On the spectral characterization of kite graphs

Sezer Sorgun, Hatice Topcu

Abstract: The Kite graph, denoted by $K_{p,q}$, is obtained by appending a complete graph $K_p$ to a pendant vertex of a path $P_q$. In this paper, firstly we show that no two non-isomorphic kite graphs are cospectral w.r.t the adjacency matrix. Let $G$ be a graph which is cospectral with $K_{p,q}$ and let $w(G)$ be the clique number of $G$. Then, it is shown that $w(G) \geq p - 2q + 1$. Also, we prove that $Kite_{p,2}$ graphs are determined by their adjacency spectrum.

2010 MSC: 05C50, 05C75

Keywords: Kite graph, Cospectral graphs, Clique number, Determined by adjacency spectrum

1. Introduction

All of the graphs considered here are simple and undirected. Let $G = (V,E)$ be a graph with vertex set $V(G) = \{v_1,v_2,\ldots,v_n\}$ and edge set $E(G)$. For a given graph $F$, if $G$ does not contain $F$ as an induced subgraph, then $G$ is called $F$–free. A complete subgraph of $G$ is a clique of $G$. The clique number of $G$ is the number of the vertices in the largest clique of $G$ and it is denoted by $w(G)$. Let $A(G)$ be the $(0,1)$-adjacency matrix of $G$ and $d_k$ denotes the degree of the vertex $v_k$. The polynomial $P_{A(G)}(\lambda) = \det(\lambda I - A(G))$ is the adjacency characteristic polynomial of $G$, where $I$ is the identity matrix. Eigenvalues of the matrix $A(G)$ are adjacency eigenvalues. Since $A(G)$ is real and symmetric matrix, adjacency eigenvalues are all real numbers and could be ordered as $\lambda_1(A(G)) \geq \lambda_2(A(G)) \geq \ldots \geq \lambda_n(A(G))$. Adjacency spectrum of the graph $G$ consists of the adjacency eigenvalues with their multiplicities. The largest absolute value of the adjacency eigenvalues of a graph is known as its adjacency spectral radius. Two graphs $G$ and $H$ are said to be cospectral if they have same spectrum (i.e., same characteristic polynomial). A graph $G$ is determined by its adjacency spectrum, shortly DAS, if every graph cospectral with $G$ w.r.t the adjacency matrix, is isomorphic to $G$. It is conjectured in [5] that almost all graphs are determined by their spectrum, DS for short. But it is difficult to show that a given
graph is DS. Up to now, some graphs are proved to be DS [1, 2, 4–11, 13, 15]. Recently, some papers have appeared in the literature that researchers focus on some special graphs (oftenly under some conditions) and prove that these special graphs are DS or non-DS [1, 2, 6–11, 13, 15]. For a recent survey, one can see [5].

The Kite graph, denoted by $Kite_{p,q}$, is obtained by appending a complete graph with $p$ vertices $K_p$ to a pendant vertex of a path graph with $q$ vertices $P_q$. If $q = 1$, it is called short kite graph.

In this paper, firstly we obtain the characteristic polynomial of kite graphs and show that no two non-isomorphic kite graphs are cospectral w.r.t the adjacency matrix. Then for a given graph $G$ which is cospectral with $Kite_{p,q}$, the clique number of $G$ is $w(G) \geq p - 2q + 1$. Also we prove that $Kite_{p,2}$ graphs are DAS for all $p$.

2. Preliminaries

First, we give some lemmas that will be used in the next sections of this paper.

**Lemma 2.1.** [8] Let $x_1$ be a pendant vertex of a graph $G$ and $x_2$ be the vertex which is adjacent to $x_1$. Let $G_1$ be the induced subgraph obtained from $G$ by deleting the vertex $x_1$. If $x_1$ and $x_2$ are deleted, the induced subgraph $G_2$ is obtained. Then,

$$P_{A(G)}(\lambda) = \lambda P_{A(G_1)}(\lambda) - P_{A(G_2)}(\lambda)$$

**Lemma 2.2.** [4] For $n \times n$ matrices $A$ and $B$, followings are equivalent:

(i) $A$ and $B$ are cospectral

(ii) $A$ and $B$ have the same characteristic polynomial

(iii) $\text{tr}(A^i) = \text{tr}(B^i)$ for $i = 1, 2, ..., n$

**Lemma 2.3.** [4] For the adjacency matrix of a graph $G$, following parameters can be deduced from the spectrum:

(i) the number of vertices

(ii) the number of edges

(iii) the number of closed walks of any fixed length.

**Theorem 2.4.** [14] If a given connected graph $G$ has the same order, same clique number and same spectral radius with $Kite_{p,q}$ then $G$ is isomorphic to $Kite_{p,q}$.

In the rest of the paper, we denote the number of subgraphs of a graph $G$ which are isomorphic to complete graph $K_3$ by $t(G)$.

**Theorem 2.5.** [14] For any integers $p \geq 3$ and $q \geq 1$, if we denote the spectral radius of $A(Kite_{p,q})$ with $\rho(Kite_{p,q})$ then

$$p - 1 + \frac{1}{p^2} + \frac{1}{p^3} < \rho(Kite_{p,q}) < p - 1 + \frac{1}{4p} + \frac{1}{p^2 - 2p}$$

**Theorem 2.6.** [12] Let $G$ be a graph with $n$ vertices, $m$ edges and spectral radius $\mu$. If $G$ is $K_{r+1}$–free, then

$$\mu \leq \sqrt{2m\left(\frac{r-1}{r}\right)}$$

**Lemma 2.7.** [8](Interlacing Lemma) If $G$ is a graph on $n$ vertices with eigenvalues $\lambda_1(G) \geq \ldots \geq \lambda_n(G)$ and $H$ is an induced subgraph on $m$ vertices with eigenvalues $\lambda_1(H) \geq \ldots \geq \lambda_m(H)$, then for $i = 1, \ldots, m$

$$\lambda_i(G) \geq \lambda_i(H) \geq \lambda_{n-m+i}(G)$$
Lemma 2.8. [3] A connected graph with the largest adjacency eigenvalue less than 2 are precisely induced subgraphs of the Smith graphs shown in Figure 1.

![Figure 1. Smith graphs]

3. Characteristic polynomial of kite graphs

We use the method similar to that given in [8] to obtain the general form of characteristic polynomials of Kite_{p,q} graphs. Obviously, if we delete the vertex with one degree from short kite graph, the induced subgraph will be the complete graph $K_p$. Then, by deleting the vertex with one degree and its adjacent vertex, we obtain the complete graph $K_{p-1}$ with $p-1$ vertices. From Lemma 2.1, we get

\[ P_{A(K_{p,1})}(\lambda) = \lambda P_{A(K_p)}(\lambda) - P_{A(K_{p-1})}(\lambda) \]
\[ = \lambda(\lambda - p + 1)(\lambda + 1)^{p-1} - [(\lambda - p + 2)(\lambda + 1)^{p-2}] \]
\[ = (\lambda + 1)^{p-2}[\lambda^2 - \lambda p + \lambda(\lambda + 1) - \lambda + p - 2] \]
\[ = (\lambda + 1)^{p-2}[\lambda^3 - (p - 2)\lambda^2 - \lambda p + p - 2]. \]

Similarly for Kite_{p,2}, induced subgraphs will be Kite_{p,1} and $K_p$ respectively. By Lemma 2.1, we get

\[ P_{A(K_{p,2})}(\lambda) = \lambda P_{A(K_{p,1})}(\lambda) - P_{A(K_p)}(\lambda) \]
\[ = \lambda(\lambda P_{A(K_{p,1})}(\lambda) - P_{A(K_{p-1})}(\lambda)) - P_{A(K_p)}(\lambda) \]
\[ = (\lambda^2 - 1)P_{A(K_p)}(\lambda) - \lambda P_{A(K_{p-1})}(\lambda). \]

By using these polynomials, we calculate the characteristic polynomial of Kite_{p,q} where $n = p + q$. Again, by Lemma 2.1 we have

\[ P_{A(K_{p,q})}(\lambda) = \lambda P_{A(K_p)}(\lambda) - P_{A(K_{p-1})}(\lambda). \]
Coefficients of above equation are $b_1 = -1$, $a_1 = \lambda$. Simultaneously, we get

$$P_A(\text{Kite}_{p,2})(\lambda) = (\lambda^2 - 1)P_A(\text{Kite}_p)(\lambda) - \lambda P_A(\text{Kite}_{p-1})(\lambda).$$

and coefficients of above equation are $b_2 = -a_1 = -\lambda$, $a_2 = \lambda a_1 - 1 = \lambda^2 - 1$. Then for $\text{Kite}_{p,3}$, we have

$$P_A(\text{Kite}_{p,3})(\lambda) = \lambda P_A(\text{Kite}_{p,2})(\lambda) - P_A(\text{Kite}_{p,1})(\lambda)$$

$$= (\lambda(\lambda^2 - 1) - \lambda)P_A(\text{Kite}_p)(\lambda) - ((\lambda^2 - 1)P_A(\text{Kite}_{p-1})(\lambda))$$

and coefficients of above equation are $b_3 = -a_2 = -(\lambda^2 - 1)$, $a_2 = \lambda a_2 - a_1 = \lambda(\lambda^2 - 1) - \lambda$. In the following steps, for $n \geq 3$, $a_n = \lambda a_{n-1} - a_{n-2}$. From this difference equation, we get

$$a_n = \sum_{k=0}^{n} \lambda^n \frac{\sqrt{\lambda^2 - 4}}{2} \left( \frac{\lambda - \sqrt{\lambda^2 - 4}}{2} \right)^{n-k}$$

Now, let $\lambda = 2 \cos \theta$ and $u = e^{i\theta}$. Then, we have

$$a_n = \sum_{k=0}^{n} u^{2k-n} = \frac{u^{-n}(1 - u^{2n+2})}{1 - u^2}$$

and by calculation the characteristic polynomial of any kite graph $\text{Kite}_{p,q}$ where $n = p + q$ is

$$P_A(\text{Kite}_{p+q})(u + u^{-1}) = a_{n-p}P_A(\text{Kite}_p)(u + u^{-1}) - a_{n-p-1}P_A(\text{Kite}_{p-1})(u + u^{-1})$$

$$= \frac{u^{-n+p}(1 - u^{2n-2p+2})}{1 - u^2}. \left( (u + u^{-1} - p + 1). (u + u^{-1} + 1)^{p-1} \right)$$

$$- \frac{u^{-n+p+1}(1 - u^{2n-2p+4})}{1 - u^2}. \left( (u + u^{-1} - p + 2). (u + u^{-1} + 1)^{p-2} \right)$$

$$= \frac{u^{-n+p}(1 + u - u^{-1})^{p-2}}{1 - u^2} \left[ (2 - p). (1 + u - u^{2p+2} - u^{2p+3}) ight.$$ \n
$$+ (u^{-2} - u^{2p+1}) \right]$$

$$= \frac{u^{-q}(1 + u - u^{-1})^{q-2}}{1 - u^2} \left[ (2 - p). (1 + u - u^{2q+2} - u^{2q+3}) ight.$$ \n
$$+ (u^{-2} - u^{2q+1}) \right].$$

**Theorem 3.1.** No two non-isomorphic kite graphs have the same adjacency spectrum.

**Proof.** Assume that there are two cospectral kite graphs with number of vertices respectively, $p_1 + q_1$ and $p_2 + q_2$. Since they are cospectral, they must have same number of vertices and same characteristic polynomials. Hence, $n = p_1 + q_1 = p_2 + q_2$ and we get

$$P_A(\text{Kite}_{p_1+q_1})(u + u^{-1}) = P_A(\text{Kite}_{p_2+q_2})(u + u^{-1})$$

i.e.,

$$\frac{u^{-n+p_1}(1 + u - u^{-1})^{p_1-2}}{1 - u^2} \left[ (2 - p_1). (1 + u - u^{2p_1+2} - u^{2p_1+3}) ight.$$ \n
$$+ (u^{-2} - u^{2p_1+1}) \right]$$

$$= \frac{u^{-n+p_2}(1 + u - u^{-1})^{p_2-2}}{1 - u^2} \left[ (2 - p_2). (1 + u - u^{2p_2+2} - u^{2p_2+3}) ight.$$ \n
$$+ (u^{-2} - u^{2p_2+1}) \right].$$
Let \( p_1 > p_2 \). It follows that \( n - p_2 > n - p_1 \). Then, we have
\[
\begin{align*}
&\left(1 + u - u^{-1}\right) \left(1 + u - u^{-1}\right) - \left(1 + u^{-1} - u^{-2n-2p_1+2} - u^{2n-2p_1+3}\right) \\
&\left(1 + u - u^{-1}\right) \left(1 + u - u^{-1}\right) - \left(1 + u^{-1} - u^{-2n-2p_2+2} - u^{2n-2p_2+3}\right)
\end{align*}
\]
By using the fact that \( u \neq 0 \) and \( 1 + u + u^{-1} \neq 0 \), we get
\[
\begin{align*}
f(u) &= [(2 - p_1). (1 + u^{-1} - u^{2n-2p_1+2} - u^{2n-2p_1+3} + (u^{-2} - u^{2n-2p_1+4})] \\
&\quad - [(2 - p_2). (1 + u^{-1} - u^{2n-2p_2+2} - u^{2n-2p_2+3} + (u^{-2} - u^{2n-2p_2+4})]
\]
\[
= 0
\]
Since \( f(u) = 0 \), the derivation of \((2n - 2p_2 + 5)\)th of \( f \) equals to zero again. Thus, we have
\[
[(p_1 - 2)(2n - 2p_2 + 4)!u^{-2n+2p_2-6}] - [(p_2 - 2)(2n - 2p_2 + 4)!u^{-2n+2p_2-6}] = 0
\]
i.e.,
\[
\begin{align*}
(p_1 - 2) - (p_2 - 2)
\end{align*}
\]
i.e.,
\[
p_1 = p_2
\]
since \( u \neq 0 \). This is a contradiction with our assumption that \( p_1 > p_2 \). For \( p_2 > p_1 \), we get the similar contradiction. So \( p_1 \) must be equal to \( p_2 \). Hence \( q_1 = q_2 \) and these graphs are isomorphic.

4. Spectral characterization of Kite\(_{p, 2}\) graphs

**Lemma 4.1.** Let \( G \) be a graph which is cospectral with \( \text{Kite}_{p,q} \). Then we get
\[
w(G) \geq p - 2q + 1.
\]

**Proof.** Since \( G \) is cospectral with \( \text{Kite}_{p,q} \), from Lemma 2.3, \( G \) has the same number of vertices, same number of edges and same spectrum with \( \text{Kite}_{p,q} \). So, if \( G \) has \( n \) vertices and \( m \) edges, \( n = p + q \) and \( m = \left(\frac{p}{2}\right) + q = \frac{p^2 - p + 2q}{2} \). Also, \( \rho(G) = \rho(\text{Kite}_{p,q}) \). From Theorem 2.6, we say that if \( \mu > \sqrt{2m(r-1)} \) then \( G \) isn’t \( K_{r+1} \)-free. It means that, \( G \) contains \( K_{r+1} \) as an induced subgraph. Now, we claim that for \( r < p - 2q \), \( \sqrt{2m(r-1)} < \rho(G) \). By Theorem 2.5, we’ve already known that \( p + 1 + \frac{1}{p^2} + \frac{1}{p^3} < \rho(G) \). Hence, we need to show that \( \sqrt{2m(r-1)} < p + 1 + \frac{1}{p^2} + \frac{1}{p^3} \), when \( r < p - 2q \). Indeed,
\[
(\sqrt{2m\left(\frac{r-1}{r}\right)})^2 - (p - 1 + \frac{1}{p^2} + \frac{1}{p^3})^2 = (p^2 - p + 2q)(r - 1) - r(p - 1 + \frac{1}{p^2} + \frac{1}{p^3})^2
\]
\[
= (p^2 - p + 2q)(r - 1) - \left(\frac{r(p^2 + p^3)}{p^3}\right)(2(p - 1) + \frac{(p^2 + p^3)}{p^3})
\]
\[
= (pr - p^2 + p + (2q - 1)r - 2q) - \left(\frac{r(p^2 + p^3)}{p^3}\right)(2(p - 1) + \frac{(p^2 + p^3)}{p^3})
\]

By the help of Mathematica, for \( r < p - 2q \) we can see
\[
(pr - p^2 + p + (2q - 1)r - 2q) - \left(\frac{r(p^2 + p^3)}{p^3}\right)(2(p - 1) + \frac{(p^2 + p^3)}{p^3}) < 0
\]
i.e.,
\[
(\sqrt{2m\left(\frac{r-1}{r}\right)})^2 - (p - 1 + \frac{1}{p^2} + \frac{1}{p^3})^2 < 0
\]
i.e.,
\[
(\sqrt{2m\left(\frac{r-1}{r}\right)})^2 < (p - 1 + \frac{1}{p^2} + \frac{1}{p^3})^2
\]
Since \(\sqrt{2m\left(\frac{r-1}{r}\right)} > 0\) and \( p - 1 + \frac{1}{p^2} + \frac{1}{p^3} > 0\), we get
\[
\sqrt{2m\left(\frac{r-1}{r}\right)} < p - 1 + \frac{1}{p^2} + \frac{1}{p^3} < \rho(G).
\]
Thus, we proved our claim and so \( G \) contains \( K_{r+1} \) as an induced subgraph such that \( r < p - 2q \). Consequently, \( \omega(G) \geq p - 2q + 1 \).

**Theorem 4.2.** Kite\(_{p,2}\) graphs are determined by their adjacency spectrum for all \( p \).

**Proof.** If \( p = 1 \) or \( p = 2 \), Kite\(_{p,2}\) graphs are actually the path graphs \( P_3 \) or \( P_4 \). Also if \( p = 3 \), then we obtain the lollipop graph \( H_{5,3} \). As is known, these graphs are already DAS [8]. Hence we will continue our proof for \( p \geq 4 \). Adjacency characteristic polynomial of Kite\(_{p,2}\) is as below,
\[
P_{\lambda(Kite_{p,2})}(\lambda) = (\lambda + 1)^{p-2}[(\lambda)^4 + (2 - p)(\lambda)^3 - (p + 1)(\lambda)^2 + (2p - 4)(\lambda) + p - 1]
\]
By calculation, for the adjacency eigenvalues of Kite\(_{p,2}\), we obtain the following facts;
\( p - 1 < \lambda_1(A(Kite_{p,2})) < p, 0 < \lambda_2(A(Kite_{p,2})) < 2, \lambda_3(A(Kite_{p,2})) < 0, \lambda_4(A(Kite_{p,2})) = \ldots = \lambda_{p+1}(A(Kite_{p,2})) = -1 \) and \( \lambda_{p+2}(A(Kite_{p,2})) < -1 \).

For a given graph \( G \) with \( n \) vertices and \( m \) edges, assume that \( G \) is cospectral with Kite\(_{p,2}\). Then by Lemma 2.3, \( n = p + 2, m = \left(\frac{p}{2}\right) + 2 = \frac{p^2 - p + 4}{2} \) and \( t(G) = t(Kite_{p,2}) = \left(\frac{p}{3}\right) = \frac{p^2 - 3p^2 + 2p}{6} \). From
Lemma 4.1, \( w(G) \geq p - 2q + 1 \). When \( q = 2 \), \( w(G) \geq p - 3 = n - 5 \). It’s well-known that complete graph \( K_n \) is DS. So \( w(G) \neq n \). If \( w(G) = n - 1 = p + 1 \), then \( G \) contains at least one clique with size \( p - 1 \). It means that the edge number of \( G \) is greater than or equal to \( \left( \frac{p + 1}{2} \right) \). But it is a contradiction since \( \left( \frac{p + 1}{2} \right) > \left( \frac{p}{2} \right) + 2 = n \). Hence, \( w(G) \neq n - 1 \). Because of these facts, we get \( p - 3 \leq w(G) \leq p \).

From interlacing lemma, \( G \) can not contain the graphs in the following figure as an induced subgraph because \( \lambda_3(G_1) = \lambda_3(G_2) = 0 \).

**Figure 2.** Graphs \( G_1 \) and \( G_2 \)

If \( G \) is disconnected, from Lemma 2.8, components of \( G \) except one of them must be induced subgraphs of Smith graphs. Clearly, this is impossible because \( G_1 \) is forbidden and any path graph (since they have symmetric eigenvalues) can not be a component of \( G \). Hence \( G \) must be a connected graph. If \( w(G) = p \), then by Theorem 2.4., \( G \cong \text{Kite}_{p, 2} \). So we continue for \( w(G) < p \). Since \( w(G) \geq p - 3 \), \( G \) contains at least one clique with size at least \( p - 3 \). This clique is denoted by \( K_{w(G)} \). There may be at most five vertices out of the clique \( K_{w(G)} \). Let us label these five vertices respectively with \( 1, 2, 3, 4, 5 \) and call the set of these five vertices with \( A \). So, we get \( |A| \leq 5 \). Moreover, \( \forall i,j \in A \) we get \( i \sim j \) since \( G_1, G_2 \) are not induced subgraphs of \( G \) and there is no isolated vertex in \( G \). Then, we can say that \( p \geq 6 \) since \( w(G) \geq p - 3 \).

For \( i \in A \), \( x_i \) denotes the number of adjacent vertices of \( i \) in \( K_{w(G)} \). By the fact that \( p - 1 \geq w(G) \geq p - 3 \), for all \( i \in A \) we say

\[
x_i \leq w(G) - |A| + 1 \tag{1}
\]

Also, \( x_{i \wedge j} \) denotes the number of common adjacent vertices in \( K_{w(G)} \) of \( i \) and \( j \) such that \( i,j \in A \) and \( i < j \). Similarly, if \( i \sim j \) then

\[
x_{i \wedge j} \leq w(G) - |A| \tag{2}
\]

Let \( d \) denotes the number of edges between the vertices of \( A \) and \( K_{w(G)} \), also \( \alpha \) denotes the number of cliques with size 3 which are not contained by \( A \) or \( K_{w(G)} \). Then, we get

\[
m = \left( \frac{p}{2} \right) + 2 = \left( \frac{w(G)}{2} \right) + \left( \frac{|A|}{2} \right) + d \tag{3}
\]

Similarly, we get

\[
t(G) = \left( \frac{p}{3} \right) = \left( \frac{w(G)}{3} \right) + \left( \frac{|A|}{3} \right) + \alpha \tag{4}
\]

On the other hand for \( \alpha \) and \( d \), we have

\[
d = \sum_{i=1}^{A} x_i \tag{5}
\]
and

\[ \alpha = \sum_{i=1}^{\lvert A \rvert} \left( \frac{x_i}{2} \right) + \sum_{i \sim j} x_{i\wedge j}. \]  

If \( w(G) = p - 3 \) then \( \lvert A \rvert = 5 \) and so \( p \geq 8 \). Thus we have

\[ d = 3p - 14 \]  

and

\[ \alpha = \left( \frac{p}{3} \right) - \left( \frac{p - 3}{3} \right) - 10 = \frac{3p^2}{2} - \frac{15p}{2}. \]  

From (1),(2),(5),(6) and (7) we have

\[ \alpha = \sum_{i=1}^{5} \left( \frac{x_i}{2} \right) + \sum_{i \sim j} x_{i\wedge j} \leq 3 \left( \frac{p - 7}{2} \right) + \left( \frac{7}{2} \right) + 2 \sum_{i=1}^{5} x_i \]
\[ = 3 \left( \frac{p - 7}{2} \right) + \left( \frac{7}{2} \right) + 6p - 28 \]
\[ = \frac{3p^2 - 33p}{2} + 77. \]

But obviously for \( p = 8 \) this result gives contradiction. Also for \( p > 8 \),

\[ \frac{3p^2 - 33p}{2} + 77 < \frac{3p^2 - 15p}{2} = \alpha. \]

So this is again a contradiction.

If \( w(G) = p - 2 \) then \( \lvert A \rvert = 4 \) and so \( p \geq 7 \). Thus we have

\[ d = 2p - 7 \]

and

\[ \alpha = \left( \frac{p}{3} \right) - \left( \frac{p - 2}{3} \right) - 4 = p^2 - 4p. \]

On the other hand we have

\[ \alpha = \sum_{i=1}^{4} \left( \frac{x_i}{2} \right) + \sum_{i \sim j} x_{i\wedge j} \leq 2 \left( \frac{p - 5}{2} \right) + \left( \frac{3}{2} \right) + 2 \sum_{i=1}^{4} x_i \]
\[ = p^2 - 7p + 19. \]

Clearly for \( p \geq 7 \),

\[ p^2 - 7p + 19 < p^2 - 4p = \alpha. \]

So this is a contradiction.

Similarly, if \( w(G) = p - 1 \) then \( \lvert A \rvert = 3 \) and so \( p \geq 6 \). Hence we have

\[ d = p - 2 \]

and

\[ \alpha = \frac{p^2 - 3p}{2}. \]
Also we have
\[
\alpha = \sum_{i=1}^{3} \left( \frac{x_i}{2} \right) + \sum_{i \sim j} x_i \land j \leq \left( \frac{p - 3}{2} \right) + p - 2
\]
\[
= \frac{p^2 - 5p}{2} + 4.
\]
Clearly for \( p \geq 6 \),
\[
\frac{p^2 - 5p}{2} + 4 < \frac{p^2 - 3p}{2} = \alpha.
\]
Again we obtain a contradiction.

By all of these facts, we can conclude that our assumption is actually false, then \( w(G) \not< p \). Hence \( w(G) = p \) and so that by Theorem 2.4., \( G \cong \text{Kite}_{p,2} \).

In the final of the paper, we give a conjecture below.

**Conjecture 4.3.** For \( q > 2 \), \( \text{Kite}_{p,q} \) graphs are DAS.

**Acknowledgment:** The authors are grateful to the referees for many suggestions which led to an improved version of this paper.

**References**
