.

2. If $k = 2$, then $\mathcal{G}$ is projective if and only if $s = 0$.
3. If $s > 0$, $k > 2$, and $n > m^{s-1}(k-1, q) + 2$ then $\mathcal{G}$ is projective.
4. For $k > 2$, if $n > s(q+1) + k - 1$ then $\mathcal{G}$ is projective.
5. If $s = 1$, $k > q$, and $n > k + 2$ then $\mathcal{G}$ is projective.

Proof. Part 1 follows from the definition of MDS codes. In $\Sigma = \text{PG}(1, q)$, points and hyperplanes coincide, so according to Definition 1.1, an $[n, 2, d]_q$ code is projective if and only if $n - d = 1$, giving part 2.

For the third part, let $\mathcal{G} = \{P_1, P_2, \dots, P_n\}$ be an $[n, k, d]_q$ A^s MDS code where, say $\mu(P_n) = m \geq 2$. Taking the quotient at P_n , the projective system $\mathcal{G}' = \mathcal{G} \setminus \{P_n\}$ corresponds to an $[n - m, k - 1, d^*]_q$ $A^{s'}$ MDS code where $d^* \geq d$ and $s' \leq s - m + 1$, giving $n \leq m^{s'}(k - 1, q) + m \leq m^{s-1}(k - 1, q) + 2$ (by Lemma 4.1).

For part 4, suppose $\mathcal{G}$ is an $[n, k, d]_q$ - A^s MDS code, and $\mu(P) = m \geq 2$. Through P , there exists a $(k - 3)$ -flat Γ with $|\Gamma \cap \mathcal{G}| = x \geq k - 1$. Each of the $q + 1$ hyperplanes through Γ meets $\mathcal{G}$ in at most $k + s - 1 - x \leq s$ points outside of Γ , giving the result.

Part 5 follows from part 3 and the fact that $m(k - 1, q) = k$ for $k - 1 \geq q$. $\square$

Remark 4.7. By part 3, an $[n, k, n - k]_q$ AMDS-code $\mathcal{G}$ is necessarily projective if $n > m(k - 1, q) + 2$. With the current state of the Main Conjecture on linear MDS codes, in many cases $m(k - 1, q) = q + 1$, whence $\mathcal{G}$ is projective if $n > q + 3$.

We may also deduce the following bound on A^s MDS codes that are not dual to an AMDS code.

Lemma 4.8. If $k \geq 3$, and $t > 1$ then

$$m_t^s(k, q) \leq m^{s-1}(t, q) + d^\perp = m^{s-1}(t, q) + k - t + 1.$$

Proof. Note that since $t > 1$, $s > 0$. Let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code. If $d^\perp = 2$ then $t = k - 1$ and $\mathcal{G}$ is not projective, so the result follows from Theorem 4.6. Let $d^\perp \geq 3$, so that in particular, $\mathcal{G}$ is projective. There exist $d^\perp$ points of $\mathcal{G}$ incident with a common $(d^\perp - 2)$ -flat, λ , and any $d^\perp - 1$ or fewer points of $\mathcal{G}$ are independent. Consider a $(d^\perp - 3)$ -flat, λ' spanned by $d^\perp - 2$ points of $\mathcal{G} \cap \lambda$. Taking the quotient by λ' yields an $[n - d^\perp + 2, k - d^\perp + 2, d^*]_q$ A^{s^*} MDS code, $\mathcal{G}^*$, with $s^* \leq s$ (Proposition 1.10). Since the point in the quotient corresponding to λ has multiplicity at least 2, $\mathcal{G}^*$ is not projective, consequently (Theorem 4.6), $n - d^\perp + 2 \leq m^{s^*-1}(k - d^\perp + 1, q) + 2 \leq m^{s-1}(k - d^\perp + 1, q) + 2$. $\square$

From Lemma 4.1, if $t \geq 3$ and $s > 0$ then $m^{s-1}(t, q) \leq m^{s-1}(3, q) + t - 3$. This bound may in turn be leveraged by bounds on $(n, s + 1)$ -arcs in the plane, such as those in Lemma 3.2, and Lemma 3.3. Specializing to the case $s = 1$: if $t \geq 2$ then $m(t, q) \leq m(2, q) + t - 2 = q + t - 1$. From part 1 of Lemma 2.1, the dual of any 2-dimensional non-MDS code is AMDS. Furthermore, if $t \geq q$, then as observed in Subsection 3.1, $m(t, q) = t + 1$. We thus have the following corollary, the second part of which appears as Theorem 3.4 of [17].

Corollary 4.9.

1. If $t \geq 3$ and $s > 0$, then $m_t^s(k, q) \leq m^{s-1}(3, q) + k - 2$
2. If $k, t \geq 2$, then $m_t^1(k, q) \leq q + k$.
3. If $t \geq q$ then $m_t^1(k, q) \leq k + 2$.

Remark 4.10. With reference to Lemma 3.1, it is often the case that $m(t, q) = q + 1$. In such cases, the above corollary provides $m_t^1(k, q) \leq q + k + 2 - t$ when $t > 1$.

4.2. Bounds on A^s MDS codes

It is noteworthy that $S(C^\perp) = t > 1$ if and only if $d^\perp < k$. Thus, for $k = 3$, all projective codes are either MDS or dual to an AMDS code. For dimensions $k > 3$, there exist projective codes that are neither MDS nor dual to an AMDS code. However, such codes are shown to necessarily be short, indicating that for dimensions three or greater, the longest codes are typically dual to AMDS codes. Ball and Hirschfeld observed in [6] that, in most cases, no examples exist of (n, r) -arcs in $PG(2, q)$ with a large n/q ratio, specifically $n/q > r - 2$. The following demonstrates that for dimensions $k > 3$, the longest codes that are neither AMDS nor dual to an AMDS code generally correspond (by shortening) to (n, r) -arcs in $PG(2, q)$ with $n \leq (r - 2)q + r$.

Remark 4.11. Recall that an $[n, k, d]_q$ A^s MDS code $\mathcal{G}$ is degenerate if and only if $d^\perp = 1$, equivalently $n = k^\perp + t$. Likewise $\mathcal{G}^\perp$ is an $[n, k^\perp, d^\perp]_q$ A^t MDS code, and is degenerate if and only if $d = 1$, in which case $n = k + s$. Thus, in our search for long non-degenerate codes we generally dismiss the case $d = 1$, or equivalently, consider only $s < n - k$.

Theorem 4.12. Let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code, with $\mathcal{G}^\perp$ an $[n, k^\perp, d^\perp]_q$ A^t MDS code, $d > 1$, and $s, t \geq 1$.

1. (a) If $t > 1$ then $n \leq s(q + 1) + k - 1$.
(b) If $s > 1$ then $k \leq t(q + 1) - 1$.
2. If $t = 1$ and $\mathcal{G}$ has a codeword of weight $d + s + 1 - \alpha$ for some $0 \leq \alpha \leq s + 1$ then $n \leq q(s + 1) + k - 2 + \alpha$.
3. If $s = 1$ then $n \leq (t + 1)(q + 1) + k^\perp - 2$, or equivalently, $k \leq (t + 1)(q + 1) - 2$.
4. If $s = t = 1$, $q > 3$, and $k \geq 3$, then $n \leq 2q + k - 2$, and if $n > k + 2$ then $k \leq 2q - 2$.
5. If $t = 1$ then $k + s - 1 \leq m(k - 1, q)$, equivalently $n \leq m(k - 1, q) + k^\perp - s + 1$.
6. If $s = 1$ then $k^\perp + t - 1 \leq m(k^\perp - 1, q)$, so $n \leq m(k^\perp - 1, q) + k - t + 1$.
7. If $s = t = 1$ then $k \leq m(k - 1, q)$; $k^\perp \leq m(k^\perp - 1, q)$.
8. $k \leq (t + 1)(q + 1) - 2$.

Remark 4.13. Part 1a: cf. Theorem 1.8; in the case $t = 2$, cf. Corollary 1 in [31]; Corollary 1.9 follows from part 1 and part 3;

Part 6: cf. Theorem 11 in [10].

Proof. Before proceeding with the proof, we first note that $d^\perp = k + 1 - t$, and from (2) it follows that $k + 1 - t$ points of $\mathcal{G}$ are incident with a common $(k - 1 - t)$ -flat, Ω , and any $k - t$ points of $\mathcal{G}$ are independent. Let H be a secant of $\mathcal{G}$, so H is a hyperplane of $PG(k - 1, q)$ with $|H \cap \mathcal{G}| = k - 1 + s$, and no hyperplane meets $\mathcal{G}$ in as many as $k + s$ points. Part 1: If $t > 1$ then $d^\perp \leq k - 1$, so $k \geq 3$, so Lemma 4.8 (which gives $m_t^s(k, q) \leq m^{s-1}(t, q) + k - t + 1$) and part 3 of Lemma 4.1 (which gives $m^{s-1}(t, q) \leq m^{s-1}(2, q) + t - 2$) give

$$m_t^s(k, q) \leq m^{s-1}(2, q) + k - 1 = s(q + 1) + k - 1.$$

Similarly, if $s > 1$ then by working with $\mathcal{G}^\perp$ and the fact that $d > 1$ provides $k^\perp \geq 3$, we obtain $n \leq t(q + 1) + k^\perp - 1$, so

$$k \leq t(q + 1) - 1.$$

Part 2: If $t = 1$ then $d^\perp = k$, so any $(k - 1)$ -subset of $\mathcal{G}$ is independent. $\mathcal{G}$ has a codeword of weight $d + s + 1 - \alpha$ for some $0 \leq \alpha \leq s + 1$ if and only if there exists some hyperplane Π meeting $\mathcal{G}$ in precisely

$k - 2 + \alpha$ points. Let Ω be the span of $k - 2$ of these points. Each hyperplane through Ω meets $\mathcal{G}$ in at most $s + 1$ further points, giving

$$n - k + 2 \leq (s + 1)(q) + \alpha. \quad (4)$$

Part 3 follows from part 2, and the existence of a word of weight $d^\perp$ in $\mathcal{G}^\perp$.

Part 4: If $t = s = 1$ then part 2 of Corollary 4.4 gives $n \leq 2q + k - 2$, and $n > k + 2$ gives $k^\perp \geq 3$, whence $n \leq 2q + k^\perp - 2$.

Part 5 is vacuous for $k = 2$; assume $k \geq 3$. If $t = 1$ then $H \cap \mathcal{G}$ is a set of $k + s - 1$ points such that every $(k - 1)$ -subset is independent. In other words $H \cap \mathcal{G}$ is a $(k + s - 1, k - 2)$ -arc in $\text{PG}(k - 2, q)$. Reasoning for part 6 is entirely similar, and applied to $\mathcal{G}^\perp$. Part 7 follows from part 5 and part 6. Part 8 follows from part 1 and part 3. $\square$

Note that codes that achieve equality in part 1 of Corollary 4.4 have no codewords of weight $d + 1$, whereas codes dual to those meeting the bound in part 8 of Theorem 4.12 have no codewords of weight $d^\perp + 1$.

Corollary 4.14. *Let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code, with $\mathcal{G}^\perp$ an $[n, k^\perp, d^\perp]_q$ A^t MDS code, $s, t \geq 1$.*

1. *If $n > s(q + 1) + k - 1$, then $s \leq m(k - 1, q) - k + 1$.*
2. *If $k > q$ and $s > 1$, then $n \leq s(q + 1) + k - 1$.*
3. *If $s > 1$ and $t = 1$ then $n \leq q + k^\perp$, so $k \leq q$.*
4. *If $t > 1$ and $s = 1$ then $n \leq q + k$, so $k^\perp \leq q$.*

Proof. Part 1 follows from part 1 and part 5 of Theorem 4.12. Part 2 clearly holds if $d = 1$, and if $d > 1$ then the result follows from part 1 of Theorem 4.12. Part 3 and part 4 are particular cases of part 1 of Theorem 4.12. $\square$

Under the hypotheses below, for $s = 1$, Theorem 4.12 gives $k \leq 2q$; if $n > k + 2$, it gives $k \leq 2q - 2$. Thus $n \leq 2q + 2$ or $n \leq 4q - 4$, respectively. For $s > 1$, it gives $k \leq q$. Combining this with Corollary 4.4, part 1 yields the following.

Corollary 4.15. *Let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code with an AMDS dual, $k \geq 3$, $q > 3$, and $d > 1$. Then*

$$n \leq \begin{cases} (s + 3)(q + 1) - 6, & \text{if } s = 1 \\ (s + 2)(q + 1) - 3, & \text{if } s > 1. \end{cases}$$

Lemma 4.16. *If $\mathcal{G}$ is an $[n, k, d]_q$ A^s MDS code and a $(k - 3)$ -flat Λ meets $\mathcal{G}$ in $k - 2 + a$ points, $a \geq 0$, then*

$$n \leq (s + 1 - a)(q + 1) + k - 2 + a$$

Proof. Each hyperplane meets $\mathcal{G}$ in at most $n - d = k + s - 1$ points of $\mathcal{G}$. Thus, by considering the pencil of hyperplanes through Λ we obtain

$$n \leq k - 2 + a + (q + 1)[k + s - 1 - (k - 2 + a)]$$

$\square$

As an immediate consequence, we have the following bound for non-projective codes.

Corollary 4.17. *If $\mathcal{G}$ is an $[n, k, d]_q$ A^s MDS code where $\mu(P_i) = 1 + x_i$, $1 \leq i \leq k - 2$, then*

$$n \leq (s + 1 - a)(q + 1) + k - 2 + a,$$

where $a = \sum_{i=1}^{k-2} x_i$.

The next four corollaries bound codes of large defect or minimum distance. Throughout these corollaries, α is a nonnegative integer.

Corollary 4.18. *If $s \geq \alpha(q + 1) - 1$ and $k \geq 3$, then*

$$m^s(k, q) \leq (s + 1 - \alpha)(q + 1) + k - 2 + \alpha.$$

In particular, if $s \geq q$, then $m^s(k, q) \leq s(q + 1) + k - 1$.

Remark 4.19. *Corollary 4.18 immediately implies both Proposition 1.6 and part 2 of Lemma 1.7.*

Proof. We shall deal with the cases $k = 3$ and $k > 3$ separately. Let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code, $k > 3$ and let H be a secant of $\mathcal{G}$. If there exists a $(k - 3)$ -flat meeting $\mathcal{G}$ in at least $k - 2 + \alpha$ points, then we are done (Lemma 4.16), so assume otherwise. Choose a $(k - 4)$ -flat $\Omega \subseteq H$ with $|\Omega \cap \mathcal{G}| = k - 3 + x$, $x \geq 0$. The assumption implies that $x < \alpha$. By considering the respective intersections of $\mathcal{G}$ with the pencil of hyperplanes of H through Ω , we obtain

$$\begin{aligned} k + s - 1 &\leq k - 3 + x + (q + 1)[(k - 3 + \alpha) - (k - 3 + x)] \\ &\Rightarrow s \leq \alpha(q + 1) - qx - 2 < \alpha(q + 1) - 1 \end{aligned}$$

giving a contradiction. If $k = 3$ then a secant of $\mathcal{G}$ is a line ℓ meeting $\mathcal{G}$ in $s + 2 \geq \alpha(q + 1) + 1$ points. It follows that some point $P \in \ell$ has multiplicity $\mu(P) \geq \alpha + 1$, hence $n \leq \alpha + 1 + (q + 1)(s + 1 - \alpha)$. $\square$

We may improve upon the bound in Corollary 4.18 when $\alpha = 1$, provided k and q are such that $m(k - 1, q) = q + 1$ (as per the Main Conjecture on MDS Codes).

Corollary 4.20. *Let k and q be such that $m(k - 1, q) = q + 1$. If $s > q + 2 - k$, then $m^s(k, q) \leq s(q + 1) + k - 1$. In particular, this bound holds if q is prime, $3 \leq k \leq q$, and $s > q + 2 - k$.*

Proof. Let k and q be as required, and let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code. If $n > s(q + 1) + k - 1$, then by part 1 of Theorem 4.12, $t = 1$, and by part 5, we obtain

$$s \leq m(k - 1, q) - k + 1 = q + 2 - k.$$

$\square$

Corollary 4.21. *If $k \geq 3$, and $d > \alpha(q^2 + q) - q$, $\alpha \geq 0$, then an $[n, k, d]_q$ A^s MDS code satisfies*

$$n \leq (s + 1 - \alpha)(q + 1) + k - 2 + \alpha.$$

In particular, if $d > q^2$, then $n \leq s(q + 1) + k - 1$.

Proof. For $k \geq 3$ let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code with $d > \alpha(q^2 + q) - q$. Let H be a secant of $\mathcal{G}$, and let $S \subseteq H$ be a $(k - 3)$ -flat with $|S \cap \mathcal{G}| \geq k - 2$. Through S there are q hyperplanes other than H . Since $d > \alpha(q^2 + q) - q$, there exists a hyperplane H' through S with $|H' \cap \mathcal{G}| \geq k - 2 + \alpha q + \alpha$. It follows that $n - d = k + s - 1 \geq k - 2 + \alpha q + \alpha$, whence $s \geq \alpha q + \alpha - 1$, and Corollary 4.18 gives the result. $\square$

We may improve slightly if $t > 1$.

Corollary 4.22. *Let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code, and $\mathcal{G}^\perp$ an $[n, k^\perp, d^\perp]_q$ A^t MDS code, $k \geq 3$. If $t > 1$, and $d > \alpha(q^2 + q) - 2q$, $\alpha \geq 0$, then*

$$n \leq (s + 1 - \alpha)(q + 1) + k - 2 + \alpha.$$

Proof. Since $t > 1$ there exists a $(k-3)$ -flat Ω with $|\Omega \cap \mathcal{G}| \geq k-1$. Let H be a hyperplane through Ω . By definition, $|H \cap \mathcal{G}| \leq n-d$, so (as in the previous proof) there exists a hyperplane H' through Ω with $|H' \cap \mathcal{G}| \geq k-1 + \alpha q + \alpha - 1$. It follows that $n-d = k+s-1 \geq k-1 + \alpha q + \alpha - 1$, whence $s \geq \alpha q + \alpha - 1$, and [Corollary 4.18](#) gives the result. $\square$

If $n > s(q+1) + k - 1$, then $d = n - k - s + 1 > sq$. The Griesmer bound gives $n \geq d + \lceil d/q \rceil + k - 2 \geq d + s + k - 1 = n$, so equality holds. Together with [Theorem 4.6](#), [part 4](#) and [Theorem 4.12](#), [part 1a](#), this gives the following; here $s > 0$ excludes an MDS dual.

Corollary 4.23. For $k \geq 3$ and $s \geq 1$, let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code with $\mathcal{G}^\perp$ an A^t MDS code. If $n > s(q+1) + k - 1$ then $\mathcal{G}$ meets the Griesmer bound, $\mathcal{G}$ is projective, and $t = 1$.

Corollary 4.24. Assume $t = 0$ when $s = 0$. The following hold.

1. If $s \in \{0, 1\}$ and $k \geq (t+1)(q+1) - 1$, then $m_t^s(k, q) = k + 1$.
2. If $s > 1$ and $k \geq t(q+1)$, no non-degenerate A^s MDS code of dimension k has an A^t MDS dual.

Proof. For $s = 0$, see Bush [13]. For $s = 1$, [Theorem 4.12](#), [part 3](#) forces $d = 1$; a parity-check vector of length $k+1$ and weight $k+1-t$ gives equality. For $s > 1$, [Theorem 4.12](#), [part 1](#) excludes $d > 1$. If $d = 1$, then $n = k + s$. Deleting the $z \geq 1$ zero coordinates of the dual leaves dimension s and defect $t-z$. [Lemma 1.7](#) ([part 1](#)) gives $n - z \leq (t - z + 1)(q+1) + s - 2$, hence $k \leq t(q+1) - 1$, a contradiction. $\square$

We next consider codes attaining the length bound

$$n \leq (s+1)(q+1) + k - 2. \quad (5)$$

By [Corollary 4.4](#) and [Corollary 4.18](#), any code of dimension $k \geq 4$ meeting (5) satisfies either

- (a) $(s+2)|q$, q even, and $2 \leq s \leq q-4$, or
- (b) $q-2 \leq s \leq q-1$.

For $k = 2$, equality in (5) holds for every $s \geq 0$. For $k = 3$, deleting a line from $PG(2, q)$ yields a $[q^2, 3, q^2 - q]_q$ code C with $S(C) = q - 2$, whereas the entire plane yields an $[q^2 + q + 1, 3, q^2]_q$ code C' , with $S(C') = q - 1$. For each $q > 2$, there exists a $(q^2 + 1)$ -cap in $PG(3, q)$ with maximal plane intersection $q+1$, and there exists a cap of size 8 in $PG(3, 2)$ (see e.g. [22]). Thus for each $q \geq 3$ there is an $[q^2 + 1, 4, d]_q$ code C with $S(C) = q - 2$, and $S(C^\perp) = 1$, and in the binary case there is an $[8, 4, 4]_2$ NMDS code ($s = q - 1$). For dimensions $k \geq 5$ we are able to show that no codes of type (b) exist for $q > 3$.

Corollary 4.25.

1. If $q > 2$, and $2 \leq k \leq 3$, or if $q = 2$ and $2 \leq k \leq 4$ then $m^{q-1}(k, q) = (s+1)(q+1) + k - 2 = q^2 + q + k - 2$.
2. If $q > 2$, and $k \geq 4$, then $m^{q-1}(k, q) \leq s(q+1) + k - 1 = q^2 + k - 2$.
3. If $q = 2$, and $k \geq 5$, then $m^{q-1}(k, q) = k + 2$.
4. If $2 \leq k \leq 4$ and $q > 2$ then $m^{q-2}(k, q) = (s+1)(q+1) + k - 2 = q^2 + k - 3$.
5. If $k \geq 5$ and $q > 3$, then $m^{q-2}(k, q) \leq s(q+1) + k - 1$.

Proof. Part 1, and part 4 follow from the preceding discussion. For the remaining parts, let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code with $\mathcal{G}^\perp$ an A^t MDS code, and $n \geq s(q+1) + k$. Note that if $d = 1$ then $n = k + s$, so $d > 1$. By part 1 of Theorem 4.12, $t = 1$, so if $k \geq 4$ then $\mathcal{G}$ is an n -cap in $\text{PG}(k-1, q)$. For part 2, take $s = q-1$ and $k = 4$. Then $n \geq q^2 + 3$ would give an n -cap in $\text{PG}(3, q)$, contradicting the bound $q^2 + 1$ of Bose [11] for odd q and Qvist [27] for even $q > 2$. Thus $m^{q-1}(4, q) \leq q^2 + 2$; Lemma 4.1, part 3 gives the result for $k > 4$.

Part 3: If $k \geq 5$ then (part 3 of Lemma 2.1), $m^1(k, q) \geq k + 2$. The result then follows from part 1a of Theorem 4.12 (which gives $n \leq s(q+1) + k - 1$ when $t > 1$), and part 4 of Theorem 1.4 (which gives $m'(k, q) = k + 1$ for $k > 2q$).

For part 5, $s = q-2$, $q > 2$. Consider $k = 5$. By part 5 of Theorem 4.12 (which gives $k+s-1 \leq m(k-1, q)$), $m(4, q) \geq q+2$. The result of Bush [13] (see introduction to Subsection 3.1) gives $m(4, 4) = 5$, so consider $q > 4$. Part 3 of Lemma 3.1 gives the contradiction $m(4, q) \leq q+1$. The bound for $k = 5$ being established, the cases $k > 5$ follow inductively from part 2 of Lemma 4.1. $\square$

Lemma 4.26. Let $k \geq 3$. If $\mathcal{G}$ is an $[n, k, d]_q$ A^s MDS code with $n = (s+1)(q+1) + k - 2$, then for each $0 \leq j \leq k-2$, $\gamma_j = \frac{\binom{n-j}{k-1-j}}{\binom{n-d-j}{k-1-j}}$ is an integer.

Proof. Suppose $\mathcal{G}$ is an $[n, k, d]_q$ A^s MDS code with $n = (s+1)(q+1) + k - 2$. Any $k-1$ or fewer points of $\mathcal{G}$ are independent: for $s = 0$ this follows from the MDS property, and for $s > 0$, Theorem 4.12 gives $d^\perp = k$. In particular, if $S \subseteq \mathcal{G}$ with $|S| = k-2$, then $\langle S \rangle$ is a $(k-3)$ -flat with $|\langle S \rangle \cap \mathcal{G}| = k-2$. Simple counting shows that each member of the pencil of $q+1$ hyperplanes through $\langle S \rangle$ meets $\mathcal{G}$ in exactly $s+k-1$ points (i.e. each is a secant of $\mathcal{G}$). Denote by γ_0 the number of secants. Counting set, hyperplane pairs $(A, \mathcal{S})$ where $A \subseteq \mathcal{G}$, $|A| = k-1$, $A \subseteq \mathcal{S}$, and $\mathcal{S}$ is a secant of $\mathcal{G}$ we obtain

$$\binom{n}{k-1} = \gamma_0 \cdot \binom{k+s-1}{k-1}, \quad (6)$$

establishing the result for $j = 0$. For $1 \leq j \leq k-2$, fix a j -subset $B \subseteq \mathcal{G}$ and let γ_j count the secants containing B . Every $(k-1)$ -subset containing B spans a unique secant, and each such secant contains $\binom{k+s-1-j}{k-1-j}$ of these subsets. Hence $\binom{n-j}{k-1-j} = \gamma_j \binom{k+s-1-j}{k-1-j}$. Since $k+s-1 = n-d$, this gives the stated formula and its integrality. $\square$

Lemma 4.27. Let $k \geq 3$. If $\mathcal{G}$ is an $[n, k, d]_q$ A^s MDS code with $n = (s+1)(q+1) + k - 3$, then for each $0 \leq j \leq k-3$, α_j , and β_j are integers, where $\alpha_j = \frac{\binom{n-j}{k-2-j}}{\binom{k+s-2-j}{k-2-j}}$ and $\beta_j = \alpha_j \cdot \frac{q(s+1)}{k+s-1-j}$.

Proof. For $s > 0$, Corollary 4.23 gives $d^\perp = k$; for $s = 0$, $\mathcal{G}$ is MDS. Thus every $k-1$ points are independent. If $S \subseteq \mathcal{G}$ has size $k-2$, its span is a $(k-3)$ -flat with $|\langle S \rangle \cap \mathcal{G}| = k-2$. Simple counting shows that the pencil of $q+1$ hyperplanes through $\langle S \rangle$ holds exactly one member meeting $\mathcal{G}$ in $s+k-2$ points (let us call such a hyperplane a tangent of $\mathcal{G}$), the remaining q members are secants of $\mathcal{G}$. Denote by α_0 the number of tangents of $\mathcal{G}$, and by β_0 the number of secants. Counting $(k-3)$ -flat, hyperplane incident pairs $(\mathcal{A}, \mathcal{T})$ where $|\mathcal{A} \cap \mathcal{G}| = k-2$, and $\mathcal{T}$ is a tangent of $\mathcal{G}$ we obtain

$$\binom{n}{k-2} \cdot 1 = \alpha_0 \cdot \binom{k+s-2}{k-2}. \quad (7)$$

Similarly, counting $(k-3)$ -flat, hyperplane incident pairs $(\mathcal{B}, \mathcal{S})$ where $|\mathcal{B} \cap \mathcal{G}| = k-2$, and $\mathcal{S}$ is a secant of $\mathcal{G}$ we obtain

$$\beta_0 = \frac{\binom{n}{k-2} \cdot q}{\binom{k+s-1}{k-2}} \left(= \alpha_0 \cdot \frac{q(s+1)}{k+s-1} \right). \quad (8)$$

For $1 \leq j \leq k-3$, fix a j -subset $B \subseteq \mathcal{G}$. Restricting both counts to $(k-2)$ -subsets containing B replaces n , $k+s-2$, $k+s-1$, and $k-2$ by their respective differences with j . The resulting counts of tangents and secants through B are exactly α_j and β_j . $\square$

Corollary 4.28. Let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code.

1. If there exists a prime p with $\prod_{i=1}^s (q(s+1) + i) \equiv 0 \pmod p$ where $s < p < k + s$, then $n \leq (s+1)(q+1) + k - 3$.
2. If there exists a prime p with $\prod_{i=1}^s (q(s+1) + i) \equiv 0 \pmod p$ where $s < p < k + s - 1$, then $n \leq (s+1)(q+1) + k - 4$.
3. If there exists a prime p with $\prod_{i=0}^s (q(s+1) + i) \equiv 0 \pmod p$, and $\gcd(p, q) = 1$ where $s+1 < p < k + s - 1$, then $n \leq (s+1)(q+1) + k - 4$.

Proof. For part 1, suppose $\mathcal{G}$ is an $[n, k, d]_q$ A^s MDS code with $n = (s+1)(q+1) + k - 2$, and let p be a prime with $\prod_{i=1}^s (d+i) \equiv 0 \pmod p$, $s < p < k + s$. By Lemma 4.26, for $0 \leq j \leq k - 2$

$$\gamma_j = \frac{\binom{n-j}{k-1-j}}{\binom{n-d-j}{k-1-j}} = \frac{(n-j)! \cdot s!}{(d+s)! \cdot (n-j-d)!} = \frac{s! \binom{n-j}{d}}{(d+1)(d+2) \cdots (d+s)} \quad (9)$$

is an integer. Taking $j = n - d - p$, we obtain

$$\gamma_j = \frac{s! \binom{d+p}{d}}{(d+1)(d+2) \cdots (d+s)} = \frac{s!(d+s+1)(d+s+2) \cdots (d+p)}{p!}. \quad (10)$$

However, since $p \mid d + a$ for some $1 \leq a \leq s$, p does not divide $d + x$ for $s+1 \leq x < a + p$, whence γ_j is not an integer. Parts 2 and 3 imply part 1 (in part 3, $p \nmid q(s+1)$), excluding full length. One below full length, $d = q(s+1) - 1$; apply the same argument to $\alpha_{k+s-2-p}$ and $\beta_{k+s-1-p}$, respectively. The indices lie in $[0, k-3]$, and Lemma 4.27 gives

$$\alpha_j = \frac{s! \binom{n-j}{d+1}}{(d+2)(d+3) \cdots (d+s+1)}, \text{ and } \beta_j = \frac{q \cdot (s+1)! \binom{n-j}{d}}{(d+1)(d+2) \cdots (d+s+1)}.$$

□

Corollary 4.28, part 1 and, one below full length, the same argument for $\beta_{k+s-1-p}$ give:

Corollary 4.29. If $0 < s < q - 2$, $(s+2) \mid q$, and there exists a prime p with $\prod_{i=1}^s (q(s+1) + i) \equiv 0 \pmod p$ where $s < p < k + s$, then

$$m^s(k, q) \leq (s+1)(q+1) + k - 4.$$

4.3. Bounds on $\kappa(s, q)$

For fixed s, q , let us denote by $\kappa(s, q)$ the maximum k such that $m^s(k, q) = (s+1)(q+1) + k - 2$. Table 1 serves to set some of our results in context.

From the bounds in Corollary 4.25, and by computing the parameters in Lemma 4.26 and Lemma 4.27 for some small values of s and q , we may leverage the bounds in Table 1 to obtain the values of $\kappa(s, q)$ in Table 2. Entries on the diagonal follow from Corollary 4.25, entries with an asterisk follow from Corollary 4.14, and remaining entries were verified numerically using Sage [30].

If $\mathcal{G}$ is an $[n, k, d]_q$ AMDS code with $n > q + k$, then by Theorem 1.5, $\mathcal{G}$ is NMDS. In the next section, we address a gap in the literature by providing sufficient conditions on n and k under which an A^s MDS code must be N^s MDS, $s \geq 1$.

Table 1. Bounds on $\kappa(s, q)$

Conditions	$\kappa(s, q)$	Ref.
$s \geq 0$	≥ 2	Lemma 2.1
$s = 0, q$ odd	$= 2$	[11]
$s = 0, q > 2$ even	$= 3$	[12]
$s = 1, q > 3$	$= 2$	[8] , [4] , [29]
q odd, $1 \leq s < q - 2$	$= 2$	[8]
q even, $(s + 2) \nmid q, 1 \leq s < q - 2$	$= 2$	[8]
q even, $(s + 2) \mid q, 1 \leq s \leq q - 2$	≥ 3	[16]
$s \geq q$	$= 2$	Corollary 4.18
$s = q - 1, q \neq 2$	$= 3$	Corollary 4.25
$s = q - 1, q = 2$	$= 4$	Corollary 4.25
$s = q - 2, q > 3$	$= 4$	Corollary 4.25

Table 2. Bounds on $\kappa = \kappa(s, q), (s + 2, q) = (2^a, 2^b)$

$q \setminus (s + 2)$	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}
2	2											
2^2	4											
2^3	3	4										
2^4	3	3	4									
2^5	≤ 5	3	3	4								
2^6	$\leq 64^*$	3	3	3	4							
2^7	3	3	3	3	3	4						
2^8	3	≤ 5	3	3	3	3	4					
2^9	≤ 27	≤ 4	3	3	3	3	3	4				
2^{10}	≤ 5	3	3	3	3	3	3	3	4			
2^{11}	3	3	≤ 5	3	3	3	3	3	3	4		
2^{12}	3	3	≤ 4	3	3	3	3	3	3	3	4	
2^{13}	≤ 5	3	3	3	3	3	3	3	3	3	3	4

4.4. More on N^s MDS codes

For $q > 3$, if an $[2q+3, 2q, 3]_q$ NMDS code were to exist, then the dual code would be an $[2q+3, 3, 2q]_q$ code, and thus, by [part 2](#) of [Corollary 4.4](#), we arrive at the contradiction $2q+3 \leq 2q+1$. By the discussion in [Section 2](#), an $[2q+2, 2, 2q]_q$ code exists and its dual is an $[2q+2, 2q, 2]_q$ NMDS code.

If $q > 3$ and $\mathcal{G}$ is an $[k+3, k, 3]_q$ NMDS code, then $\mathcal{G}^\perp$ is an $[k+3, 3, k]_q$ NMDS code, whence $k+3 \leq 2q+1$, so $k \leq 2q-2$. From the discussion in [Section 2](#), the dual code of a $[k+2, 2, k]_q$ code, $1 \leq k \leq 2q$, is an $[k+2, k, 2]_q$ NMDS code. We thus have the following:

Lemma 4.30. *If $3 \leq k \leq 2q$ then $m'(k, q) \geq k+2$ with equality if $q > 3$ and $k = 2q-1, 2q$.*

Remark 4.31. *cf. [part 5](#) and [part 6](#) of [Theorem 1.4](#).*

With reference to [Theorem 4.12](#) we obtain the following.

Corollary 4.32. *Let $\mathcal{G}$ be an $[n, k, d]_q$ N^s MDS code.*

1. If $s = 1$ then $n \leq 2q + k$; if also $q > 3$ and $n \geq k + 3$, then $k \leq 2q - 2$.

2. If $s > 1$ then $n \leq \min\{s(q+1) + k - 1, s(q+1) + k^\perp - 1\}$, and $k, k^\perp \leq s(q+1) - 1$.

The dual of an MDS code is also MDS, and if $\mathcal{G}$ is an $[n, k, n-k]_q$ AMDS code with $n > q + k$ then (Theorem 4.12, part 1) $\mathcal{G}^\perp$ is also AMDS, so $\mathcal{G}$ is NMDS. For $s > 1$, Liao and Liao ([25], Theorem 3.6) provide lower bounds on n and k sufficient for an $[n, k, d]_q$ A^s MDS code to be N^s MDS, so too does Tong ([31], Theorem 4). However, in both works the authors require $n > s(q+1) + k - 1$, so according to part 1 of Theorem 4.12 their results hold vacuously. This observation identifies a gap in the existing literature, which we address by establishing feasible bounds within which such codes demonstrably exist. Towards this end, we first provide the following lemma.

Lemma 4.33. *Let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code, $s, d > 1$. If $k \geq (s-1)(q+1)$, and every $(k-s)$ -subset of $\mathcal{G}$ is independent, then $\mathcal{G}$ is N^s MDS.*

Proof. Write $S(\mathcal{G}^\perp) = t$. Since $s > 0$, the dual is not MDS, so $t \geq 1$. Theorem 4.12 (part 1b) gives $k \leq t(q+1) - 1$. Together with $k \geq (s-1)(q+1)$, this forces $t \geq s$. On the other hand, independence of every $(k-s)$ -subset gives $d^\perp = k + 1 - t \geq k - s + 1$, hence $t \leq s$. Thus $t = s$, as required. $\square$

The previous Lemma provides some easily obtained examples of N^2 MDS binary codes.

Example 4.34. *Let $\mathcal{G}$ be a binary $[n, k, d]_2$ A^2 MDS code, where $\mathcal{G}^\perp$ is an A^t MDS code.*

1. If $k = 3$, then (since $\mathcal{G}$ is assumed to be non-degenerate), $\mathcal{G}$ is N^2 MDS.
2. If $k = 4$, and $\mathcal{G}$ is projective, then $\mathcal{G}$ is N^2 MDS.
3. If $k = 5$, and $\mathcal{G}$ is projective and contains no line, then $\mathcal{G}$ is N^2 MDS.

We now present some bounds providing sufficiency conditions under which an A^s MDS code is necessarily an N^s MDS code.

Theorem 4.35. *Let $\mathcal{G}$ be an $[n, k, d]_q$ A^s MDS code, $1 < s < q - 1$, $(s+1, q) \neq (2^e, 2^h)$. If $k > (s-1)(q+1) - 1$ and $n > s(q+1) + k - 3$ then $\mathcal{G}^\perp$ is an A^s MDS code, so $\mathcal{G}$ is an N^s MDS code.*

Proof. Write $S(\mathcal{G}^\perp) = t \geq 1$, since $s > 0$. The length hypothesis gives $n > k + s$, so $d > 1$. Theorem 4.12 (part 1b) gives $k \leq t(q+1) - 1$; hence $k > (s-1)(q+1) - 1$ forces $t \geq s$. Suppose $t > s$. If $d^\perp = 2$, set $\mathcal{G}' = \mathcal{G}$ and $s' = s$. Otherwise, choose $d^\perp$ dependent points spanning a $(d^\perp - 2)$ -flat Λ , and let Ω be the span of $d^\perp - 2$ of them. By minimality of $d^\perp$, Ω is a $(d^\perp - 3)$ -flat containing exactly these points of $\mathcal{G}$. Quotienting at Ω gives $\mathcal{G}'$, in which the point corresponding to Λ has multiplicity at least 2. In either case, $\mathcal{G}'$ is a non-projective $[n - d^\perp + 2, k - d^\perp + 2, d']_q$ $A^{s'}$ MDS code, with $d' \geq d$ and $s' \leq s$. Since $t > s$ gives $k - d^\perp = t - 1 \geq s \geq 2$, Theorem 4.6, part 3 yields

$$n - d^\perp + 2 \leq m^{s'-1}(k - d^\perp + 1, q) + 2 \quad (11)$$

$$\leq m^{s-1}(k - d^\perp + 1, q) + 2 \quad (12)$$

$$\leq s(q+1) + k - d^\perp - 1, \quad (13)$$

where (11) follows from part 3 of Theorem 4.6 (if $n > m^{s'-1}(k - 1, q) + 2$ then $\mathcal{G}$ is projective), (12) follows from part 1 of Lemma 4.1 (which gives $m^{s'-1}(k, q) \leq m^{s-1}(k, q)$ since $s' \leq s$), and (13) follows from part 2 of Corollary 4.4 (which gives $m^{s-1}(k, q) \leq s(q+1) + k - 4$ when $(s+1, q) \neq (2^e, 2^h)$). Since this contradicts $n > s(q+1) + k - 3$, it must be the case that $t = s$. $\square$

We note that there are codes meeting the condition of the previous theorem, for example, both $[13, 5, 7]_4$, and $[14, 5, 8]_4$ A^2 MDS codes exist (see e.g. [20]).

5. Conclusion

Here, we have explored bounds on the lengths of codes with non-zero Singleton defects within the framework of projective systems. We introduced the parameters $m^s(k, q)$, denoting the maximum length of a (non-degenerate) $[n, k, d]_q$ A^sMDS code, and $m_t^s(k, q)$ denoting the maximum length of a (non-degenerate) $[n, k, d]_q$ A^sMDS code $\mathcal{G}$ such that $\mathcal{G}^\perp$ is an A^tMDS code. Table 3, and Table 4 may help place our results in context.

Table 3. Summary of bounds on $m^s(k, q)$, $\alpha \in \mathbb{Z}_{\geq 0}$; $k \geq 3$ unless stated otherwise

Conditions	$m^s(k, q)$	Ref.
$s = q - 1$; $q > 2$, and $2 \leq k \leq 3$, or $q = 2$ and $2 \leq k \leq 4$	$= (s + 1)(q + 1) + k - 2$	Corollary 4.25
$s = q - 2$; $2 \leq k \leq 4$ and $q > 2$		
$k \geq 3$	$\leq (s + 1)(q + 1) + k - 2$	Corollary 4.4
prime p divides $\prod_{i=1}^s (q(s + 1) + i)$, $s < p < k + s$	$\leq (s + 1)(q + 1) + k - 3$	Corollary 4.28
prime p divides $\prod_{i=1}^s (q(s + 1) + i)$, $s < p < k + s - 1$		
prime p divides $\prod_{i=0}^s (q(s + 1) + i)$, $\gcd(p, q) = 1$, $s + 1 < p < k + s - 1$	$\leq (s + 1)(q + 1) + k - 4$	
prime p divides $\prod_{i=1}^s (q(s + 1) + i)$, $s < p < k + s$, $0 < s < q - 2$, $(s + 2) \mid q$		
$k \geq 3$, $0 < s < q - 2$, and $(s + 2, q) \neq (2^e, 2^h)$		Corollary 4.4
$q > 3$, and $s = 1$		
q is prime, and $s + 2 \leq (q + 3)/2$	$\leq q(s + 1) + k - 2$	Corollary 4.4
q odd, $s + 2$ divides q , $s < \frac{1}{4}\sqrt{q} - 2$		
$s > 1$ and $k > q$	$\leq s(q + 1) + k - 1$	Corollary 4.14
$s \geq q$, and $k \geq 3$		Corollary 4.18
$m(k - 1, q) = q + 1$, and $s > q + 2 - k$		Corollary 4.20
$m(k - 1, q) = q + 1$, $s > q + 2 - k$, q is prime, and $3 \leq k \leq q$		
$k \geq 3$, and $d > q^2$		Corollary 4.21
$s = q - 1$; $q > 2$, and $k \geq 4$		Corollary 4.25
$s = q - 2$; $k \geq 5$ and $q > 3$		
$s \geq \alpha(q + 1) - 1$, and $k \geq 3$	$\leq (s + 1 - \alpha)(q + 1) + k - 2 + \alpha$	Corollary 4.18
$k \geq 3$, and $d > \alpha(q^2 + q) - q$, $\alpha \geq 0$		Corollary 4.21
$s = q - 1$; $q = 2$, and $k \geq 5$	$\leq k + 2$	Corollary 4.25

Our results provide new insights that both extend and refine existing knowledge in the literature, with the bounds obtained through these methods often subsuming those previously established.

The findings presented highlight the effectiveness of interpreting linear codes as projective systems. This perspective not only allows us to derive tighter bounds on code lengths but also to gain insight into the structural properties of these codes. For example, we easily see that for fixed Singleton defect, dimension, and field, codes of reasonable length must meet the Griesmer bound, must be projective, and must be dual to an AMDS code.

The results and methods presented here have potential implications for the design and analysis of error-correcting codes, particularly in scenarios where maximizing code length is critical. Based on the findings and observations, we propose the following questions and conjectures for further investigation.

Table 4. Bounds on $m_t^s(k, q)$, $k \geq 3$, $s < n - k$, $\alpha \in \mathbb{Z}_{\geq 0}$

Conditions	$m_t^s(k, q)$	Ref.
$s > 1$	$\leq t(q+1) + k^\perp - 1$	Theorem 4.12
$t > 1$	$\leq s(q+1) + k - 1$	
$s, t > 1$	$\leq (s+t)(q+1) - 2$	
$t = 1$	$\leq m(k-1, q) + k^\perp - s + 1$	
$s = 1$	$\leq (t+1)(q+1) + k^\perp - 2$	
	$\leq m(k^\perp - 1, q) + k - t + 1$	
$s = t = 1, q > 3$	$\leq 2q + k - 2$	Corollary 4.15
	$\leq (s+3)(q+1) - 6$	
$s > 1, t = 1, q > 3$	$\leq (s+2)(q+1) - 3$	
$s > 1$ and $t = 1$	$\leq q + k^\perp$	Corollary 4.14
$t > 1$, and $d > \alpha(q^2 + q) - 2q$, $\alpha \geq 0$	$\leq (s+1-\alpha)(q+1) + k - 2 + \alpha$	Corollary 4.22
$t > 1$ and $s = 1$	$\leq m(t, q) + k - t + 1$	Lemma 4.8
$t > 2$ and $s > 0$	$\leq m^{s-1}(3, q) + k - 2$	Corollary 4.9
$1 < t \leq q$, $s = 1$, and q is an odd prime	$\leq q + k - (t-2)$	Remark 4.10
$t > 1$, and $s > 0$	$\leq m^{s-1}(t, q) + k - t + 1$	Lemma 4.8
$s \in \{0, 1\}$, and $k \geq (t+1)(q+1) - 1$	$\leq k + 1$	Corollary 4.24
$s > 1$, and $k \geq t(q+1)$	No such code	
$s = t = 1, q > 3$, and $k = 2q - 1, 2q$	$\leq k + 2$	Lemma 4.30
$t = s = 1$	$\leq 2q + k$	Corollary 4.32

Question 1: Codes meeting the hypothesis of Theorem 4.35 form an extremely restricted family (especially in view of Corollary 4.9). Can the bounds be widened, or otherwise relaxed to include more codes?

Question 2: For which values of k, s , and q is the inequality in part 2 of Lemma 4.1 strict? For example, when $s = 0$ and $k = 3$, the inequality is strict only if q is odd.

The dual of the perfect ternary Golay code is a projective two-weight $[11, 5, 6]_3$ code underlying the Berlekamp–van Lint–Seidel strongly regular graph $\text{SRG}(243, 22, 1, 2)$. This code is length-maximal, with weights 6 and 9. The self-dual extended ternary Golay code $[12, 6, 6]_3$ is a three-weight length-maximal code, with weights 6, 9 and 12. We suspect that, up to equivalence, these are the only linear length-maximal codes of non-zero defect in dimensions $k > 4$.

Conjecture 5.1. (Weak $\kappa(s, q)$ conjecture) Linear length-maximal codes of dimension 5 or more do not exist if $q > 3$. That is: $\kappa(s, q) \leq 4$ for $q > 3$.

Conjecture 5.2. (Strong $\kappa(s, q)$ conjecture) For $q > 3$, the maximum dimension of a length-maximal code is given by

$$\kappa(s, q) = \begin{cases} 4 & \text{if } s = q - 2; \\ 3 & \text{if } s = q - 1, \text{ or if } q \text{ is even, } s < q - 2, \text{ and } (s + 2) \mid q; \\ 2 & \text{otherwise.} \end{cases}$$

Declarations

Author contributions

Both authors contributed to the research and preparation of the manuscript and approve the final version.

Conflict of interest

The authors declare that they have no conflict of interest.

Data availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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Generative AI and AI-assisted technologies

The author(s) declare that no generative AI or AI-assisted technologies were used in the preparation of this manuscript, except for tools used solely for spelling, grammar, language editing, reference formatting, translating to journal template style, or similar routine editorial corrections.

References

- [1] S. Ball, [Multiple blocking sets and arcs in finite planes](#), J. London Math. Soc. (2) 54(3) (1996) 581–593.
- [2] S. Ball, [On sets of vectors of a finite vector space in which every subset of basis size is a basis](#), J. Eur. Math. Soc. 14(3) (2012) 733–748.
- [3] S. Ball, [Finite Geometry and Combinatorial Applications](#), vol. 82 in London Mathematical Society Student Texts, Cambridge University Press, Cambridge (2015).
- [4] S. Ball, A. Blokhuis, and F. Mazzocca, [Maximal arcs in Desarguesian planes of odd order do not exist](#), Combinatorica 17(1) (1997) 31–41.
- [5] S. Ball and J. De Beule, [On sets of vectors of a finite vector space in which every subset of basis size is a basis II](#), Des. Codes Cryptogr. 65(1-2) (2012) 5–14.
- [6] S. Ball and J. W. P. Hirschfeld, [Bounds on \$\(n, r\)\$ -arcs and their application to linear codes](#), Finite Fields Appl. 11(3) (2005) 326–336.
- [7] S. Ball and M. Lavrauw, [Arcs in finite projective spaces](#), EMS Surv. Math. Sci. 6(1-2) (2019) 133–172.
- [8] A. Barlotti, Sui $\{k; n\}$ -archi di un piano lineare finito, Boll. Un. Mat. Ital. (3) 11(4) (1956) 553–556.

- [9] A. Blokhuis, [On multiple nuclei and a conjecture of Lunelli and Sce](#), Bull. Belg. Math. Soc. Simon Stevin 1(3) (1994) 349–353.
- [10] M. A. de Boer, [Almost MDS codes](#), Des. Codes Cryptogr. 9 (1996) 143–155.
- [11] R. C. Bose, Mathematical theory of the symmetrical factorial design, Sankhyā 8 (1947) 107–166.
- [12] A. A. Bruen and R. Silverman, [On the nonexistence of certain M.D.S. codes and projective planes](#), Math. Z. 183 (1983) 171–175.
- [13] K. A. Bush, [Orthogonal arrays of index unity](#), Ann. Math. Statist. 23 (1952) 426–434.
- [14] L. R. A. Casse, A solution to Beniamino Segre’s “Problem $I_{r,q}$ ” for q even, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) 46 (1969) 13–20.
- [15] L. R. A. Casse and D. G. Glynn, [The solution to Beniamino Segre’s problem \$I_{r,q}\$, \$r = 3\$, \$q = 2^h\$](#) , Geom. Dedicata 13 (1982) 157–163.
- [16] R. H. F. Denniston, [Some maximal arcs in finite projective planes](#), J. Combinatorial Theory 6 (1969) 317–319.
- [17] S. Dodunekov and I. Landgev, [On near-MDS codes](#), J. Geom. 54 (1995) 30–43.
- [18] S. M. Dodunekov and I. N. Landjev, [Near-MDS codes over some small fields](#), Discrete Math. 213(1-3) (2000) 55–65.
- [19] A. Faldum and W. Willems, [Codes of small defect](#), Des. Codes Cryptogr. 10 (1997) 341–350.
- [20] M. Grassl, Bounds on the minimum distance of linear codes and quantum codes, Online available at <http://www.codetables.de>, Accessed on 2026-09-12 (2007).
- [21] B. R. Gulati and E. G. Kounias, [Maximal sets of points in finite projective space, no \$t\$ -linearly dependent](#), J. Combinatorial Theory Ser. A 15 (1973) 54–65.
- [22] J. W. P. Hirschfeld, Finite Projective Spaces of Three Dimensions, Oxford Mathematical Monographs, Oxford University Press, New York (1985).
- [23] J. W. P. Hirschfeld, [Projective Geometries over Finite Fields](#), 2nd ed., Oxford Mathematical Monographs, Oxford University Press, New York (1998).
- [24] J. W. P. Hirschfeld and L. Storme, [The Packing Problem in Statistics, Coding Theory and Finite Projective Spaces: Update 2001](#), Springer US, Boston, MA (2001) 201–246.
- [25] Q. Liao and H. Liao, [On \$m\$ -MDS codes over finite fields](#), Int. J. Comput. Math. 91(5) (2014) 863–871.
- [26] F. J. MacWilliams and N. J. A. Sloane, The Theory of Error-Correcting Codes. I, vol. 16 in North-Holland Mathematical Library, North-Holland, Amsterdam (1977).
- [27] B. Qvist, Some remarks concerning curves of the second degree in a finite plane, Ann. Acad. Sci. Fennicae Ser. A. I. Math.-Phys. 1952(134) (1952) 27.
- [28] B. Segre, [Curve razionali normali e \$k\$ -archi negli spazi finiti](#), Ann. Mat. Pura Appl. 39 (1955) 357–379.
- [29] J. A. Thas, [Some results concerning \$\{\(q+1\)\(n-1\); n\}\$ -arcs and \$\{\(q+1\)\(n-1\)+1; n\}\$ -arcs in finite projective planes of order \$q\$](#) , J. Combinatorial Theory Ser. A 19(2) (1975) 228–232.
- [30] The SageMath Developers, [SageMath \(Version 10.3\) \[Computer software\]](#), Zenodo (2024).
- [31] H. Tong, [NNMDS codes](#), J Syst Sci Complex. 25 (2012) 617–624.
- [32] M. Tsfasman, S. Vlăduț, and D. Nogin, [Algebraic Geometric Codes: Basic Notions](#), American Mathematical Society (2007).
- [33] Z. Weiner, [On \$\(k, p^e\)\$ -arcs in Desarguesian planes](#), Finite Fields Appl. 10(3) (2004) 390–404.