

Classification of four-dimensional anti-dendriform algebras whose associated associative algebra has a one-dimensional annihilator

Research Article

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Abstract: This paper is devoted to the classification of four-dimensional anti-dendriform algebras over the field of complex numbers. Such algebras are defined by the presence of two binary operations whose sum determines an associative multiplication. The algebra obtained with respect to this multiplication is called the associated associative algebra, which is necessarily nilpotent. In the present paper, we obtain a classification of all four-dimensional anti-dendriform algebras associated with four-dimensional associative algebras whose annihilator is one-dimensional.

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1. Introduction

The idea of studying algebras endowed with two or more multiplications is not new, although it has attracted renewed attention in recent years. For instance, associative dialgebras, defined as vector spaces equipped with two associative operations, were investigated in [10, 16, 34].

A dual analogue of Poisson algebras, obtained by interchanging the roles of the two binary operations in the Leibniz rule that defines a Poisson algebra, was introduced by Bai, Bai, Guo, and Wu (see the references in [1, 33]). In [13], a new family of modified double Poisson brackets and the corresponding notion of mixed double Poisson algebras were introduced.

In [18], the authors defined dendriform di- and trialgebras in an arbitrary variety Var of binary algebras (such as associative, commutative, Poisson, etc.). They proved that every dendriform dialgebra can be embedded into a Rota–Baxter algebra of weight zero in the same variety, while every dendriform trialgebra can be embedded into a Rota–Baxter algebra of nonzero weight.

Various other classes of algebras defined by several multiplications satisfying specific identities have also been studied in the literature; see, for example, [7, 22, 25, 28, 32].

The notion of anti-dendriform algebras, proposed as a new way of splitting the associativity condition, was introduced by Gao, Liu, and Bai in [17], motivated by the concept of dendriform algebras from [27]. Anti-dendriform algebras are defined as algebras equipped with two binary operations whose sum is associative, and such that the negatives of the left and right multiplication operators form bimodules over the resulting associative algebra. This characterization explains the terminology by comparison with the analogous characterization of dendriform algebras.

Note that there exists an anti-structure corresponding to pre-Lie algebras, namely the notion of anti-pre-Lie algebras introduced in [26]. These algebras are characterized as Lie-admissible algebras. Motivated by the study of anti-pre-Lie algebras, a new approach to splitting operations was proposed.

Moreover, the following commutative diagram holds, which is obtained from the classical diagram by replacing dendriform and pre-Lie algebras with anti-dendriform and anti-pre-Lie algebras, respectively:

$$\begin{array}{ccc} \text{anti-dendriform algebras} & \longrightarrow & \text{anti-pre-Lie algebras} \\ \downarrow & & \downarrow \\ \text{associative algebras} & \longrightarrow & \text{Lie algebras} \end{array}$$

The algebraic classification (up to isomorphism) of n -dimensional algebras belonging to a variety defined by a family of polynomial identities is a classical problem in the theory of non-associative algebras. Numerous results concerning the algebraic classification of low-dimensional algebras in various varieties of associative and non-associative algebras are known (see, for example, [3, 4, 9, 11, 12, 14, 15, 29]). Algebraic classification also provides a basis for obtaining the geometric classification [20]. In particular, algebraic classifications have been obtained for 2-dimensional algebras [21, 30], 3-dimensional evolution algebras [8], 3-dimensional anticommutative algebras [23, 24], 3-dimensional diassociative algebras [31], as well as for non-isomorphic complex 3-dimensional transposed Poisson algebras [6]. See [19] for an overview of classification results. The present paper is devoted to the classification of four-dimensional anti-dendriform algebras under the certain conditions.

The paper is organized as follows. In Section 2 we recall the necessary definitions and establish several preliminary results. In particular, to every anti-dendriform algebra there corresponds, in a unique way, an associative algebra, which is known to be nilpotent (see Proposition 2.2 and Corollary 2.5). We then recall the classification of all four-dimensional complex nilpotent indecomposable associative algebras, stated in Theorem 2.8, which was obtained in [5]. In Section 3, working over the field of complex numbers, we obtain a classification of all four-dimensional anti-dendriform algebras associated with four-dimensional associative algebras having a one-dimensional annihilator (see Theorem 2.9 together with Theorems 3.1–3.8). In future, we will complete the classification of all four-dimensional anti-dendriform algebras whose associated associative algebra is indecomposable.

2. Preliminaries

Throughout the paper, all vector spaces and algebras are finite-dimensional and over the complex field $\mathbb{C}$ unless otherwise stated. Let A be a vector space with two bilinear operations

$$\triangleright : A \otimes A \rightarrow A, \quad \triangleleft : A \otimes A \rightarrow A.$$

Define a bilinear operation $\cdot$ on A as follows:

$$x \cdot y = x \triangleright y + x \triangleleft y, \quad \forall x, y \in A. \quad (1)$$

If $(A, \cdot)$ is an associative algebra, then we call the triple $(A, \triangleright, \triangleleft)$ an *associative admissible algebra* and $(A, \cdot)$ is called *the associated associative algebra of $(A, \triangleright, \triangleleft)$* .

Definition 2.1 (Gao, Liu, Bai [17]). *Let A be a vector space with two bilinear operations $\triangleright$ and $\triangleleft$. The triple $(A, \triangleright, \triangleleft)$ is called an *anti-dendriform algebra* if the following identities are satisfied:*

$$(x \triangleright y) \triangleleft z = x \triangleright (y \triangleleft z), \quad (2)$$

$$x \triangleright (y \triangleright z) = -(x \cdot y) \triangleright z, \quad (3)$$

$$x \triangleright (y \triangleleft z) = -x \triangleleft (y \cdot z), \quad (4)$$

$$x \triangleright (y \triangleright z) = (x \triangleleft y) \triangleleft z, \quad (5)$$

$$(x \cdot y) \triangleright z = x \triangleleft (y \cdot z), \quad (6)$$

$$-(x \cdot y) \triangleright z = (x \triangleleft y) \triangleleft z, \quad (7)$$

$$-x \triangleleft (y \cdot z) = (x \triangleleft y) \triangleleft z. \quad (8)$$

Proposition 2.2 ([17]). *Let $(A, \triangleright, \triangleleft)$ be an anti-dendriform algebra. Define a bilinear operation $\cdot$ by (1). Then $(A, \cdot)$ is an associative algebra, called **the associated associative algebra** of $(A, \triangleright, \triangleleft)$. Furthermore, $(A, \triangleright, \triangleleft)$ is called **a compatible anti-dendriform algebra structure** on $(A, \cdot)$.*

Recall that an associative algebra $(A, \cdot)$ is 2-nilpotent if $(x \cdot y) \cdot z = x \cdot (y \cdot z) = 0$ for all $x, y, z \in A$. Similarly, an anti-dendriform algebra $(A, \triangleright, \triangleleft)$ is 2-nilpotent if $(x *_{i_1} y) *_{i_2} z = x *_{i_3} (y *_{i_4} z) = 0$ for all $i_1, i_2, i_3, i_4 \in \{\triangleright, \triangleleft\}$ and $x, y, z \in A$. Using identities (2)–(8), it was shown in [2] that the anti-dendriform algebras associated with the associative abelian algebra are 2-nilpotent.

The following sets are called the annihilators of associative and anti-dendriform algebras, respectively:

$$Z_{As}(A) = \{x \in A \mid x \cdot y = y \cdot x = 0, \forall y \in A\},$$

$$Z_{AD}(A) = \{x \in A \mid x \triangleright y = x \triangleleft y = y \triangleright x = y \triangleleft x = 0, \forall y \in A\}.$$

An *ideal* I of an anti-dendriform algebra A is a subspace of A such that

$$x \triangleright y, x \triangleleft y, y \triangleright x, y \triangleleft x \in I, \quad \text{for all } x \in A, y \in I.$$

It is obvious that the annihilator of an arbitrary anti-dendriform algebra is an ideal.

Proposition 2.3 ([2]). *Let $(A, \cdot)$ be an associative algebra and let $(A, \triangleright, \triangleleft)$ be a compatible anti-dendriform algebra structure on $(A, \cdot)$. If the annihilator $(Z(A), \cdot)$ of $(A, \cdot)$ is the annihilator $(Z(A), \triangleright, \triangleleft)$ of $(A, \triangleright, \triangleleft)$, then the quotient $(A/Z(A), \triangleright, \triangleleft)$ is a compatible anti-dendriform algebra structure on $(A/Z(A), \cdot)$.*

Proposition 2.4 ([17]). *Let $(A, \cdot)$ be an associative algebra with a non-zero idempotent e , that is, $e \cdot e = e$. Then there does not exist a compatible anti-dendriform algebra structure on $(A, \cdot)$.*

It is known that any finite-dimensional associative algebra without a non-zero idempotent element is nilpotent. Therefore we have the following conclusion.

Corollary 2.5. *The associated associative algebra of any anti-dendriform algebra is nilpotent.*

For an algebra A of an arbitrary variety, we consider the series

$$A^1 = A, \quad A^{i+1} = \sum_{k=1}^i A^k A^{i+1-k}, \quad i \geq 1.$$

We say that an algebra A is *nilpotent* if $A^i = \{0\}$ for some $i \in \mathbb{N}$. The smallest integer satisfying $A^i = \{0\}$ is called the *index of nilpotency* of A . Obviously, for an n -dimensional nilpotent algebra A we have $A^{n+1} = \{0\}$, i.e. the index of nilpotency of an n -dimensional nilpotent algebra is not greater than $n + 1$. An n -dimensional algebra A is called *maximum nilpotent* if the nilpotency index of the algebra is equal to $n + 1$.

From now we use the following notations: As_n^q and AD_n^q denote n -dimensional associative and anti-dendriform algebra structures associated with nilpotent associative algebras, where q is an index for numbering.

In what follows, we assume that $\{e_1, \dots, e_n\}$ is a basis of the n -dimensional algebra A . When presenting the multiplication table of A , all products of basis elements that are equal to zero are omitted. We denote by $\langle v_1, \dots, v_k \rangle$ the vector space spanned by the vectors $v_1, \dots, v_k$.

By Corollary 2.5, in order to present the known classification of arbitrary three-dimensional anti-dendriform algebras over the field of complex numbers, we first recall the classification of all three-dimensional nilpotent associative algebras over $\mathbb{C}$.

Theorem 2.6 ([24]). *Any three-dimensional nilpotent associative algebra over the field of complex numbers is isomorphic to exactly one of the following pairwise non-isomorphic associative algebras:*

- As_3^1 : Abelian;
- As_3^2 : $e_1 e_2 = e_3, e_2 e_1 = -e_3$;
- As_3^3 : $e_1 e_1 = e_3$;
- As_3^4 : $e_1 e_2 = e_3$;
- $As_3^5(\lambda)$: $e_1 e_1 = e_3, e_1 e_2 = \lambda e_3, e_2 e_2 = e_3, \lambda \in \mathbb{C}$;
- As_3^6 : $e_1 e_1 = e_2, e_1 e_2 = e_3, e_2 e_1 = e_3$.

We now present the classification of all three-dimensional anti-dendriform algebras whose associated associative algebra is one of the above three-dimensional associative algebras.

Theorem 2.7 ([2]). *Any three-dimensional anti-dendriform algebra over the field of complex numbers is isomorphic to one of the following anti-dendriform algebras:*

- AD_3^1 : $e_1 \triangleright e_1 = \frac{1}{2}e_2, e_1 \triangleleft e_1 = \frac{1}{2}e_2, e_1 \triangleright e_2 = e_2 \triangleleft e_1 = 2e_3, e_2 \triangleright e_1 = e_1 \triangleleft e_2 = -e_3$;
- AD_3^2 : $e_1 \triangleright e_1 = \frac{1}{2}e_2 + e_3, e_1 \triangleleft e_1 = \frac{1}{2}e_2 - e_3, e_1 \triangleright e_2 = e_2 \triangleleft e_1 = 2e_3, e_2 \triangleright e_1 = e_1 \triangleleft e_2 = -e_3$;
- AD_3^3 : is trivial, that is, all products are zero;
- AD_3^4 : $e_1 \triangleright e_2 = e_3, e_2 \triangleright e_1 = -e_3, e_1 \triangleleft e_2 = -e_3, e_2 \triangleleft e_1 = e_3$;
- AD_3^5 : $e_1 \triangleright e_1 = e_3, e_1 \triangleleft e_1 = -e_3$;
- AD_3^6 : $e_1 \triangleright e_2 = e_3, e_1 \triangleleft e_2 = -e_3$;
- $AD_3^7(\lambda)$: $\begin{cases} e_1 \triangleright e_1 = e_3, e_1 \triangleright e_2 = \lambda e_3, e_2 \triangleright e_2 = e_3, \\ e_1 \triangleleft e_1 = -e_3, e_1 \triangleleft e_2 = -\lambda e_3, e_2 \triangleleft e_2 = -e_3; \end{cases}$
- $AD_3^8(\alpha, \beta)$: $\begin{cases} e_1 \triangleright e_1 = e_3, e_1 \triangleright e_2 = \alpha e_3, e_2 \triangleright e_1 = \beta e_3, \\ e_1 \triangleleft e_1 = -e_3, e_1 \triangleleft e_2 = (1 - \alpha)e_3, e_2 \triangleleft e_1 = (-1 - \beta)e_3; \end{cases}$

$$AD_3^9(\alpha) : e_1 \triangleright e_2 = \alpha e_3, e_2 \triangleright e_1 = -\alpha e_3, e_1 \triangleleft e_2 = (1 - \alpha)e_3, e_2 \triangleleft e_1 = (-1 + \alpha)e_3;$$

$$AD_3^{10} : e_1 \triangleright e_1 = e_2, e_2 \triangleright e_1 = -e_3, e_1 \triangleleft e_1 = -e_2, e_1 \triangleleft e_2 = e_3;$$

$$AD_3^{11}(\alpha, \beta) : e_1 \triangleright e_2 = \alpha e_3, e_2 \triangleright e_1 = \beta e_3, e_1 \triangleleft e_2 = (1 - \alpha)e_3, e_2 \triangleleft e_1 = -\beta e_3;$$

$$AD_3^{12}(\alpha, \beta) : \begin{cases} e_1 \triangleright e_2 = \alpha e_3, e_2 \triangleright e_1 = \beta e_3, e_2 \triangleright e_2 = e_3, \\ e_1 \triangleleft e_2 = (1 - \alpha)e_3, e_2 \triangleleft e_1 = -\beta e_3, e_2 \triangleleft e_2 = -e_3; \end{cases}$$

$$AD_3^{13}(\alpha, \beta, \gamma) : \begin{cases} e_1 \triangleright e_1 = e_3, e_1 \triangleright e_2 = \alpha e_3, e_2 \triangleright e_1 = \beta e_3, e_2 \triangleright e_2 = \gamma e_3, \\ e_1 \triangleleft e_1 = -e_3, e_1 \triangleleft e_2 = (1 - \alpha)e_3, e_2 \triangleleft e_1 = -\beta e_3, e_2 \triangleleft e_2 = -\gamma e_3; \end{cases}$$

$$AD_3^{14} : e_1 \triangleright e_1 = e_2, e_1 \triangleleft e_1 = -e_2, e_1 \triangleleft e_2 = e_3;$$

$$AD_3^{15}(\alpha, \beta, \gamma, \lambda) : \begin{cases} e_1 \triangleright e_1 = \alpha e_3, e_1 \triangleright e_2 = \beta e_3, e_2 \triangleright e_1 = \gamma e_3, \alpha \neq 0, \\ e_1 \triangleleft e_1 = (1 - \alpha)e_3, e_1 \triangleleft e_2 = (\lambda - \beta)e_3, e_2 \triangleleft e_1 = -\gamma e_3, e_2 \triangleleft e_2 = e_3; \end{cases}$$

$$AD_3^{16}(\alpha, \lambda) : \begin{cases} e_1 \triangleright e_2 = \alpha e_3, e_2 \triangleright e_1 = -\alpha e_3, \\ e_1 \triangleleft e_1 = e_3, e_1 \triangleleft e_2 = (\lambda - \alpha)e_3, e_2 \triangleleft e_1 = -\alpha e_3, e_2 \triangleleft e_2 = e_3; \end{cases}$$

$$AD_3^{17}(\lambda) : e_1 \triangleright e_1 = e_2, e_1 \triangleleft e_1 = -e_2 + e_3, e_1 \triangleleft e_2 = \lambda e_3, e_2 \triangleleft e_2 = e_3;$$

$$AD_3^{18}(\alpha) : e_1 \triangleright e_1 = \alpha e_3, e_1 \triangleleft e_1 = (1 - \alpha)e_3;$$

$$AD_3^{19} : e_2 \triangleright e_1 = e_3, e_1 \triangleleft e_1 = e_3, e_2 \triangleleft e_1 = -e_3;$$

$$AD_3^{20}(\alpha) : \begin{cases} e_1 \triangleright e_1 = \alpha e_3, e_1 \triangleright e_2 = e_3, e_2 \triangleright e_1 = -e_3, \\ e_1 \triangleleft e_1 = (1 - \alpha)e_3, e_1 \triangleleft e_2 = -e_3, e_2 \triangleleft e_1 = e_3; \end{cases}$$

$$AD_3^{21}(\alpha) : \begin{cases} e_1 \triangleright e_2 = e_3, e_2 \triangleright e_1 = \alpha e_3, \\ e_1 \triangleleft e_1 = e_3, e_1 \triangleleft e_2 = -e_3, e_2 \triangleleft e_1 = -\alpha e_3, \alpha \neq -1; \end{cases}$$

$$AD_3^{22}(\alpha, \beta) : \begin{cases} e_1 \triangleright e_1 = \alpha e_3, e_2 \triangleright e_1 = \beta e_3, e_2 \triangleright e_2 = e_3, \\ e_1 \triangleleft e_1 = (1 - \alpha)e_3, e_2 \triangleleft e_1 = -\beta e_3, e_2 \triangleleft e_2 = -e_3; \end{cases}$$

$$AD_3^{23} : e_1 \triangleright e_1 = e_2, e_1 \triangleleft e_1 = -e_2 + e_3;$$

where $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$.

The only isomorphisms between algebras from the above list are the following:

$$AD_3^8(\alpha, \beta) \cong AD_3^8(-\beta, -\alpha), \quad AD_3^{15}(\alpha, \beta, \gamma, 0) \cong AD_3^{15}(\alpha, -\beta, -\gamma, 0), \quad AD_3^{21}(-1) \cong AD_3^{20}(0).$$

From this theorem, the following observations can be made:

- the anti-dendriform algebras AD_3^1 and AD_3^2 are compatible anti-dendriform algebra structures on As_3^6 ;
- the anti-dendriform algebras AD_3^3 – $AD_3^7(\lambda)$ are compatible anti-dendriform algebra structures on As_3^1 ;
- the anti-dendriform algebras $AD_3^8(\alpha, \beta)$ – AD_3^{10} are compatible anti-dendriform algebra structures on As_3^2 ;
- the anti-dendriform algebras $AD_3^{11}(\alpha, \beta)$ – AD_3^{14} are compatible anti-dendriform algebra structures on As_3^4 ;
- the anti-dendriform algebras $AD_3^{15}(\alpha, \beta, \gamma, \lambda)$ – $AD_3^{17}(\lambda)$ are compatible anti-dendriform algebra structures on $As_3^5(\lambda)$;
- the anti-dendriform algebras $AD_3^{18}(\alpha)$ – AD_3^{23} are compatible anti-dendriform algebra structures on As_3^3 .

In order to classify arbitrary four-dimensional anti-dendriform algebras over the field of complex numbers, we first recall the classification of all four-dimensional nilpotent associative algebras over $\mathbb{C}$. We use Theorem 4.1 from [5] and keep the same notation as in that reference. Recall that an algebra is called *decomposable* if it can be written as a direct sum of two subalgebras, each of positive dimension.

Theorem 2.8 (Theorem 4.1 from [5]). *Any four-dimensional complex nilpotent indecomposable associative algebra is isomorphic to one of the following pairwise non-isomorphic algebras:*

<i>Algebra</i>	<i>Table of multiplication</i>	<i>Automorphisms</i>	<i>Annihilator</i>
As_4^2	$e_1e_2 = e_3, e_2e_1 = e_4$	$\varphi_1(e_1) = ae_1 + be_3 + de_4, \varphi_1(e_2) = ae_2 + ce_3 + ee_4,$ $\varphi_1(e_3) = a^2e_3, \varphi_1(e_4) = a^2e_4,$ $\varphi_2(e_1) = be_2 + ce_3 + ee_4, \varphi_2(e_2) = ae_1 + de_3 + fe_4,$ $\varphi_2(e_3) = abe_4, \varphi_2(e_4) = abe_3,$	$\langle e_3, e_4 \rangle$
As_4^3	$e_1e_2 = e_4, e_3e_1 = e_4$	$\varphi(e_1) = ae_1 + ce_2 - ce_3 + de_4, \varphi(e_2) = be_2 + ee_4,$ $\varphi(e_3) = be_3 + fe_4, \varphi(e_4) = abe_4,$	$\langle e_4 \rangle$
As_4^4	$e_1e_2 = e_3, e_2e_1 = e_4,$ $e_2e_2 = -e_3$	$\varphi(e_1) = ae_1 + be_3 + de_4, \varphi(e_2) = ae_2 + ce_3 + ee_4,$ $\varphi(e_3) = a^2e_3, \varphi(e_4) = a^2e_4,$	$\langle e_3, e_4 \rangle$
As_4^5	$e_1e_2 = e_3, e_2e_2 = e_4,$ $e_2e_1 = -e_3$	$\varphi(e_1) = ae_1 + de_3 + fe_4, \varphi(e_2) = ce_1 + be_2 + ee_3 +$ $ge_4,$ $\varphi(e_3) = abe_3, \varphi(e_4) = b^2e_4,$	$\langle e_3, e_4 \rangle$
As_4^6	$e_1e_2 = e_4, e_3e_3 = e_4,$ $e_2e_1 = -e_4$	$\varphi_1(e_1) = ae_1 + ce_2 + de_4, \varphi_1(e_2) = -\frac{b^2}{c}e_1 + ee_4,$ $\varphi_1(e_3) = be_3 + fe_4, \varphi_1(e_4) = b^2e_4$ $\varphi_2(e_1) = \frac{d^2+ab}{c}e_1 + be_2 + ee_4, \varphi_2(e_2) = ae_1 +$ $ce_2 + fe_4,$ $\varphi_2(e_3) = de_3 + he_4, \varphi_2(e_4) = d^2e_4,$	$\langle e_4 \rangle$
$As_4^7(\alpha)$	$e_1e_2 = e_4, e_2e_2 = e_3,$ $e_2e_1 = \frac{1+\alpha}{1-\alpha}e_4, \alpha \neq 1$	$\varphi(e_1) = ae_1 + de_3 + fe_4, \varphi(e_2) = ce_1 + be_2 + ee_3 +$ $ge_4,$ $\varphi(e_3) = b^2e_3 + bc(1+\alpha)e_4, \varphi(e_4) = abe_4,$	$\langle e_3, e_4 \rangle$
As_4^8	$e_1e_1 = e_3, e_1e_3 = e_4,$ $e_2e_2 = -e_4, e_3e_1 = e_4$	$\varphi(e_1) = \sqrt[3]{a^2}e_1 + be_2 + ce_3 + de_4,$ $\varphi(e_2) = ae_2 + b\sqrt[3]{ae_3} + ee_4,$ $\varphi(e_3) = \sqrt[3]{a^4}e_3 + (2c\sqrt[3]{a^2} - b^2)e_4, \varphi(e_4) = a^2e_4,$	$\langle e_4 \rangle$
As_4^9	$e_1e_1 = e_3, e_2e_1 = e_4,$ $e_1e_3 = -e_4, e_3e_1 = -e_4$	$\varphi(e_1) = ae_1 + be_2 + ce_3 + ee_4, \varphi(e_2) = a^2e_2 + de_4,$ $\varphi(e_3) = a^2e_3 + a(b-2c)e_4, \varphi(e_4) = a^3e_4,$	$\langle e_4 \rangle$
As_4^{10}	$e_1e_1 = e_4, e_1e_2 = e_4,$ $e_2e_1 = -e_4, e_3e_3 = e_4$	$\varphi_1(e_1) = ae_1 + be_4, \varphi_1(e_2) = ae_2 + ce_4,$ $\varphi_1(e_3) = ae_3 + de_4, \varphi_1(e_4) = a^2e_4,$ $\varphi_2(e_1) = ae_1 + be_2 + de_4, \varphi_2(e_2) = \frac{c^2}{a}e_2 + ee_4,$ $\varphi_2(e_3) = ce_3 + fe_4, \varphi_2(e_4) = c^2e_4,$	$\langle e_4 \rangle$
As_4^{11}	$e_1e_1 = e_4, e_1e_2 = e_3,$ $e_2e_1 = -e_3, e_2e_2 =$ $-2e_3 + e_4$	$\varphi_1(e_1) = ae_1 + be_3 + de_4, \varphi_1(e_2) = ae_2 + ce_3 + ee_4,$ $\varphi_1(e_3) = a^2e_3, \varphi_1(e_4) = a^2e_4,$ $\varphi_2(e_1) = ae_2 + be_3 + de_4, \varphi_2(e_2) = ae_1 + ce_3 + ee_4,$ $\varphi_2(e_3) = -a^2e_3, \varphi_2(e_4) = -2a^2e_3 + a^2e_4,$	$\langle e_3, e_4 \rangle$
$As_4^{12}(\alpha)$	$e_1e_1 = e_4, e_1e_2 = e_3,$ $e_2e_1 = -\alpha e_4, e_2e_2 =$ $-e_3$	$\varphi_1(e_1) = ae_1 + be_3 + de_4, \varphi_1(e_2) = ae_2 + ce_3 + ee_4,$ $\varphi_1(e_3) = a^2e_3, \varphi_1(e_4) = a^2e_4,$ $\varphi_2(e_1) = ae_2 + be_3 + de_4, \varphi_2(e_2) = ae_1 + ce_3 + ee_4,$ $\varphi_2(e_3) = -a^2e_3, \varphi_2(e_4) = -2a^2e_3 + a^2e_4,$	$\langle e_3, e_4 \rangle$

<i>Algebra</i>	<i>Table of multiplication</i>	<i>Automorphisms</i>	<i>Annihilator</i>
As_4^{13}	$e_1e_1 = e_3, e_1e_3 = -e_4,$ $e_2e_1 = e_4, e_2e_2 = e_4,$ $e_3e_1 = -e_4$	$\varphi(e_1) = e_1 + ae_2 + be_3 + ce_4, \varphi(e_2) = e_2 + ae_3 + de_4,$ $\varphi(e_3) = e_3 + (a^2 + a - 2b)e_4, \varphi(e_4) = e_4,$	$\langle e_4 \rangle$
As_4^{14}	$e_1e_2 = e_4, e_1e_3 = e_4,$ $e_2e_1 = -e_4, e_2e_2 = e_4,$ $e_3e_1 = e_4$	$\varphi(e_1) = ae_1 + be_2 - \frac{b^2}{2a}e_3 + ce_4, \varphi(e_2) = ae_2 -$ $be_3 + de_4,$ $\varphi(e_3) = ae_3 + ee_4, \varphi(e_4) = a^2e_4,$	$\langle e_4 \rangle$
$As_4^{15}(\alpha)$	$e_1e_1 = e_4, e_1e_2 = \alpha e_4,$ $e_2e_1 = -\alpha e_4, e_2e_2 = e_4,$ $e_3e_3 = e_4$	$\varphi(e_1) = ae_1 + be_2 + ce_3 + de_4,$ $\varphi(e_2) = ee_1 + fe_2 + ge_3 + he_4,$ $\varphi(e_3) = ie_1 + je_2 + ke_3 + le_4, \varphi(e_4) = (af - be)e_4,$	$\langle e_4 \rangle$
As_4^{16}	$e_1e_1 = e_2, e_1e_2 = e_3,$ $e_1e_3 = e_4, e_2e_1 = e_3,$ $e_2e_2 = e_4, e_3e_1 = e_4$	$\varphi(e_1) = ae_1 + be_2 + ce_3 + de_4,$ $\varphi(e_2) = a^2e_2 + 2abe_3 + (2ac + b^2)e_4,$ $\varphi(e_3) = a^3e_3 + 3a^2be_4, \varphi(e_4) = a^4e_4,$	$\langle e_4 \rangle$

where $a, b, c, d, e, f, g, h, i, j, k, l, \alpha \in \mathbb{C}$.

In the following section, we classify all four-dimensional anti-dendriform algebras associated with indecomposable associative algebras having a one-dimensional annihilator. Clearly, if a nilpotent associative algebra is decomposable, then its annihilator has dimension greater than one. Consequently, we obtain the following result.

Theorem 2.9. *All four-dimensional anti-dendriform algebras over the field of complex numbers whose associated associative algebra has a one-dimensional annihilator are classified.*

3. One-dimensional annihilator

In this section, we classify four-dimensional anti-dendriform algebras whose associated associative algebras are indecomposable and have a one-dimensional annihilator. Applying Theorem 2.8, we obtain the following list of all four-dimensional indecomposable associative algebras with a one-dimensional annihilator:

$$As_4^3, As_4^6, As_4^8, As_4^9, As_4^{10}, As_4^{13}, As_4^{14}, As_4^{15}(\alpha), As_4^{16}.$$

It should be noted that the associative algebra As_4^{16} is a four-dimensional null-filiform algebra, and by Theorem 3.1 in [2] there exists no compatible anti-dendriform algebra structure on As_4^{16} .

Theorem 3.1. *Any four-dimensional anti-dendriform algebra over the field of complex numbers whose associated associative algebra is As_4^3 is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{aligned}
 AD_4^1 : & e_1 \triangleright e_2 = e_4, e_3 \triangleright e_1 = e_4; \\
 AD_4^2 : & e_1 \triangleright e_2 = e_4, e_2 \triangleright e_1 = -e_4, e_3 \triangleright e_3 = e_4, e_2 \triangleleft e_1 = e_4, e_3 \triangleleft e_1 = e_4, e_3 \triangleleft e_3 = -e_4; \\
 AD_4^3 : & \begin{cases} e_1 \triangleright e_1 = e_4, e_1 \triangleright e_2 = e_4, e_2 \triangleright e_1 = -e_4, e_3 \triangleright e_3 = e_4, \\ e_1 \triangleleft e_1 = -e_4, e_2 \triangleleft e_1 = e_4, e_3 \triangleleft e_1 = e_4, e_3 \triangleleft e_3 = -e_4; \end{cases} \\
 AD_4^4 : & \begin{cases} e_1 \triangleright e_2 = e_4, e_1 \triangleright e_3 = e_4, e_2 \triangleright e_1 = -e_4, e_2 \triangleright e_2 = e_4, e_3 \triangleright e_1 = e_4 \\ e_1 \triangleleft e_3 = -e_4, e_2 \triangleleft e_1 = e_4, e_2 \triangleleft e_2 = -e_4; \end{cases} \\
 AD_4^5(\alpha) : & \begin{cases} e_1 \triangleright e_1 = e_4, e_1 \triangleright e_2 = \alpha e_4, e_2 \triangleright e_1 = -\alpha e_4, e_2 \triangleright e_2 = e_4, e_3 \triangleright e_3 = e_4, \alpha \in \mathbb{C}, \\ e_1 \triangleleft e_1 = -e_4, e_1 \triangleleft e_2 = (1 - \alpha)e_4, e_2 \triangleleft e_1 = \alpha e_4, \\ e_2 \triangleleft e_2 = -e_4, e_3 \triangleleft e_1 = e_4, e_3 \triangleleft e_3 = -e_4; \end{cases}
 \end{aligned}$$

$$AD_4^6 : \begin{cases} e_1 \triangleright e_1 = e_4, & e_1 \triangleright e_2 = e_3 + e_4, & e_2 \triangleright e_1 = -e_3, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = -e_4, & e_1 \triangleleft e_2 = -e_3, & e_2 \triangleleft e_1 = e_3; \end{cases}$$

$$AD_4^7(\alpha) : \begin{cases} e_1 \triangleright e_2 = e_3 + e_4, & e_2 \triangleright e_1 = -e_3 + \alpha e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_2 = -e_3, & e_2 \triangleleft e_1 = e_3 - \alpha e_4, & \alpha \in \mathbb{C}; \end{cases}$$

$$AD_4^8(\alpha) : \begin{cases} e_1 \triangleright e_1 = \alpha e_4, & e_1 \triangleright e_2 = e_3 + e_4, & e_2 \triangleright e_1 = -e_3, & e_2 \triangleright e_2 = e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = -\alpha e_4, & e_1 \triangleleft e_2 = -e_3, & e_2 \triangleleft e_1 = e_3, & e_2 \triangleleft e_2 = -e_4; \end{cases}$$

$$AD_4^9(\alpha, \beta) : \begin{cases} e_1 \triangleright e_1 = e_3, & e_1 \triangleright e_2 = (1 + \alpha)e_4, & e_2 \triangleright e_1 = \beta e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = -e_3, & e_1 \triangleleft e_2 = -\alpha e_4, & e_2 \triangleleft e_1 = -\beta e_4; \end{cases}$$

$$AD_4^{10}(\alpha) : \begin{cases} e_1 \triangleright e_1 = e_3, & e_1 \triangleright e_2 = (1 + \alpha)e_4, & e_2 \triangleright e_2 = e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = -e_3, & e_1 \triangleleft e_2 = -\alpha e_4, & e_2 \triangleleft e_2 = -e_4; \end{cases}$$

$$AD_4^{11}(\alpha, \delta) : \begin{cases} e_1 \triangleright e_1 = \delta e_4, & e_1 \triangleright e_2 = e_3 + e_4, & e_2 \triangleright e_1 = \alpha e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = -\delta e_4, & e_1 \triangleleft e_2 = -e_3, & e_2 \triangleleft e_1 = -\alpha e_4, & \delta \in \mathbb{C}; \end{cases}$$

$$AD_4^{12}(\alpha, \beta) : \begin{cases} e_1 \triangleright e_1 = \alpha e_4, & e_1 \triangleright e_2 = e_3 + e_4, & e_2 \triangleright e_1 = \beta e_4, & e_2 \triangleright e_2 = e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = -\alpha e_4, & e_1 \triangleleft e_2 = -e_3, & e_2 \triangleleft e_1 = -\beta e_4, & e_2 \triangleleft e_2 = -e_4. \end{cases}$$

$$AD_4^{13}[\lambda](\alpha, \beta, \gamma) : \begin{cases} e_1 \triangleright e_1 = e_3, & e_1 \triangleright e_2 = \lambda e_3 + (1 + \alpha)e_4, & e_2 \triangleright e_1 = \beta e_4, \\ e_2 \triangleright e_2 = e_3 + \gamma e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = -e_3, & e_1 \triangleleft e_2 = -\lambda e_3 - \alpha e_4, & e_2 \triangleleft e_1 = -\beta e_4, & e_2 \triangleleft e_2 = -e_3 - \gamma e_4, \end{cases}$$

where if $\lambda \neq 0$, then $\gamma \in \mathbb{C}$ and if $\lambda = 0$, then $\gamma \geq 0$.

Proof. Considering identity (6) for the following triples

$$\{e_1, e_2, e_1\}, \{e_1, e_1, e_2\}, \{e_1, e_2, e_2\}, \{e_2, e_1, e_2\}, \{e_3, e_3, e_1\}, \{e_3, e_1, e_3\}, \{e_4, e_3, e_1\},$$

we obtain $e_4 \triangleright e_1 = 0$, $e_1 \triangleleft e_4 = 0$, $e_4 \triangleright e_2 = 0$, $e_2 \triangleleft e_4 = 0$, $e_3 \triangleleft e_4 = 0$, $e_4 \triangleright e_3 = 0$, $e_4 \triangleleft e_4 = 0$, respectively.

Since $e_4 e_i = e_4 \triangleright e_i + e_4 \triangleleft e_i = 0$ and $e_i e_4 = e_i \triangleright e_4 + e_i \triangleleft e_4 = 0$, we have $e_4 \triangleleft e_i = 0$ and $e_i \triangleright e_4 = 0$. Then it is easy to see that $\langle e_4 \rangle$ is the annihilator of the algebras $(As_4^3, \cdot)$ and $(As_4^3, \triangleright, \triangleleft)$. The algebra $As_4^3 / \langle e_4 \rangle$ is a three-dimensional associative Abelian algebra, and we know that any three-dimensional complex anti-dendriform algebra associated with the Abelian algebra is isomorphic to one of the pairwise non-isomorphic algebras: $AD_3^3 - AD_3^7(\lambda)$.

Case $(As_4^3 / \langle e_4 \rangle, \triangleright, \triangleleft) \cong AD_3^3$

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11} e_4, \\ e_1 \triangleright e_2 = (1 + \alpha_{12}) e_4, \\ e_1 \triangleright e_3 = \alpha_{13} e_4, \\ e_2 \triangleright e_1 = \alpha_{21} e_4, \\ e_2 \triangleright e_2 = \alpha_{22} e_4, \\ e_2 \triangleright e_3 = \alpha_{23} e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31}) e_4, \\ e_3 \triangleright e_2 = \alpha_{32} e_4, \\ e_3 \triangleright e_3 = \alpha_{33} e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11} e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12} e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13} e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21} e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22} e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23} e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31} e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32} e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33} e_4. \end{cases}$$

It is not difficult to see that $(As_4^3, \triangleright, \triangleleft)$ is 2-nilpotent and has three generators. Thus we have $(x \triangleright y) \triangleright z = x \triangleright (y \triangleright z) = 0$ which implies that $(As_4^3, \triangleright)$ is also an associative 2-nilpotent algebra and it has three

generator elements. By Theorem 2.8, there are five non-isomorphic four-dimensional indecomposable associative 2-nilpotent three generated algebras. Hence, we obtain $AD_4^1 - AD_4^5(\alpha)$ algebras.

It is possible to consider the multiplication $\triangleleft$ in the same manner as above. However, it is not difficult to show that the constructed algebras are isomorphic.

Case $(As_4^3/\langle e_4 \rangle, \triangleright, \triangleleft) \cong AD_3^4$

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = e_3 + (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = -e_3 + \alpha_{21}e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Changing the basis $e'_3 = e_3 + \alpha_{12}e_4$, we can rewrite

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = e_3 + e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = -e_3 + \alpha_{21}e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Consider identity (7) for the triple $\{e_1, e_2, e_i\}$:

$$-(e_1e_2) \triangleright e_i = (e_1 \triangleleft e_2) \triangleleft e_i \Leftrightarrow -e_4 \triangleright e_i = -e_3 \triangleleft e_i \Leftrightarrow e_3 \triangleleft e_i = 0.$$

Hence, $\alpha_{3i} = 0$, $i \in \{1, 2, 3\}$. Similarly, considering identity (4) for the triple $\{e_i, e_1, e_2\}$, we obtain that $\alpha_{i3} = 0$, $i \in \{1, 2, 3\}$. We rewrite

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \quad e_1 \triangleright e_2 = e_3 + e_4, \quad e_2 \triangleright e_1 = -e_3 + \alpha_{21}e_4, \quad e_2 \triangleright e_2 = \alpha_{22}e_4, \quad e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = -\alpha_{11}e_4, \quad e_1 \triangleleft e_2 = -e_3, \quad e_2 \triangleleft e_1 = e_3 - \alpha_{21}e_4, \quad e_2 \triangleleft e_2 = -\alpha_{22}e_4. \end{cases}$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = ae_1 + ce_2 - ce_3 + de_4, \quad \varphi(e_2) = be_2 + ee_4, \quad \varphi(e_3) = be_3 + fe_4, \quad \varphi(e_4) = abe_4,$$

where $ab \neq 0$.

We express the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)\}$ via the basis elements $\{e_1, e_2, e_3, e_4\}$. Verifying all the multiplications of the algebra in the new basis, we obtain the relations between the parameters $\{\alpha'_{11}, \alpha'_{21}, \alpha'_{22}\}$ and $\{\alpha_{11}, \alpha_{21}, \alpha_{22}\}$:

$$\begin{aligned}
 \varphi(e_1) \triangleright \varphi(e_2) &= \varphi(e_3) + \varphi(e_4) & \Rightarrow a = 1, f = bc\alpha_{22}, \\
 \varphi(e_1) \triangleright \varphi(e_1) &= \alpha'_{11}\varphi(e_4) & \Rightarrow \alpha'_{11} = \frac{\alpha_{11} + c\alpha_{21} + c^2\alpha_{22}}{b}, \\
 \varphi(e_2) \triangleright \varphi(e_1) &= -\varphi(e_3) + \alpha'_{21}\varphi(e_4) & \Rightarrow \alpha'_{21} = \alpha_{21} + 2c\alpha_{22}, \\
 \varphi(e_2) \triangleright \varphi(e_2) &= \alpha'_{22}\varphi(e_4) & \Rightarrow \alpha'_{22} = b\alpha_{22}.
 \end{aligned}$$

We have the following cases:

- Let $\alpha_{22} = 0$. Then we have $\alpha'_{22} = 0$, $\alpha'_{21} = \alpha_{21}$, $\alpha'_{11} = \frac{\alpha_{11} + c\alpha_{21}}{b}$. If $\alpha_{21} = 0$, we obtain the algebra $(0, 0, 0)$ and $(1, 0, 0)$ when $\alpha_{11} = 0$ and $\alpha_{11} \neq 0$ with choosing $b = \alpha_{11}$, respectively, i.e. $AD_4^6(\delta)$, $\delta \in \{0, 1\}$. If $\alpha_{21} \neq 0$, then choosing $c = -\frac{\alpha_{11}}{\alpha_{21}}$, we obtain the algebra $(0, \alpha, 0)$, $\alpha \neq 0$ i.e. $AD_4^7(\alpha)$. Hence, in this case, we obtain the algebras AD_4^6 and $AD_4^7(\alpha)$, $\alpha \in \mathbb{C}$.
- Let $\alpha_{22} \neq 0$. Then choosing $b = \frac{1}{\alpha_{22}}$ and $c = -\frac{\alpha_{21}}{2\alpha_{22}}$, we obtain the algebra $AD_4^8(\alpha)$.

Case $(As_4^3/\langle e_4 \rangle, \triangleright, \triangleleft) \cong AD_3^5$

$$\begin{cases} e_1 \triangleright e_1 = e_3 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = \alpha_{21}e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

By the basis change $e'_3 = e_3 + \alpha_{11}e_4$, we can obtain

$$\begin{cases} e_1 \triangleright e_1 = e_3, \\ e_1 \triangleright e_2 = (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = \alpha_{21}e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_3, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Consider identity (7) for the triple $\{e_1, e_1, e_i\}$:

$$-(e_1e_1) \triangleright e_i = (e_1 \triangleleft e_1) \triangleleft e_i \Leftrightarrow e_3 \triangleleft e_i = 0.$$

Hence, $\alpha_{3i} = 0$, $i \in \{1, 2, 3\}$. Similarly, considering identity (4) for the triple $\{e_i, e_1, e_1\}$, we obtain $\alpha_{i3} = 0$, $i \in \{1, 2, 3\}$. We rewrite

$$\begin{cases} e_1 \triangleright e_1 = e_3, e_1 \triangleright e_2 = (1 + \alpha_{12})e_4, e_2 \triangleright e_1 = \alpha_{21}e_4, e_2 \triangleright e_2 = \alpha_{22}e_4, e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = -e_3, e_1 \triangleleft e_2 = -\alpha_{12}e_4, e_2 \triangleleft e_1 = -\alpha_{21}e_4, e_2 \triangleleft e_2 = -\alpha_{22}e_4. \end{cases}$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = ae_1 + ce_2 - ce_3 + de_4, \quad \varphi(e_2) = be_2 + ee_4, \quad \varphi(e_3) = be_3 + fe_4, \quad \varphi(e_4) = abe_4,$$

where $ab \neq 0$.

We express the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)\}$ via the basis elements $\{e_1, e_2, e_3, e_4\}$. By verifying all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha'_{12}, \alpha'_{21}, \alpha'_{22}\}$ and $\{\alpha_{12}, \alpha_{21}, \alpha_{22}\}$:

$$\begin{aligned} \varphi(e_1) \triangleright \varphi(e_1) &= \varphi(e_3) & \Rightarrow b &= a^2, \quad f = ac\alpha_{12} + ac\alpha_{21} + c^2\alpha_{22}, \\ \varphi(e_1) \triangleright \varphi(e_2) &= (1 + \alpha'_{12})\varphi(e_4) & \Rightarrow \alpha'_{12} &= \alpha_{12} + \frac{c}{a}\alpha_{22}, \\ \varphi(e_2) \triangleright \varphi(e_1) &= -\varphi(e_3) + \alpha'_{21}\varphi(e_4) & \Rightarrow \alpha'_{21} &= \alpha_{21} + \frac{c}{a}\alpha_{22}, \\ \varphi(e_2) \triangleright \varphi(e_2) &= \alpha'_{22}\varphi(e_4) & \Rightarrow \alpha'_{22} &= a\alpha_{22}. \end{aligned}$$

Then we have the following cases:

- Let $\alpha_{22} = 0$. Then we obtain the algebra $AD_4^9(\alpha, \beta)$.
- Let $\alpha_{22} \neq 0$. Then, choosing $a = \frac{1}{\alpha_{22}}$ and $c = -\frac{\alpha_{21}}{\alpha_{22}}$, we obtain the algebra $AD_4^{10}(\alpha)$.

Case $(As_4^3/\langle e_4 \rangle, \triangleright, \triangleleft) \cong AD_3^6$

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = e_3 + (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = \alpha_{21}e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

By the basis change $e'_3 = e_3 + \alpha_{12}e_4$, we can obtain

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = e_3 + e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = \alpha_{21}e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Consider identity (7) for the triple $\{e_1, e_2, e_i\}$:

$$-(e_1e_2) \triangleright e_i = (e_1 \triangleleft e_2) \triangleleft e_i \Leftrightarrow -e_4 \triangleright e_i = -e_3 \triangleleft e_i, \Leftrightarrow e_3 \triangleleft e_i = 0.$$

Hence, $\alpha_{3i} = 0$, $i \in \{1, 2, 3\}$. Similarly, considering identity (4) for the triple $\{e_i, e_1, e_2\}$, we obtain $\alpha_{i3} = 0$, $i \in \{1, 2, 3\}$. Thus, we have the following table of multiplications:

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, & e_1 \triangleright e_2 = e_3 + e_4, & e_2 \triangleright e_1 = \alpha_{21}e_4, & e_2 \triangleright e_2 = \alpha_{22}e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = -\alpha_{11}e_4, & e_1 \triangleleft e_2 = -e_3, & e_2 \triangleleft e_1 = -\alpha_{21}e_4, & e_2 \triangleleft e_2 = -\alpha_{22}e_4. \end{cases}$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = ae_1 + ce_2 - ce_3 + de_4, \quad \varphi(e_2) = be_2 + ee_4, \quad \varphi(e_3) = be_3 + fe_4, \quad \varphi(e_4) = abe_4,$$

where $ab \neq 0$.

We express the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)\}$ via the basis elements $\{e_1, e_2, e_3, e_4\}$. By verifying all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha'_{11}, \alpha'_{21}, \alpha'_{22}\}$ and $\{\alpha_{11}, \alpha_{21}, \alpha_{22}\}$:

$$\begin{aligned} \varphi(e_1) \triangleright \varphi(e_2) &= \varphi(e_3) + \varphi(e_4) \Rightarrow a = 1, \quad f = bc\alpha_{22}, \\ \varphi(e_1) \triangleright \varphi(e_1) &= \alpha'_{11}\varphi(e_4) \Rightarrow c = 0, \quad \alpha'_{11} = \frac{1}{b}\alpha_{11}, \\ \varphi(e_2) \triangleright \varphi(e_1) &= \alpha'_{21}\varphi(e_4) \Rightarrow \alpha'_{21} = \alpha_{21}, \\ \varphi(e_2) \triangleright \varphi(e_2) &= \alpha'_{22}\varphi(e_4) \Rightarrow \alpha'_{22} = b\alpha_{22}. \end{aligned}$$

Thus, we have the following cases:

- Let $\alpha_{22} = 0$. If $\alpha_{11} = 0$, then we obtain the algebra $AD_4^{11}(\alpha, 0)$. If $\alpha_{11} \neq 0$, choosing $b = \frac{1}{\alpha_{11}}$, we obtain the algebra $AD_4^{11}(\alpha, 1)$.
- Let $\alpha_{22} \neq 0$. Then choosing $b = \frac{1}{\alpha_{22}}$ we obtain the algebra $AD_4^{12}(\alpha, \beta)$.

Case $(As_4^3/\langle e_4 \rangle, \triangleright, \triangleleft) \cong AD_3^7(\lambda)$

$$\begin{cases} e_1 \triangleright e_1 = e_3 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = \lambda e_3 + (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = \alpha_{21}e_4, \\ e_2 \triangleright e_2 = e_3 + \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\lambda e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -e_3 - \alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

where $\lambda \in \mathbb{C}$.

By the basis change $e'_3 = e_3 + \alpha_{11}e_4$, we can obtain

$$\begin{cases} e_1 \triangleright e_1 = e_3, \\ e_1 \triangleright e_2 = \lambda e_3 + (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = \alpha_{21}e_4, \\ e_2 \triangleright e_2 = e_3 + \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_3, \\ e_1 \triangleleft e_2 = -\lambda e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -e_3 - \alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Consider identity (7) for the triple $\{e_1, e_1, e_i\}$:

$$-(e_1 e_1) \triangleright e_i = (e_1 \triangleleft e_1) \triangleleft e_i \Leftrightarrow e_3 \triangleleft e_i = 0.$$

Hence, $\alpha_{3i} = 0$, $i \in \{1, 2, 3\}$. Similarly, considering identity (4) for the triple $\{e_i, e_1, e_1\}$, we obtain $\alpha_{i3} = 0$, $i \in \{1, 2, 3\}$. We rewrite

$$\begin{cases} e_1 \triangleright e_1 = e_3, & e_1 \triangleright e_2 = \lambda e_3 + (1 + \alpha_{12})e_4, & e_2 \triangleright e_1 = \alpha_{21}e_4, & e_2 \triangleright e_2 = e_3 + \alpha_{22}e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = -e_3, & e_1 \triangleleft e_2 = -\lambda e_3 - \alpha_{12}e_4, & e_2 \triangleleft e_1 = -\alpha_{21}e_4, & e_2 \triangleleft e_2 = -e_3 - \alpha_{22}e_4. \end{cases}$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = ae_1 + ce_2 - ce_3 + de_4, \quad \varphi(e_2) = be_2 + ee_4, \quad \varphi(e_3) = be_3 + fe_4, \quad \varphi(e_4) = abe_4,$$

where $ab \neq 0$.

We express the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)\}$ via the basis elements $\{e_1, e_2, e_3, e_4\}$. By verifying all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha'_{12}, \alpha'_{21}, \alpha'_{22}\}$ and $\{\alpha_{12}, \alpha_{21}, \alpha_{22}\}$:

$$\begin{aligned} \varphi(e_1) \triangleright \varphi(e_1) &= \varphi(e_3) & \Rightarrow & b = a^2 + ac\lambda + c^2, \quad f = ac\alpha_{12} + ac\alpha_{21} + c^2\alpha_{22}, \\ \varphi(e_1) \triangleright \varphi(e_2) &= \lambda\varphi(e_3) + (1 + \alpha'_{12})\varphi(e_4) & \Rightarrow & a\lambda + c = \lambda, \quad ab\alpha_{12} + bc\alpha_{22} = f\lambda + aba'_{12}, \\ \varphi(e_2) \triangleright \varphi(e_1) &= \alpha'_{21}\varphi(e_4) & \Rightarrow & c = 0, \quad \alpha'_{21} = \alpha_{21}, \\ \varphi(e_2) \triangleright \varphi(e_2) &= \varphi(e_3) + \alpha'_{22}\varphi(e_4) & \Rightarrow & b = 1, \quad \alpha'_{22} = a\alpha_{22}. \end{aligned}$$

From this, we obtain

$$a^2 = b = 1, \quad c = f = 0, \quad \alpha'_{12} = \alpha_{12}, \quad \alpha'_{21} = \alpha_{21}, \quad \alpha'_{22} = a\alpha_{22}.$$

Hence, we have $AD_4^{13}[\lambda](\alpha, \beta, \gamma)$ for $\lambda \neq 0$ and $AD_4^{13}[0](\alpha, \beta, \gamma)$, where $\gamma \geq 0$. □

Theorem 3.2. *Any four-dimensional anti-dendriform algebra over the field of complex numbers whose associated associative algebra is As_4^6 is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$AD_4^{14}: e_1 \triangleright e_2 = e_4, \quad e_3 \triangleright e_1 = e_4, \quad e_2 \triangleleft e_1 = -e_4, \quad e_3 \triangleleft e_1 = -e_4, \quad e_3 \triangleleft e_3 = e_4;$$

$$AD_4^{15}: e_1 \triangleright e_2 = e_4, \quad e_2 \triangleright e_1 = -e_4, \quad e_3 \triangleright e_3 = e_4;$$

$$AD_4^{16}: e_1 \triangleright e_1 = e_4, \quad e_1 \triangleright e_2 = e_4, \quad e_2 \triangleright e_1 = -e_4, \quad e_3 \triangleright e_3 = e_4, \quad e_1 \triangleleft e_1 = -e_4;$$

$$AD_4^{17}: \begin{cases} e_1 \triangleright e_2 = e_4, & e_1 \triangleright e_3 = e_4, & e_2 \triangleright e_1 = -e_4, & e_2 \triangleright e_2 = e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_3 = -e_4, & e_2 \triangleleft e_2 = -e_4, & e_3 \triangleleft e_1 = -e_4, & e_3 \triangleleft e_3 = e_4; \end{cases}$$

$$AD_4^{18}(\alpha): \begin{cases} e_1 \triangleright e_1 = e_4, & e_1 \triangleright e_2 = \alpha e_4, & e_2 \triangleright e_1 = -\alpha e_4, & e_2 \triangleright e_2 = e_4, & e_3 \triangleright e_3 = e_4, & \alpha \in \mathbb{C}, \\ e_1 \triangleleft e_1 = -e_4, & e_1 \triangleleft e_2 = (1 - \alpha)e_4, & e_2 \triangleleft e_1 = (\alpha - 1)e_4, & e_2 \triangleleft e_2 = -e_4. \end{cases}$$

Proof. Considering identity (6) for the following triples

$$\{e_1, e_1, e_2\}, \quad \{e_1, e_2, e_1\}, \quad \{e_3, e_3, e_2\}, \quad \{e_2, e_3, e_3\}, \quad \{e_1, e_2, e_3\}, \quad \{e_3, e_1, e_2\}, \quad \{e_3, e_3, e_4\},$$

we obtain $e_1 \triangleleft e_4 = 0$, $e_4 \triangleright e_1 = 0$, $e_4 \triangleright e_2 = 0$, $e_2 \triangleleft e_4 = 0$, $e_4 \triangleright e_3 = 0$, $e_3 \triangleleft e_4 = 0$, $e_4 \triangleright e_4 = 0$, respectively.

By $e_4 e_i = e_4 \triangleright e_i + e_4 \triangleleft e_i = 0$ and $e_i e_4 = e_i \triangleright e_4 + e_i \triangleleft e_4 = 0$ these imply $e_4 \triangleleft e_i = 0$ and $e_i \triangleright e_4 = 0$. Then it is easy to see that $\langle e_4 \rangle$ is the annihilator of the algebras $(As_4^6, \cdot)$ and $(As_4^6, \triangleright, \triangleleft)$. If we take the algebra $As_4^6 / \langle e_4 \rangle$, this algebra is a three-dimensional associative Abelian algebra, and we know that any three-dimensional complex anti-dendriform algebra associated with the Abelian algebra is isomorphic to one of the pairwise non-isomorphic algebras: $AD_3^3 - AD_3^7(\lambda)$.

Case $(As_4^6 / \langle e_4 \rangle, \triangleright, \triangleleft) \cong AD_3^3$

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11} e_4, \\ e_1 \triangleright e_2 = (1 + \alpha_{12}) e_4, \\ e_1 \triangleright e_3 = \alpha_{13} e_4, \\ e_2 \triangleright e_1 = (-1 + \alpha_{21}) e_4, \\ e_2 \triangleright e_2 = \alpha_{22} e_4, \\ e_2 \triangleright e_3 = \alpha_{23} e_4, \\ e_3 \triangleright e_1 = \alpha_{31} e_4, \\ e_3 \triangleright e_2 = \alpha_{32} e_4, \\ e_3 \triangleright e_3 = (1 + \alpha_{33}) e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11} e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12} e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13} e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21} e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22} e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23} e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31} e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32} e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33} e_4. \end{cases}$$

Similarly, it is not difficult to see that $(As_4^6, \triangleright, \triangleleft)$ are 2-nilpotent and have three generator elements. Thus we have $(x \triangleright y) \triangleright z = x \triangleright (y \triangleright z) = 0$ which implies that $(As_4^6, \triangleright)$ is also associative 2-nilpotent algebra and has three generator elements. According to Theorem 2.8 there are 3 non-isomorphic four-dimensional indecomposable associative 2-nilpotent and the three generated algebras. Hence, we derive the algebras $AD_4^{14} - AD_4^{18}(\alpha)$.

It is possible to consider the multiplication $\triangleleft$ as above. However, it is not difficult to show that the constructed algebras are isomorphic.

Case $(As_4^6 / \langle e_4 \rangle, \triangleright, \triangleleft) \cong AD_3^4$

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11} e_4, \\ e_1 \triangleright e_2 = e_3 + (1 + \alpha_{12}) e_4, \\ e_1 \triangleright e_3 = \alpha_{13} e_4, \\ e_2 \triangleright e_1 = -e_3 + (-1 + \alpha_{21}) e_4, \\ e_2 \triangleright e_2 = \alpha_{22} e_4, \\ e_2 \triangleright e_3 = \alpha_{23} e_4, \\ e_3 \triangleright e_1 = \alpha_{31} e_4, \\ e_3 \triangleright e_2 = \alpha_{32} e_4, \\ e_3 \triangleright e_3 = (1 + \alpha_{33}) e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11} e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12} e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13} e_4, \\ e_2 \triangleleft e_1 = e_3 - \alpha_{21} e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22} e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23} e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31} e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32} e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33} e_4. \end{cases}$$

Considering identity (5) for the triples $\{e_2, e_1, e_3\}$ and $\{e_3, e_1, e_2\}$, we obtain the following restrictions:

$$e_2 \triangleright (e_1 \triangleright e_3) \stackrel{(5)}{=} (e_2 \triangleleft e_1) \triangleleft e_3 \Leftrightarrow \alpha_{13}(e_2 \triangleright e_4) = (e_3 - \alpha_{21} e_4) \triangleleft e_3 \Leftrightarrow \alpha_{33} = 0,$$

$$e_3 \triangleright (e_1 \triangleright e_2) \stackrel{(5)}{=} (e_3 \triangleleft e_1) \triangleleft e_2 \Leftrightarrow e_3 \triangleright (e_3 + (1 + \alpha_{12}) e_4) = -\alpha_{31} e_4 \triangleleft e_2 \Leftrightarrow \alpha_{33} = -1.$$

Thus, we have a contradiction. Hence, this case cannot occur.

Case $(As_4^6/\langle e_4 \rangle, \triangleright, \triangleleft) \cong AD_3^5$

$$\begin{cases} e_1 \triangleright e_1 = e_3 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = \alpha_{31}e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = (1 + \alpha_{33})e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (5) for the triples $\{e_1, e_1, e_3\}$ and $\{e_3, e_1, e_1\}$, we obtain the following restrictions on structure constants:

$$\alpha_{33} = 0, \quad \alpha_{33} = -1.$$

Thus, we have a contradiction. Hence, there is no anti-dendriform algebra associated to the algebra As_4^6 and satisfying the condition $(As_4^6/\langle e_4 \rangle, \triangleright, \triangleleft) \cong AD_3^5$.

Case $(As_4^6/\langle e_4 \rangle, \triangleright, \triangleleft) \cong AD_3^6$

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = e_3 + (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = \alpha_{31}e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = (1 + \alpha_{33})e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (5) for the triples $\{e_1, e_2, e_3\}$ and $\{e_3, e_1, e_2\}$, we obtain the restrictions on structure constants as follows:

$$e_1 \triangleright (e_2 \triangleright e_3) = (e_1 \triangleleft e_2) \triangleleft e_3 \Leftrightarrow \alpha_{23}(e_1 \triangleright e_4) = (-e_3 - \alpha_{12}e_4) \triangleleft e_3 \Leftrightarrow \alpha_{33} = 0,$$

$$e_3 \triangleright (e_1 \triangleright e_2) = (e_3 \triangleleft e_1) \triangleleft e_2 \Leftrightarrow e_3 \triangleright (e_3 + (1 + \alpha_{12})e_4) = -\alpha_{31}e_4 \triangleleft e_2 \Leftrightarrow \alpha_{33} = -1.$$

Thus, we have a contradiction. Hence, this case cannot occur.

Case $(As_4^6/\langle e_4 \rangle, \triangleright, \triangleleft) \cong AD_3^7(\lambda)$

$$\begin{cases} e_1 \triangleright e_1 = e_3 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = \lambda e_3 + (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = e_3 + \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = \alpha_{31}e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = (1 + \alpha_{33})e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\lambda e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -e_3 - \alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (5) for the triples $\{e_1, e_1, e_3\}$ and $\{e_3, e_1, e_1\}$, we obtain the following restrictions, respectively:

$$\alpha_{33} = 0, \quad \alpha_{33} = -1.$$

Thus, we have a contradiction. Hence, there is no a compatible anti-dendriform algebra structure on As_4^6 and satisfying the condition $(As_4^6/\langle e_4 \rangle, \triangleright, \triangleleft) \cong AD_3^7(\lambda)$. $\square$

Theorem 3.3. *Any four-dimensional anti-dendriform algebra over the field of complex numbers whose associated associative algebra is As_4^8 is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{aligned} AD_4^{19}(\alpha) : & \begin{cases} e_1 \triangleright e_1 = \frac{1}{2}e_3, & e_1 \triangleright e_3 = 2e_4, & e_2 \triangleright e_2 = (-1 + \alpha)e_4, & e_3 \triangleright e_1 = -e_4, \\ e_1 \triangleleft e_1 = \frac{1}{2}e_3, & e_1 \triangleleft e_3 = -e_4, & e_2 \triangleleft e_2 = -\alpha e_4, & e_3 \triangleleft e_1 = 2e_4; \end{cases} \\ AD_4^{20}(\alpha) : & \begin{cases} e_1 \triangleright e_1 = \frac{1}{2}e_3, & e_1 \triangleright e_3 = 2e_4, & e_2 \triangleright e_1 = e_4, & e_2 \triangleright e_2 = (-1 + \alpha)e_4, & e_3 \triangleright e_1 = -e_4, \\ e_1 \triangleleft e_1 = \frac{1}{2}e_3, & e_1 \triangleleft e_3 = -e_4, & e_2 \triangleleft e_1 = -e_4, & e_2 \triangleleft e_2 = -\alpha e_4, & e_3 \triangleleft e_1 = 2e_4, & \alpha \neq -1; \end{cases} \\ AD_4^{21}(\alpha, \beta) : & \begin{cases} e_1 \triangleright e_1 = \frac{1}{2}e_3 + e_4, & e_1 \triangleright e_3 = 2e_4, & e_2 \triangleright e_1 = \alpha e_4, \\ e_2 \triangleright e_2 = (-1 + \beta)e_4, & e_3 \triangleright e_1 = -e_4, \\ e_1 \triangleleft e_1 = \frac{1}{2}e_3 - e_4, & e_1 \triangleleft e_3 = -e_4, & e_2 \triangleleft e_1 = -\alpha e_4, \\ e_2 \triangleleft e_2 = -\beta e_4, & e_3 \triangleleft e_1 = 2e_4, & \beta \neq -1; \end{cases} \\ AD_4^{22} : & \begin{cases} e_1 \triangleright e_1 = \frac{1}{2}e_3, & e_1 \triangleright e_2 = e_4, & e_1 \triangleright e_3 = 2e_4, & e_2 \triangleright e_2 = -2e_4, & e_3 \triangleright e_1 = -e_4, \\ e_1 \triangleleft e_1 = \frac{1}{2}e_3, & e_1 \triangleleft e_2 = -e_4, & e_1 \triangleleft e_3 = -e_4, & e_2 \triangleleft e_2 = e_4, & e_3 \triangleleft e_1 = 2e_4; \end{cases} \\ AD_4^{23}(\alpha) : & \begin{cases} e_1 \triangleright e_1 = \frac{1}{2}e_3 + e_4, & e_1 \triangleright e_2 = \alpha e_4, & e_1 \triangleright e_3 = 2e_4, & e_2 \triangleright e_2 = -2e_4, & e_3 \triangleright e_1 = -e_4, \\ e_1 \triangleleft e_1 = \frac{1}{2}e_3 - e_4, & e_1 \triangleleft e_2 = -\alpha e_4, & e_1 \triangleleft e_3 = -e_4, & e_2 \triangleleft e_2 = e_4, & e_3 \triangleleft e_1 = 2e_4. \end{cases} \end{aligned}$$

Proof. Considering identity (6) for the following triples

$$\{e_1, e_2, e_2\}, \{e_2, e_2, e_1\}, \{e_1, e_3, e_2\}, \{e_2, e_1, e_3\}, \{e_2, e_2, e_3\}, \{e_3, e_2, e_2\}, \{e_2, e_2, e_4\},$$

we obtain $e_1 \triangleleft e_4 = 0$, $e_4 \triangleright e_1 = 0$, $e_4 \triangleright e_2 = 0$, $e_2 \triangleleft e_4 = 0$, $e_4 \triangleright e_3 = 0$, $e_3 \triangleleft e_4 = 0$, $e_4 \triangleright e_4 = 0$, respectively.

By $e_4 e_i = e_4 \triangleright e_i + e_4 \triangleleft e_i = 0$ and $e_i e_4 = e_i \triangleright e_4 + e_i \triangleleft e_4 = 0$ these imply $e_4 \triangleleft e_i = 0$ and $e_i \triangleright e_4 = 0$. Then it is easy to see that $\langle e_4 \rangle$ is the annihilator of the algebras $(As_4^8, \cdot)$ and $(As_4^8, \triangleright, \triangleleft)$. If we take the algebra $As_4^8/\langle e_4 \rangle$, this algebra is a three-dimensional associative algebra As_3^3 with multiplication $e_1 e_1 = e_3$, and we know that any three-dimensional complex anti-dendriform algebra associated with the algebra As_3^3 is isomorphic to one of the pairwise non-isomorphic algebras: $AD_3^{18}(\alpha) - AD_3^{23}$.

Case $(As_4^8, \triangleright, \triangleleft)/\langle e_4 \rangle \cong AD_3^{18}(\alpha)$

$$\begin{cases} e_1 \triangleright e_1 = \alpha e_3 + \alpha_{11} e_4, \\ e_1 \triangleright e_2 = \alpha_{12} e_4, \\ e_1 \triangleright e_3 = (1 + \alpha_{13}) e_4, \\ e_2 \triangleright e_1 = \alpha_{21} e_4, \\ e_2 \triangleright e_2 = (-1 + \alpha_{22}) e_4, \\ e_2 \triangleright e_3 = \alpha_{23} e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31}) e_4, \\ e_3 \triangleright e_2 = \alpha_{32} e_4, \\ e_3 \triangleright e_3 = \alpha_{33} e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = (1 - \alpha) e_3 - \alpha_{11} e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12} e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13} e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21} e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22} e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23} e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31} e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32} e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33} e_4. \end{cases}$$

Considering identity (3) for the triples $\{e_1, e_1, e_2\}$ and $\{e_1, e_1, e_3\}$, we obtain $\alpha_{32} = 0$ and $\alpha_{33} = 0$.

Considering identities (3) and (4) for the triple $\{e_2, e_1, e_1\}$, we obtain $\alpha_{23} = 0$.

Considering identities (2)-(5) for the triple $\{e_1, e_1, e_1\}$, we obtain the following system:

$$\begin{cases} (1 + \alpha_{13})(-1 + \alpha) - \alpha_{31}\alpha = 0, \\ (1 + \alpha_{31}) + (1 + \alpha_{13})\alpha = 0, \\ -\alpha_{13} + (1 + \alpha_{13})\alpha = 0, \\ -\alpha_{31}(-1 + \alpha) + (1 + \alpha_{13})\alpha = 0. \end{cases}$$

Then the system has only one solution i.e. $\alpha_{13} = 1, \alpha_{31} = -2, \alpha = \frac{1}{2}$.

Rewrite the multiplication table

$$\begin{cases} e_1 \triangleright e_1 = \frac{1}{2}e_3 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = \alpha_{12}e_4, \\ e_1 \triangleright e_3 = 2e_4, \\ e_2 \triangleright e_1 = \alpha_{21}e_4, \\ e_2 \triangleright e_2 = (-1 + \alpha_{22})e_4, \\ e_3 \triangleright e_1 = -e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = \frac{1}{2}e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_3 \triangleleft e_1 = 2e_4. \end{cases}$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = \sqrt[3]{a^2}e_1 + be_2 + ce_3 + de_4, \quad \varphi(e_2) = ae_2 + b\sqrt[3]{a}e_3 + ee_4, \quad \varphi(e_3) = \sqrt[3]{a^4}e_3 + (2c\sqrt[3]{a^2} - b^2)e_4, \quad \varphi(e_4) = a^2e_4,$$

where $a \neq 0$.

We express the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)\}$ via the basis elements $\{e_1, e_2, e_3, e_4\}$. By verifying all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha'_{11}, \alpha'_{12}, \alpha'_{21}, \alpha'_{22}\}$ and $\{\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}\}$:

$$\begin{aligned} \varphi(e_1) \triangleright \varphi(e_1) &= \frac{1}{2}\varphi(e_3) + \alpha'_{11}\varphi(e_4) \Rightarrow a^2\alpha'_{11} = \sqrt[3]{a^4}\alpha_{11} + b\sqrt[3]{a^2}(\alpha_{12} + \alpha_{21}) + b^2(\alpha_{22} - \frac{1}{2}), \\ \varphi(e_1) \triangleright \varphi(e_2) &= \alpha'_{12}\varphi(e_4) \Rightarrow a\alpha'_{12} = \sqrt[3]{a^2}\alpha_{12} + b(1 + \alpha_{22}), \\ \varphi(e_2) \triangleright \varphi(e_1) &= \alpha'_{21}\varphi(e_4) \Rightarrow a\alpha'_{21} = \sqrt[3]{a^2}\alpha_{21} + b(-2 + \alpha_{22}), \\ \varphi(e_2) \triangleright \varphi(e_2) &= (-1 + \alpha'_{22})\varphi(e_4) \Rightarrow \alpha'_{22} = \alpha_{22}. \end{aligned}$$

Thus, we have the following cases:

(a) Let $\alpha_{22} = 2$. Then by choosing b we obtain following restrictions:

$$a\alpha'_{11} = \sqrt[3]{a}\alpha_{11}, \quad \alpha'_{12} = 0, \quad a\alpha'_{21} = \sqrt[3]{a^2}\alpha_{21}, \quad \alpha'_{22} = 2.$$

– Let $\alpha_{11} = 0$. If $\alpha_{21} = 0$, then we obtain the algebras $AD_4^{19}(2)$. If $\alpha_{21} \neq 0$, then by putting $a = \sqrt[3]{\alpha_{21}}$ we obtain the algebra $AD_4^{20}(2)$.

– Let $\alpha_{11} \neq 0$. Then applying $a = \sqrt[3]{\alpha_{11}^2}$ we obtain the algebra $AD_4^{21}(\alpha, 2)$.

(b) Let $\alpha_{22} = -1$. Then by choosing b we obtain following restrictions:

$$a\alpha'_{11} = \sqrt[3]{a}\alpha_{11}, \quad a\alpha'_{12} = \sqrt[3]{a^2}\alpha_{12}, \quad \alpha'_{21} = 0, \quad \alpha'_{22} = -1.$$

– Let $\alpha_{11} = 0$. If $\alpha_{12} = 0$, then we obtain the algebras $AD_4^{19}(-1)$. If $\alpha_{12} \neq 0$, then by putting $a = \sqrt[3]{\alpha_{12}}$ we obtain the algebra AD_4^{22} .

– Let $\alpha_{11} \neq 0$. Then applying $a = \sqrt[3]{\alpha_{11}^2}$ we obtain the algebra $AD_4^{23}(\alpha)$.

(c) Let $\alpha_{22} \notin \{-1, 2\}$. Then by choosing b we obtain following restrictions:

$$a\alpha'_{11} = \sqrt[3]{a}\alpha_{11}, \alpha'_{12} = 0, a\alpha'_{21} = \sqrt[3]{a^2}\alpha_{12}, \alpha'_{22} = \alpha_{22}.$$

– Let $\alpha_{11} = 0$. If $\alpha_{21} = 0$, then we obtain the algebras $AD_4^{19}(\alpha)$, where $\alpha \notin \{-1, 2\}$. If $\alpha_{21} \neq 0$, then by putting $a = \sqrt[3]{\alpha_{21}}$ we obtain the algebra $AD_4^{20}(\alpha)$, where $\alpha \notin \{-1, 2\}$.

– Let $\alpha_{11} \neq 0$. Then applying $a = \sqrt[3]{\alpha_{11}^2}$ we obtain the algebra $AD_4^{21}(\alpha, \beta)$, where $\beta \notin \{-1, 2\}$.

Case $(As_4^8, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{19}$

$$\left\{ \begin{array}{l} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = \alpha_{12}e_4, \\ e_1 \triangleright e_3 = (1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = e_3 + \alpha_{21}e_4, \\ e_2 \triangleright e_2 = (-1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{array} \right. \quad \left\{ \begin{array}{l} e_1 \triangleleft e_1 = e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{array} \right.$$

Considering identity (4) for the triples $\{e_1, e_1, e_1\}$ and $\{e_1, e_2, e_1\}$, we obtain

$$\alpha_{13} = 0, \alpha_{13} = -1.$$

Thus, we have a contradiction. Hence, there is no anti-dendriform algebra associated to the algebra As_4^8 and satisfying the condition $(As_4^8, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{19}$.

Case $(As_4^8, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{20}(\alpha)$

$$\left\{ \begin{array}{l} e_1 \triangleright e_1 = \alpha e_3 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = e_3 + \alpha_{12}e_4, \\ e_1 \triangleright e_3 = (1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = -e_3 + \alpha_{21}e_4, \\ e_2 \triangleright e_2 = (-1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{array} \right. \quad \left\{ \begin{array}{l} e_1 \triangleleft e_1 = (1 - \alpha)e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{array} \right.$$

Considering identity (4) for the triples $\{e_1, e_1, e_1\}$ and $\{e_1, e_1, e_2\}$, we obtain the following restrictions:

$$\alpha(1 + \alpha_{13}) = -\alpha_{13}, \alpha_{13} = -1$$

Thus, we have a contradiction. Hence, this case cannot occur.

Case $(As_4^8, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{21}(\alpha)$

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = e_3 + \alpha_{12}e_4, \\ e_1 \triangleright e_3 = (1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = \alpha e_3 + \alpha_{21}e_4, \\ e_2 \triangleright e_2 = (-1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

where $\alpha \neq -1$.

Considering identity (4) for the triples $\{e_1, e_1, e_1\}$ and $\{e_1, e_1, e_2\}$, we obtain the restrictions on structure constants as follows:

$$e_1 \triangleright (e_1 \triangleright e_1) = -e_1 \triangleleft (e_1 \cdot e_1) \Leftrightarrow \alpha_{11}e_1 \triangleright e_4 = e_1 \triangleleft e_3 \Leftrightarrow \alpha_{13} = 0,$$

$$e_1 \triangleright (e_1 \triangleright e_2) = -e_1 \triangleleft (e_1 \cdot e_2) \Leftrightarrow e_1 \triangleright (e_3 + \alpha_{12}e_4) = 0 \Leftrightarrow \alpha_{13} = -1.$$

Thus, we have a contradiction. Hence, this case cannot occur.

Case $(As_4^8, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{22}(\alpha, \beta)$

$$\begin{cases} e_1 \triangleright e_1 = \alpha e_3 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = \alpha_{12}e_4, \\ e_1 \triangleright e_3 = (1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = \beta e_3 + \alpha_{21}e_4, \\ e_2 \triangleright e_2 = e_3 + (-1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = (1 - \alpha)e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\beta e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -e_3 - \alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (4) for the triples $\{e_1, e_1, e_1\}$ and $\{e_1, e_2, e_2\}$, we obtain the following restrictions on structure constants:

$$\alpha(1 + \alpha_{13}) = -\alpha_{13}, \quad \alpha_{13} = -1.$$

Thus, we have a contradiction. Hence, it means that there is no anti-dendriform algebra associated to the algebra As_4^8 and satisfying the condition $(As_4^8, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{22}(\alpha, \beta)$.

Case $(As_4^8, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{23}$

$$\begin{cases} e_1 \triangleright e_1 = e_2 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = \alpha_{12}e_4, \\ e_1 \triangleright e_3 = (1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = \alpha_{21}e_4, \\ e_2 \triangleright e_2 = (-1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_2 + e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering the some equations for the triples $\{e_1, e_1, e_2\}$, $\{e_2, e_1, e_1\}$ and $\{e_1, e_1, e_2\}$ we obtain the following restrictions:

$$e_1 \triangleright (e_1 \triangleright e_2) \stackrel{(3)}{=} -(e_1 \cdot e_1) \triangleright e_2 \Leftrightarrow \alpha_{12}e_1 \triangleright e_4 = -e_3 \triangleright e_2 \Leftrightarrow \alpha_{32} = 0,$$

$$e_2 \triangleright (e_1 \triangleright e_1) \stackrel{(3)}{=} -(e_2 \cdot e_1) \triangleright e_1 \Leftrightarrow e_2 \triangleright (e_2 + \alpha_{11}e_4) = 0 \Leftrightarrow \alpha_{22} = 1,$$

$$e_1 \triangleright (e_1 \triangleright e_2) \stackrel{(5)}{=} -(e_1 \triangleleft e_1) \triangleleft e_2 \Leftrightarrow \alpha_{12}e_1 \triangleright e_4 = (-e_2 + e_3 - \alpha_{11}e_4) \triangleleft e_2 \Leftrightarrow \alpha_{22} = \alpha_{32}.$$

Thus, we have a contradiction. Hence, this case cannot occur. $\square$

Theorem 3.4. *Any four-dimensional anti-dendriform algebra over the field of complex numbers whose associated associative algebra is As_4^9 is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{aligned} AD_4^{24}(\alpha, \beta) : & \begin{cases} e_1 \triangleright e_1 = \frac{1}{2}e_3, & e_1 \triangleright e_2 = \alpha e_4, & e_1 \triangleright e_3 = -2e_4, & e_2 \triangleright e_1 = (1 + \beta)e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = \frac{1}{2}e_3, & e_1 \triangleleft e_2 = -\alpha e_4, & e_1 \triangleleft e_3 = e_4, & e_2 \triangleleft e_1 = -\beta e_4, & e_3 \triangleleft e_1 = -2e_4; \end{cases} \\ AD_4^{25}(\alpha, \beta) : & \begin{cases} e_1 \triangleright e_1 = \frac{1}{2}e_3, & e_1 \triangleright e_2 = \alpha e_4, & e_1 \triangleright e_3 = -2e_4, \\ e_2 \triangleright e_1 = (1 + \beta)e_4, & e_2 \triangleright e_2 = e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = \frac{1}{2}e_3, & e_1 \triangleleft e_2 = -\alpha e_4, & e_1 \triangleleft e_3 = e_4, \\ e_2 \triangleleft e_1 = -\beta e_4, & e_2 \triangleleft e_2 = -e_4, & e_3 \triangleleft e_1 = -2e_4; \end{cases} \\ AD_4^{26}(\alpha) : & \begin{cases} e_1 \triangleright e_1 = e_2, & e_1 \triangleright e_2 = \alpha e_4, & e_1 \triangleright e_3 = (-1 + \alpha)e_4, & e_2 \triangleright e_1 = 2e_4, & e_3 \triangleright e_1 = -\alpha e_4, \\ e_1 \triangleleft e_1 = -e_2 + e_3, & e_1 \triangleleft e_2 = -\alpha e_4, & e_1 \triangleleft e_3 = -\alpha e_4, \\ e_2 \triangleleft e_1 = -e_4, & e_3 \triangleleft e_1 = (-1 + \alpha)e_4. \end{cases} \end{aligned}$$

Proof. Considering identity (6) for the following triples

$$\{e_1, e_2, e_1\}, \{e_1, e_3, e_1\}, \{e_2, e_1, e_2\}, \{e_2, e_3, e_1\}, \{e_4, e_3, e_1\}, \{e_1, e_1, e_4\}, \{e_4, e_1, e_1\},$$

we obtain $e_1 \triangleleft e_4 = 0$, $e_4 \triangleright e_1 = 0$, $e_4 \triangleright e_2 = 0$, $e_2 \triangleleft e_4 = 0$, $e_4 \triangleleft e_4 = 0$, $e_3 \triangleright e_4 = 0$, $e_4 \triangleleft e_3 = 0$, respectively.

By $e_4e_i = e_4 \triangleright e_i + e_4 \triangleleft e_i = 0$ and $e_i e_4 = e_i \triangleright e_4 + e_i \triangleleft e_4 = 0$ these imply $e_4 \triangleleft e_i = 0$ and $e_i \triangleright e_4 = 0$. Then it is easy to see that $\langle e_4 \rangle$ is the annihilator of the algebras $(As_4^9, \cdot)$ and $(As_4^9, \triangleright, \triangleleft)$. If we take the algebra $As_4^9 / \langle e_4 \rangle$, this algebra is a three-dimensional associative algebra As_3^3 with multiplication $e_1e_1 = e_3$, and we know that any three-dimensional complex anti-dendriform algebra associated with the algebra As_3^3 is isomorphic to one of the pairwise non-isomorphic algebras $AD_3^{18}(\alpha) - AD_3^{23}$.

Case $(As_4^9, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{18}(\alpha)$

$$\begin{cases} e_1 \triangleright e_1 = \alpha e_3 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = \alpha_{12}e_4, \\ e_1 \triangleright e_3 = (-1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = (1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (-1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = (1 - \alpha)e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identities (3) and (4) for the triple $\{e_1, e_1, e_1\}$, we obtain the following system of equations:

$$\begin{cases} -1 + \alpha_{31} + (-1 + \alpha_{13})\alpha = 0, \\ -\alpha_{13} + (-1 + \alpha_{13})\alpha = 0. \end{cases}$$

From this system, we obtain that $\alpha_{31} = 1 - \alpha_{13}$.

Similarly, considering identity (5) for the triple $\{e_1, e_1, e_1\}$, we obtain the relation $(\alpha_{13} - 1)(2\alpha - 1) = 0$. So we have the following system of equations:

$$\begin{cases} (\alpha_{13} - 1)(2\alpha - 1) = 0, \\ \alpha_{13}(\alpha - 1) - \alpha = 0. \end{cases}$$

If we assume $\alpha \neq \frac{1}{2}$, then from the first equation of the system we obtain $\alpha_{13} = 1$, and if we put this value into the second equation of the system, we obtain a contradiction $-1 = 0$. So $\alpha = \frac{1}{2}$. From the above system, we find $\alpha_{13} = -1$ and $\alpha_{31} = 2$.

Considering identity (3) for the triples $\{e_1, e_1, e_2\}$, $\{e_1, e_1, e_3\}$ and $\{e_2, e_1, e_1\}$, we obtain

$$\alpha_{23} = \alpha_{32} = \alpha_{33} = 0.$$

So, the multiplication table of the algebra has the following form:

$$\begin{cases} e_1 \triangleright e_1 = \frac{1}{2}e_3 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = \alpha_{12}e_4, \\ e_1 \triangleright e_3 = -2e_4, \\ e_2 \triangleright e_1 = (1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_3 \triangleright e_1 = e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = \frac{1}{2}e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_3 \triangleleft e_1 = -2e_4. \end{cases}$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = ae_1 + be_2 + ce_3 + ee_4, \quad \varphi(e_2) = a^2e_2 + de_4, \quad \varphi(e_3) = a^2e_3 + a(b - 2c)e_4, \quad \varphi(e_4) = a^3e_4,$$

where $a \neq 0$.

We express the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)\}$ via the basis elements $\{e_1, e_2, e_3, e_4\}$. By verifying all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha'_{11}, \alpha'_{12}, \alpha'_{21}, \alpha'_{22}\}$ and $\{\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}\}$:

$$\begin{aligned} \varphi(e_1) \triangleright \varphi(e_1) &= \frac{1}{2}\varphi(e_3) + \alpha'_{11}\varphi(e_4) \Rightarrow a^3\alpha'_{11} = a^2\alpha_{11} + ab(\frac{1}{2} + \alpha_{12} + \alpha_{21}) + b^2\alpha_{22}, \\ \varphi(e_1) \triangleright \varphi(e_2) &= \alpha'_{12}\varphi(e_4) \Rightarrow a\alpha'_{12} = a\alpha_{12} + b\alpha_{22}, \\ \varphi(e_2) \triangleright \varphi(e_1) &= (1 + \alpha'_{21})\varphi(e_4) \Rightarrow a\alpha'_{21} = a\alpha_{21} + b\alpha_{22}, \\ \varphi(e_2) \triangleright \varphi(e_2) &= \alpha'_{22}\varphi(e_4) \Rightarrow \alpha'_{22} = a\alpha_{22}. \end{aligned}$$

We have the following cases:

- Let $\alpha_{22} = 0$. Then if $\alpha_{11} \neq 0$, with choosing $a = -\frac{b(1+2\alpha_{12}+2\alpha_{21})}{2\alpha_{11}}$ and if $\alpha_{11} = 0$ with choosing $b = 0$, we can conclude $\alpha'_{11} = 0$. So, we obtain the algebra $AD_4^{24}(\alpha, \beta)$.
- Let $\alpha_{22} \neq 0$. Choosing $a = \frac{1}{\alpha_{22}}$ and the suitable value of b we derive $\alpha'_{11} = 0$ and $\alpha'_{22} = 0$, obtain the algebra $AD_4^{25}(\alpha, \beta)$.

Now if $(As_4^9, \triangleright, \triangleleft) / \langle e_4 \rangle$ anti-dendriform algebra is isomorphic to one of the algebras AD_3^{19} , $AD_3^{20}(\alpha)$, $AD_3^{21}(\alpha)$ and $AD_3^{22}(\alpha, \beta)$, then we will proof there is not a four-dimensional complex anti-dendriform algebra associated to the associative algebra As_4^9 .

Case $(As_4^9, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{19}$

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = \alpha_{12}e_4, \\ e_1 \triangleright e_3 = (-1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = e_3 + (1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (-1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (4) for the triples $\{e_1, e_1, e_1\}$ and $\{e_1, e_2, e_1\}$, we obtain $\alpha_{13} = 0$ and $\alpha_{13} = 1$; a contradiction. Therefore, this case cannot occur.

Case $(As_4^9, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{20}(\alpha)$

$$\begin{cases} e_1 \triangleright e_1 = \alpha e_3 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = e_3 + \alpha_{12}e_4, \\ e_1 \triangleright e_3 = (-1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = -e_3 + (1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (-1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = (1 - \alpha)e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (4) for the triples $\{e_1, e_1, e_2\}$ and $\{e_1, e_1, e_1\}$, we obtain $\alpha_{13} = 1$ and $\alpha_{13} = 0$, which yields a contradiction. Hence, this case cannot occur.

Case $(As_4^9, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{21}(\alpha)$

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = e_3 + \alpha_{12}e_4, \\ e_1 \triangleright e_3 = (-1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = \alpha e_3 + (1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (-1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

where $\alpha \neq -1$.

Considering identity (4) for the triples $\{e_1, e_1, e_1\}$ and $\{e_1, e_1, e_2\}$, we obtain the following restrictions, respectively:

$$\alpha_{13} = 0, \quad \alpha_{13} = 1.$$

Hence, we have a contradiction. Hence, there is no a compatible anti-dendriform algebra structure on As_4^9 .

Case $(As_4^9, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{22}(\alpha, \beta)$

$$\left\{ \begin{array}{l} e_1 \triangleright e_1 = \alpha e_3 + \alpha_{11} e_4, \\ e_1 \triangleright e_2 = \alpha_{12} e_4, \\ e_1 \triangleright e_3 = (-1 + \alpha_{13}) e_4, \\ e_2 \triangleright e_1 = \beta e_3 + (1 + \alpha_{21}) e_4, \\ e_2 \triangleright e_2 = e_3 + \alpha_{22} e_4, \\ e_2 \triangleright e_3 = \alpha_{23} e_4, \\ e_3 \triangleright e_1 = (-1 + \alpha_{31}) e_4, \\ e_3 \triangleright e_2 = \alpha_{32} e_4, \\ e_3 \triangleright e_3 = \alpha_{33} e_4, \end{array} \right. \quad \left\{ \begin{array}{l} e_1 \triangleleft e_1 = (1 - \alpha) e_3 - \alpha_{11} e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12} e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13} e_4, \\ e_2 \triangleleft e_1 = -\beta e_3 - \alpha_{21} e_4, \\ e_2 \triangleleft e_2 = -e_3 - \alpha_{22} e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23} e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31} e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32} e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33} e_4. \end{array} \right.$$

Similarly, considering identity (4) for the triples $\{e_1, e_2, e_2\}$ and $\{e_1, e_1, e_1\}$, we conclude $\alpha_{13} = 1$ and $\alpha_{13} = 0$, which yields a contradiction. Hence, this case cannot occur.

Case $(As_4^9, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{23}$

$$\left\{ \begin{array}{l} e_1 \triangleright e_1 = e_2 + \alpha_{11} e_4, \\ e_1 \triangleright e_2 = \alpha_{12} e_4, \\ e_1 \triangleright e_3 = (-1 + \alpha_{13}) e_4, \\ e_2 \triangleright e_1 = (1 + \alpha_{21}) e_4, \\ e_2 \triangleright e_2 = \alpha_{22} e_4, \\ e_2 \triangleright e_3 = \alpha_{23} e_4, \\ e_3 \triangleright e_1 = (-1 + \alpha_{31}) e_4, \\ e_3 \triangleright e_2 = \alpha_{32} e_4, \\ e_3 \triangleright e_3 = \alpha_{33} e_4, \end{array} \right. \quad \left\{ \begin{array}{l} e_1 \triangleleft e_1 = -e_2 + e_3 - \alpha_{11} e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12} e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13} e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21} e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22} e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23} e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31} e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32} e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33} e_4. \end{array} \right.$$

Considering the same identity, we obtain the following restrictions on structure constants:

Triple	Identity	restrictions
$\{e_1, e_1, e_2\}$	(3)	$\alpha_{32} = 0,$
$\{e_1, e_1, e_3\}$	(3)	$\alpha_{33} = 0,$
$\{e_2, e_1, e_1\}$	(3)	$\alpha_{22} = 0,$
$\{e_1, e_1, e_1\}$	(4)	$\alpha_{13} = \alpha_{12},$
$\{e_2, e_1, e_1\}$	(4)	$\alpha_{23} = 0,$
$\{e_1, e_1, e_1\}$	(2)	$\alpha_{21} = 1,$
$\{e_1, e_1, e_1\}$	(5)	$\alpha_{31} = 1 - \alpha_{12}.$

By the basis change $e'_2 = e_2 + \alpha_{11} e_4$ we can put $\alpha_{11} = 0$. Thus, the table of multiplications of the algebra has the form:

$$\left\{ \begin{array}{l} e_1 \triangleright e_1 = e_2, \quad e_1 \triangleright e_2 = \alpha_{12} e_4, \quad e_1 \triangleright e_3 = (-1 + \alpha_{12}) e_4, \quad e_2 \triangleright e_1 = 2e_4, \quad e_3 \triangleright e_1 = -\alpha_{12} e_4, \\ e_1 \triangleleft e_1 = -e_2 + e_3, \quad e_1 \triangleleft e_2 = -\alpha_{12} e_4, \quad e_1 \triangleleft e_3 = -\alpha_{12} e_4, \quad e_2 \triangleleft e_1 = -e_4, \quad e_3 \triangleleft e_1 = (-1 + \alpha_{12}) e_4. \end{array} \right.$$

By using the general change in the algebra, we can show that $\alpha'_{12} = \alpha_{12}$. Hence, we obtain the algebra $AD_4^{26}(\alpha)$. $\square$

Theorem 3.5. *Any four-dimensional anti-dendriform algebra over the field of complex numbers whose associated associative algebra is As_4^{10} is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$AD_4^{27} : e_1 \triangleright e_2 = e_4, e_3 \triangleright e_1 = e_4, e_1 \triangleleft e_1 = e_4, e_2 \triangleleft e_1 = -e_4, e_3 \triangleleft e_1 = -e_4, e_3 \triangleleft e_3 = e_4;$$

$$AD_4^{28} : e_1 \triangleright e_2 = e_4, e_3 \triangleright e_3 = e_4, e_2 \triangleright e_1 = -e_4, e_1 \triangleleft e_1 = e_4;$$

$$AD_4^{29} : e_1 \triangleright e_1 = e_4, e_1 \triangleright e_2 = e_4, e_2 \triangleright e_1 = -e_4, e_3 \triangleright e_3 = e_4;$$

$$AD_4^{30} : \begin{cases} e_1 \triangleright e_2 = e_4, e_1 \triangleright e_3 = e_4, e_2 \triangleright e_1 = -e_4, e_2 \triangleright e_2 = e_4, e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = e_4, e_1 \triangleleft e_3 = -e_4, e_2 \triangleleft e_2 = -e_4, e_3 \triangleleft e_1 = -e_4, e_3 \triangleleft e_3 = e_4; \end{cases}$$

$$AD_4^{31}(\alpha) : \begin{cases} e_1 \triangleright e_1 = e_4, e_1 \triangleright e_2 = \alpha e_4, e_2 \triangleright e_1 = -\alpha e_4, e_2 \triangleright e_2 = e_4, e_3 \triangleright e_3 = e_4, \\ e_1 \triangleleft e_2 = (1 - \alpha)e_4, e_2 \triangleleft e_1 = (\alpha - 1)e_4, e_2 \triangleleft e_2 = -e_4. \end{cases}$$

Proof. Considering identity (6) for the following triples

$$\{e_1, e_3, e_3\}, \{e_3, e_3, e_1\}, \{e_3, e_3, e_2\}, \{e_2, e_3, e_3\}, \{e_1, e_1, e_3\}, \{e_3, e_1, e_1\}, \{e_3, e_3, e_4\},$$

we obtain $e_1 \triangleleft e_4 = 0$, $e_4 \triangleright e_1 = 0$, $e_4 \triangleright e_2 = 0$, $e_2 \triangleleft e_4 = 0$, $e_4 \triangleright e_3 = 0$, $e_3 \triangleleft e_4 = 0$, $e_4 \triangleright e_4 = 0$, respectively.

By $e_4 e_i = e_4 \triangleright e_i + e_4 \triangleleft e_i = 0$ and $e_i e_4 = e_i \triangleright e_4 + e_i \triangleleft e_4 = 0$ these imply $e_4 \triangleleft e_i = 0$ and $e_i \triangleright e_4 = 0$. Then it is easy to see that $\langle e_4 \rangle$ is the annihilator of the algebras $(As_4^{10}, \cdot)$ and $(As_4^{10}, \triangleright, \triangleleft)$. If we take the algebra $As_4^{10}/\langle e_4 \rangle$, this algebra is a three-dimensional associative Abelian algebra, and we know that any three-dimensional complex anti-dendriform algebra associated with the Abelian algebra is isomorphic to one of the pairwise non-isomorphic algebras $AD_3^3 - AD_3^7(\lambda)$.

Case $(As_4^{10}, \triangleright, \triangleleft)/\langle e_4 \rangle \cong AD_3^3$

$$\begin{cases} e_1 \triangleright e_1 = (1 + \alpha_{11})e_4, \\ e_1 \triangleright e_2 = (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = \alpha_{31}e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = (1 + \alpha_{33})e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

It is not difficult to see that $(As_4^{10}, \triangleright, \triangleleft)$ are 2-nilpotent and have three generator elements. Thus we have $(x \triangleright y) \triangleright z = x \triangleright (y \triangleright z) = 0$ which implies that $(As_4^{10}, \triangleright)$ is also associative 2-nilpotent algebra and has three generator elements. According to Theorem 2.8 there are five non-isomorphic four-dimensional indecomposable associative 2-nilpotent and the three generated algebras. Hence, we obtain $AD_4^{27} - AD_4^{31}(\alpha)$ algebras.

It is possible to consider the multiplication $\triangleleft$ as above. However, it is not difficult to show that the constructed algebras are isomorphic.

Now if $(As_4^{10}, \triangleright, \triangleleft)/\langle e_4 \rangle$ anti-dendriform algebra is isomorphic to one of the algebras $AD_3^4 - AD_3^7(\alpha)$, then we will proof there is not a four-dimensional complex anti-dendriform algebra associated to the associative algebra As_4^{10} .

Case $(As_4^{10}, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^4$

$$\begin{cases} e_1 \triangleright e_1 = (1 + \alpha_{11})e_4, \\ e_1 \triangleright e_2 = e_3 + (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = -e_3 + (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = \alpha_{31}e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = (1 + \alpha_{33})e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (5) for the triples $\{e_1, e_2, e_3\}$ and $\{e_3, e_1, e_2\}$, we obtain $\alpha_{33} = 0$ and $\alpha_{33} = -1$, which yields a contradiction. Hence, this case cannot occur.

Case $(As_4^{10}, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^5$

$$\begin{cases} e_1 \triangleright e_1 = e_3 + (1 + \alpha_{11})e_4, \\ e_1 \triangleright e_2 = (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = \alpha_{31}e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = (1 + \alpha_{33})e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (5) for the triples $\{e_1, e_1, e_3\}$ and $\{e_3, e_1, e_1\}$, we obtain the following restrictions on structure constants:

$$\alpha_{33} = 0, \quad \alpha_{33} = -1,$$

which yields a contradiction. Hence, this case cannot occur.

Case $(As_4^{10}, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^6$

$$\begin{cases} e_1 \triangleright e_1 = (1 + \alpha_{11})e_4, \\ e_1 \triangleright e_2 = e_3 + (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = \alpha_{31}e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = (1 + \alpha_{33})e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (5) for the triples $\{e_1, e_2, e_3\}$ and $\{e_3, e_1, e_2\}$, we obtain the following restrictions, respectively:

$$\alpha_{33} = 0, \quad \alpha_{33} = -1.$$

Thus, we have a contradiction. Hence, there is no a compatible anti-dendriform algebra structure on As_4^{10} .

Case $(As_4^{10}, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^7(\lambda)$

$$\begin{cases} e_1 \triangleright e_1 = e_3 + (1 + \alpha_{11})e_4, \\ e_1 \triangleright e_2 = \lambda e_3 + (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = e_3 + \alpha_{22}e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = \alpha_{31}e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = (1 + \alpha_{33})e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\lambda e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -e_3 - \alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (5) for the triples $\{e_1, e_1, e_3\}$ and $\{e_3, e_1, e_1\}$, we obtain $\alpha_{33} = 0$ and $\alpha_{33} = -1$; a contradiction. Therefore, there this case cannot occur. $\square$

Theorem 3.6. Any four-dimensional anti-dendriform algebra over the field of complex numbers whose associated associative algebra is As_4^{13} is isomorphic to one of the following pairwise non-isomorphic algebras:

$$AD_4^{32}(\alpha, \beta, \gamma) : \begin{cases} e_1 \triangleright e_1 = \frac{1}{2}e_3 + \alpha e_4, & e_1 \triangleleft e_1 = \frac{1}{2}e_3 - \alpha e_4, \\ e_1 \triangleright e_2 = \beta e_4, & e_1 \triangleleft e_2 = -\beta e_4, \\ e_1 \triangleright e_3 = -2e_4, & e_1 \triangleleft e_3 = e_4, \\ e_2 \triangleright e_1 = (1 + \beta)e_4, & e_2 \triangleleft e_1 = -\beta e_4, \\ e_2 \triangleright e_2 = (1 + \gamma)e_4, & e_2 \triangleleft e_2 = -\gamma e_4, \\ e_3 \triangleright e_1 = e_4, & e_3 \triangleleft e_1 = -2e_4. \end{cases}$$

Proof. Considering identity (6) for the following triples

$$\{e_1, e_2, e_2\}, \{e_1, e_2, e_1\}, \{e_1, e_3, e_2\}, \{e_2, e_3, e_1\}, \{e_1, e_2, e_4\}, \{e_4, e_1, e_2\}, \{e_1, e_1, e_4\},$$

we obtain $e_1 \triangleleft e_4 = 0$, $e_4 \triangleright e_1 = 0$, $e_4 \triangleright e_2 = 0$, $e_2 \triangleleft e_4 = 0$, $e_4 \triangleright e_4 = 0$, $e_4 \triangleleft e_3 = 0$, $e_3 \triangleright e_4 = 0$, respectively.

By $e_4 e_i = e_4 \triangleright e_i + e_4 \triangleleft e_i = 0$ and $e_i e_4 = e_i \triangleright e_4 + e_i \triangleleft e_4 = 0$ these imply $e_4 \triangleleft e_i = 0$ and $e_i \triangleright e_4 = 0$. Then it is easy to see that $\langle e_4 \rangle$ is the annihilator of the algebras $(As_4^{13}, \cdot)$ and $(As_4^{13}, \triangleright, \triangleleft)$. If we take the algebra $As_4^{13} / \langle e_4 \rangle$, this algebra is a three-dimensional associative algebra As_3^3 with multiplication $e_1 e_1 = e_3$, and we know that any three-dimensional complex anti-dendriform algebra associated with the algebra As_3^3 is isomorphic to one of the pairwise non-isomorphic algebras $AD_3^{18}(\alpha) - AD_3^{23}$.

Case $(As_4^{13}, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{18}(\alpha)$

$$\begin{cases} e_1 \triangleright e_1 = \alpha e_3 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = \alpha_{12}e_4, \\ e_1 \triangleright e_3 = (-1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = (1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (-1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = (1 - \alpha)e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identities (2)-(5) for the triple $\{e_1, e_1, e_1\}$, we obtain the following system of equations:

$$\begin{cases} (-1 + \alpha_{13})(-1 + \alpha) - \alpha_{31}\alpha = 0, \\ (-1 + \alpha_{31}) + (-1 + \alpha_{13})\alpha = 0, \\ -\alpha_{13} + (-1 + \alpha_{13})\alpha = 0, \\ -\alpha_{31}(-1 + \alpha) + (-1 + \alpha_{13})\alpha = 0. \end{cases}$$

When $\alpha \neq \frac{1}{2}$, we obtain a contradiction. Hence, we can assume $\alpha = \frac{1}{2}$. Then the system has only one solution $\alpha = \frac{1}{2}$, $\alpha_{13} = -1$, $\alpha_{31} = 2$.

Considering identity (3) for the triples $\{e_1, e_1, e_2\}$, $\{e_1, e_1, e_3\}$ and $\{e_2, e_1, e_1\}$, we obtain the restrictions

$$\alpha_{32} = \alpha_{33} = \alpha_{23} = 0.$$

Thus, the multiplication table of the algebra has the following form:

$$\begin{cases} e_1 \triangleright e_1 = \frac{1}{2}e_3 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = \alpha_{12}e_4, \\ e_1 \triangleright e_3 = -2e_4, \\ e_2 \triangleright e_1 = (1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_3 \triangleright e_1 = e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = \frac{1}{2}e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_3 \triangleleft e_1 = -2e_4. \end{cases}$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = e_1 + ae_2 + be_3 + ce_4, \quad \varphi(e_2) = e_2 + ae_3 + de_4, \quad \varphi(e_3) = e_3 + (a^2 + a - 2b)e_4, \quad \varphi(e_4) = e_4.$$

We express the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)\}$ via the basis elements $\{e_1, e_2, e_3, e_4\}$. By verifying all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha'_{11}, \alpha'_{12}, \alpha'_{21}, \alpha'_{22}\}$ and $\{\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}\}$:

$$\begin{aligned} \varphi(e_1) \triangleright \varphi(e_1) &= \frac{1}{2}\varphi(e_3) + \alpha'_{11}\varphi(e_4) \Rightarrow \alpha'_{11} = \alpha_{11} + a(\frac{1}{2} + \alpha_{12} + \alpha_{21}) + a^2(\frac{1}{2} + \alpha_{22}), \\ \varphi(e_1) \triangleright \varphi(e_2) &= \alpha'_{12}\varphi(e_4) \Rightarrow \alpha'_{12} = \alpha_{12} + a(\alpha_{22} - 1), \\ \varphi(e_2) \triangleright \varphi(e_1) &= (1 + \alpha'_{21})\varphi(e_4) \Rightarrow \alpha'_{21} = \alpha_{21} + a(\alpha_{22} + 2), \\ \varphi(e_2) \triangleright \varphi(e_2) &= (1 + \alpha'_{22})\varphi(e_4) \Rightarrow \alpha'_{22} = \alpha_{22}. \end{aligned}$$

If choosing $a = \frac{\alpha_{12} - \alpha_{21}}{3}$, then we can reduce $\alpha'_{12} = \alpha'_{21}$ we obtain the algebra $AD_4^{32}(\alpha, \beta, \gamma)$.

Now, if the anti-dendriform algebra $(As_4^{13}, \triangleright, \triangleleft) / \langle e_4 \rangle$ is isomorphic to one of the algebras $AD_3^{19} - AD_3^{23}$, then we proof that there is no a four-dimensional complex anti-dendriform algebra associated to the associative algebra As_4^{13} .

Case $(As_4^{13}, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{19}$

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = \alpha_{12}e_4, \\ e_1 \triangleright e_3 = (-1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = e_3 + (1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (-1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (4) for the triples $\{e_1, e_1, e_1\}$ and $\{e_1, e_2, e_1\}$, we obtain the following restrictions, respectively:

$$\alpha_{13} = 0, \quad \alpha_{13} = 1.$$

Thus, we have a contradiction. Hence, there is no a compatible anti-dendriform algebra structure on As_4^{13} .

Case $(As_4^{13}, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{20}(\alpha)$

$$\begin{cases} e_1 \triangleright e_1 = \alpha e_3 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = e_3 + \alpha_{12}e_4, \\ e_1 \triangleright e_3 = (-1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = -e_3 + (1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (-1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = (1 - \alpha)e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (4) for the triples $\{e_1, e_1, e_2\}$ and $\{e_1, e_1, e_1\}$, we obtain $\alpha_{13} = 1$ and $\alpha_{13} = 0$, which yields a contradiction. Hence, this case cannot occur.

Case $(As_4^{13}, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{21}(\alpha)$

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = e_3 + \alpha_{12}e_4, \\ e_1 \triangleright e_3 = (-1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = \alpha e_3 + (1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (-1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

where $\alpha \neq -1$.

Considering identity (4) for the triples $\{e_1, e_1, e_1\}$ and $\{e_1, e_1, e_2\}$, we $\alpha_{13} = 0$ and $\alpha_{13} = 1$; a contradiction. Therefore, there this case cannot occur.

Case $(As_4^{13}, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{22}(\alpha, \beta)$

$$\begin{cases} e_1 \triangleright e_1 = \alpha e_3 + \alpha_{11} e_4, \\ e_1 \triangleright e_2 = \alpha_{12} e_4, \\ e_1 \triangleright e_3 = (-1 + \alpha_{13}) e_4, \\ e_2 \triangleright e_1 = \beta e_3 + (1 + \alpha_{21}) e_4, \\ e_2 \triangleright e_2 = e_3 + (1 + \alpha_{22}) e_4, \\ e_2 \triangleright e_3 = \alpha_{23} e_4, \\ e_3 \triangleright e_1 = (-1 + \alpha_{31}) e_4, \\ e_3 \triangleright e_2 = \alpha_{32} e_4, \\ e_3 \triangleright e_3 = \alpha_{33} e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = (1 - \alpha) e_3 - \alpha_{11} e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12} e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13} e_4, \\ e_2 \triangleleft e_1 = -\beta e_3 - \alpha_{21} e_4, \\ e_2 \triangleleft e_2 = -e_3 - \alpha_{22} e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23} e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31} e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32} e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33} e_4. \end{cases}$$

Considering identity (4) for the triples $\{e_1, e_2, e_2\}$ and $\{e_1, e_1, e_1\}$, we obtain

$$\alpha_{13} = 1, \alpha_{13} = 0.$$

Thus, we have a contradiction. Hence, there is no anti-dendriform algebra associated to the algebra As_4^{13} and satisfying the condition $(As_4^{13}, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{22}(\alpha, \beta)$.

Case $(As_4^{13}, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^{23}$

$$\begin{cases} e_1 \triangleright e_1 = e_2 + \alpha_{11} e_4, \\ e_1 \triangleright e_2 = \alpha_{12} e_4, \\ e_1 \triangleright e_3 = (-1 + \alpha_{13}) e_4, \\ e_2 \triangleright e_1 = (1 + \alpha_{21}) e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22}) e_4, \\ e_2 \triangleright e_3 = \alpha_{23} e_4, \\ e_3 \triangleright e_1 = (-1 + \alpha_{31}) e_4, \\ e_3 \triangleright e_2 = \alpha_{32} e_4, \\ e_3 \triangleright e_3 = \alpha_{33} e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_2 + e_3 - \alpha_{11} e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12} e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13} e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21} e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22} e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23} e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31} e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32} e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33} e_4. \end{cases}$$

Considering identity (5) for the triples $\{e_3, e_1, e_1\}$, $\{e_2, e_1, e_1\}$ and $\{e_1, e_1, e_2\}$, we obtain

$$\alpha_{32} = 0, \alpha_{22} = -1, 0 = -1,$$

which yields a contradiction. Hence, this case cannot occur. $\square$

Theorem 3.7. *Any four-dimensional anti-dendriform algebra over the field of complex numbers whose associated associative algebra is As_4^{14} is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$AD_4^{33} : e_1 \triangleright e_2 = e_4, e_3 \triangleright e_1 = e_4, e_1 \triangleleft e_3 = e_4, e_2 \triangleleft e_1 = -e_4, e_2 \triangleleft e_2 = e_4;$$

$$AD_4^{34} : \begin{cases} e_1 \triangleright e_2 = e_4, e_2 \triangleright e_1 = -e_4, e_3 \triangleright e_3 = e_4, \\ e_1 \triangleleft e_3 = e_4, e_2 \triangleleft e_2 = e_4, e_3 \triangleleft e_1 = e_4, e_3 \triangleleft e_3 = -e_4; \end{cases}$$

$$AD_4^{35} : \begin{cases} e_1 \triangleright e_1 = e_4, e_1 \triangleright e_2 = e_4, e_2 \triangleright e_1 = -e_4, e_3 \triangleright e_3 = e_4, \\ e_1 \triangleleft e_1 = -e_4, e_1 \triangleleft e_3 = e_4, e_2 \triangleleft e_2 = e_4, e_3 \triangleleft e_1 = e_4, e_3 \triangleleft e_3 = -e_4; \end{cases}$$

$$AD_4^{36} : e_1 \triangleright e_2 = e_4, e_1 \triangleright e_3 = e_4, e_2 \triangleright e_1 = -e_4, e_2 \triangleright e_2 = e_4, e_3 \triangleright e_1 = e_4;$$

$$AD_4^{37}(\alpha) : \begin{cases} e_1 \triangleright e_1 = e_4, e_1 \triangleright e_2 = \alpha e_4, e_2 \triangleright e_1 = -\alpha e_4, e_2 \triangleright e_2 = e_4, e_3 \triangleright e_3 = e_4, \alpha \in \mathbb{C} \\ e_1 \triangleleft e_1 = -e_4, e_1 \triangleleft e_2 = (1 - \alpha) e_4, e_1 \triangleleft e_3 = e_4, \\ e_2 \triangleleft e_1 = (\alpha - 1) e_4, e_3 \triangleleft e_1 = e_4, e_3 \triangleleft e_3 = -e_4; \end{cases}$$

$$\begin{aligned}
 AD_4^{38}(\alpha) : & \begin{cases} e_1 \triangleright e_2 = e_3 + e_4, & e_2 \triangleright e_1 = -e_3 + (-1 + \alpha)e_4, & e_2 \triangleright e_2 = \frac{1}{2}e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_2 = -e_3, & e_1 \triangleleft e_3 = e_4, & e_2 \triangleleft e_1 = e_3 - \alpha e_4, & e_2 \triangleleft e_2 = \frac{1}{2}e_4, & \alpha \neq 0; \end{cases} \\
 AD_4^{39}(\alpha, \beta) : & \begin{cases} e_1 \triangleright e_1 = \alpha e_4, & e_1 \triangleright e_2 = e_3 + e_4, & e_2 \triangleright e_1 = -e_3 - e_4, \\ e_2 \triangleright e_2 = (1 + \beta)e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = -\alpha e_4, & e_1 \triangleleft e_2 = -e_3, & e_1 \triangleleft e_3 = e_4, & e_2 \triangleleft e_1 = e_3, & e_2 \triangleleft e_2 = -\beta e_4; \end{cases} \\
 AD_4^{40}(\alpha, \beta) : & \begin{cases} e_1 \triangleright e_1 = e_3, & e_1 \triangleright e_2 = (1 + \alpha)e_4, & e_2 \triangleright e_1 = (-1 + \alpha)e_4, \\ e_2 \triangleright e_2 = (1 + \beta)e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = -e_3, & e_1 \triangleleft e_2 = -\alpha e_4, & e_1 \triangleleft e_3 = e_4, & e_2 \triangleleft e_1 = -\alpha e_4, & e_2 \triangleleft e_2 = -\beta e_4; \end{cases} \\
 AD_4^{41}(\alpha, \beta, \gamma) : & \begin{cases} e_1 \triangleright e_1 = \alpha e_4, & e_1 \triangleright e_2 = e_3 + e_4, & e_2 \triangleright e_1 = (-1 + \beta)e_4, \\ e_2 \triangleright e_2 = (1 + \gamma)e_4, & e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = -\alpha e_4, & e_1 \triangleleft e_2 = -e_3, & e_1 \triangleleft e_3 = e_4, & e_2 \triangleleft e_1 = -\beta e_4, & e_2 \triangleleft e_2 = -\gamma e_4; \end{cases} \\
 AD_4^{42}[\lambda](\alpha, \beta, \gamma) : & \begin{cases} e_1 \triangleright e_1 = e_3, & e_1 \triangleright e_2 = \lambda e_3 + (1 + \alpha)e_4, & e_2 \triangleright e_1 = (-1 + \beta)e_4, \\ e_2 \triangleright e_2 = e_3 + (1 + \gamma)e_4, & e_3 \triangleright e_1 = e_4, & e_1 \triangleleft e_1 = -e_3, & e_1 \triangleleft e_2 = -\lambda e_3 - \alpha e_4, \\ e_1 \triangleleft e_3 = e_4, & e_2 \triangleleft e_1 = -\beta e_4, & e_2 \triangleleft e_2 = -e_3 - \gamma e_4. \end{cases}
 \end{aligned}$$

where $AD_4^{38}(0) = AD_4^{39}(0, -\frac{1}{2})$.

Proof. Considering identity (6) for the following triples

$$\{e_3, e_1, e_1\}, \{e_1, e_2, e_2\}, \{e_1, e_3, e_2\}, \{e_2, e_3, e_1\}, \{e_2, e_2, e_3\}, \{e_3, e_2, e_2\}, \{e_2, e_2, e_4\},$$

we obtain $e_4 \triangleright e_1 = 0$, $e_1 \triangleleft e_4 = 0$, $e_4 \triangleright e_2 = 0$, $e_2 \triangleleft e_4 = 0$, $e_4 \triangleright e_3 = 0$, $e_3 \triangleleft e_4 = 0$, $e_4 \triangleright e_4 = 0$, respectively.

By $e_4 e_i = e_4 \triangleright e_i + e_4 \triangleleft e_i = 0$ and $e_i e_4 = e_i \triangleright e_4 + e_i \triangleleft e_4 = 0$ these imply $e_4 \triangleleft e_i = 0$ and $e_i \triangleright e_4 = 0$. Then it is easy to see that $\langle e_4 \rangle$ is the annihilator of the algebras $(As_4^{14}, \cdot)$ and $(As_4^{14}, \triangleright, \triangleleft)$. If we take the algebra $As_4^{14}/\langle e_4 \rangle$, this algebra is a three-dimensional associative Abelian algebra, and we know that any three-dimensional complex anti-dendriform algebra associated with the abelian algebra is isomorphic to one of the pairwise non-isomorphic algebras $AD_3^3 - AD_3^7(\lambda)$.

Case $(As_4^{14}, \triangleright, \triangleleft)/\langle e_4 \rangle \cong AD_3^3$

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = (1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

It is not difficult to see that $(As_4^{14}, \triangleright, \triangleleft)$ are 2-nilpotent and have three generator elements. Thus we have $(x \triangleright y) \triangleright z = x \triangleright (y \triangleright z) = 0$ which implies that $(As_4^{14}, \triangleright)$ is also associative 2-nilpotent algebra and has three generator elements. According to Theorem 2.8 there are five non-isomorphic four-dimensional indecomposable associative 2-nilpotent and the three generated algebras. Hence, we obtain $AD_4^{33} - AD_4^{37}(\alpha)$ algebras.

Case $(As_4^{14}, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^4$

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = e_3 + (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = (1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = -e_3 + (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (5) for the triples $\{e_1, e_1, e_2\}$, $\{e_2, e_2, e_1\}$ and $\{e_1, e_2, e_i\}$, $i = 1, 2, 3$, we obtain

$$\alpha_{13} = -1, \alpha_{23} = 0, \alpha_{3i} = 0, i = 1, 2, 3.$$

By the basis change $e'_3 = e_3 + \alpha_{12}e_4$, the multiplication table of the algebra has the following form:

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = e_3 + e_4, \\ e_2 \triangleright e_1 = -e_3 + (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_3 \triangleright e_1 = e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3, \\ e_1 \triangleleft e_3 = e_4, \\ e_2 \triangleleft e_1 = e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4. \end{cases}$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = ae_1 + be_2 - \frac{b^2}{2a}e_3 + ce_4, \quad \varphi(e_2) = ae_2 - be_3 + de_4, \quad \varphi(e_3) = ae_3 + ee_4, \quad \varphi(e_4) = a^2e_4,$$

where $a \neq 0$.

We express the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)\}$ via the basis elements $\{e_1, e_2, e_3, e_4\}$. By verifying all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha'_{11}, \alpha'_{21}, \alpha'_{22}\}$ and $\{\alpha_{11}, \alpha_{21}, \alpha_{22}\}$:

$$\begin{aligned} \varphi(e_1) \triangleright \varphi(e_2) &= \varphi(e_3) + \varphi(e_4) & \Rightarrow a^2 &= a, \quad b(1 + \alpha_{22}) = e, \\ \varphi(e_1) \triangleright \varphi(e_1) &= \alpha'_{11}\varphi(e_4) & \Rightarrow \alpha'_{11} &= \alpha_{11} + b\alpha_{21} + b^2(\frac{1}{2} + \alpha_{22}), \\ \varphi(e_1) \triangleright \varphi(e_2) &= -\varphi(e_3) + (-1 + \alpha'_{21})\varphi(e_4) & \Rightarrow \alpha'_{21} &= \alpha_{21} + b(1 + 2\alpha_{22}), \\ \varphi(e_2) \triangleright \varphi(e_2) &= (1 + \alpha'_{22})\varphi(e_4) & \Rightarrow \alpha'_{22} &= \alpha_{22}. \end{aligned}$$

We have the following cases:

(a) Let $\alpha_{22} = -\frac{1}{2}$. Then

If $\alpha_{21} = 0$, we obtain the algebra $AD_4^{39}(\alpha, -\frac{1}{2})$.

If $\alpha_{21} \neq 0$, then by choosing $b = -\frac{\alpha_{11}}{\alpha_{21}}$ we obtain $\alpha'_{11} = 0$ and we obtain the algebra $AD_4^{38}(\alpha)$, $\alpha \neq 0$.

(b) Let $\alpha_{22} \neq -\frac{1}{2}$. Then choosing $b = -\frac{\alpha_{21}}{1+2\alpha_{22}}$ we derive $\alpha'_{21} = 0$ and we obtain the algebra $AD_4^{39}(\alpha, \beta)$, $\beta \neq -\frac{1}{2}$.

Case $(As_4^{14}, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^5$

$$\begin{cases} e_1 \triangleright e_1 = e_3 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = (1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (3) for the triples $\{e_i, e_1, e_1\}$, $i = 1, 2, 3$, we obtain $\alpha_{13} = -1$, $\alpha_{23} = \alpha_{33} = 0$. From identity (5) for the triples $\{e_1, e_1, e_i\}$, $i = 1, 2$, we obtain $\alpha_{31} = \alpha_{32} = 0$. By the basis change $e'_3 = e_3 + \alpha_{11}e_4$, we obtain $\alpha_{11} = 0$.

Thus, we have the algebra

$$\begin{cases} e_1 \triangleright e_1 = e_3, \\ e_1 \triangleright e_2 = (1 + \alpha_{12})e_4, \\ e_2 \triangleright e_1 = (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_3 \triangleright e_1 = e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_3, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4. \end{cases}$$

Let us consider the general change and by verifying all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha'_{12}, \alpha'_{21}, \alpha'_{22}\}$ and $\{\alpha_{12}, \alpha_{21}, \alpha_{22}\}$:

$$\begin{aligned} \varphi(e_1) \triangleright \varphi(e_1) &= \varphi(e_3) &\Rightarrow a^2 = a, \quad b(\alpha_{12} + \alpha_{21}) + b^2(\tfrac{1}{2} + \alpha_{22}) = e, \\ \varphi(e_1) \triangleright \varphi(e_2) &= (1 + \alpha'_{12})\varphi(e_4) &\Rightarrow \alpha'_{12} = \alpha_{12} + b(1 + \alpha_{22}), \\ \varphi(e_2) \triangleright \varphi(e_1) &= (-1 + \alpha'_{21})\varphi(e_4) &\Rightarrow \alpha'_{21} = \alpha_{21} + b\alpha_{22}, \\ \varphi(e_2) \triangleright \varphi(e_2) &= (1 + \alpha'_{22})\varphi(e_4) &\Rightarrow \alpha'_{22} = \alpha_{22}. \end{aligned}$$

If choosing $b = \alpha_{21} - \alpha_{12}$, then we can reduce $\alpha'_{12} = \alpha'_{21}$ we obtain the algebra $AD_4^{40}(\alpha, \beta)$.

Case $(As_4^{14}, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^6$

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = e_3 + (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = (1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

By the basis change $e'_3 = e_3 + \alpha_{12}e_4$, we can put $\alpha_{12} = 0$. Further considering identity (2) for the triples $\{e_1, e_1, e_2\}$, $\{e_2, e_1, e_2\}$ and $\{e_1, e_2, e_i\}$, $i = 1, 2, 3$, we obtain the following restrictions:

$$\alpha_{13} = -1, \quad \alpha_{23} = 0, \quad \alpha_{3i} = 0, \quad i = 1, 2, 3.$$

Thus, the multiplication table of the algebra has the following form:

$$\begin{cases} e_1 \triangleright e_1 = \alpha_{11}e_4, \\ e_1 \triangleright e_2 = e_3 + e_4, \\ e_2 \triangleright e_1 = (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_3 \triangleright e_1 = e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3, \\ e_1 \triangleleft e_3 = e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4. \end{cases}$$

If we consider the general change of the basis in this algebra to study the behavior of these structure constants of the algebra, then we have the following relations:

$$\alpha'_{11} = \alpha_{11}, \alpha'_{21} = \alpha_{21}, \alpha'_{22} = \alpha_{22}$$

Hence, we have the algebra $AD_4^{41}(\alpha, \beta, \gamma)$.

Case $(As_4^{14}, \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^7(\lambda)$

$$\begin{cases} e_1 \triangleright e_1 = e_3 + \alpha_{11}e_4, \\ e_1 \triangleright e_2 = \lambda e_3 + (1 + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = (1 + \alpha_{13})e_4, \\ e_2 \triangleright e_1 = (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = e_3 + (1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = (1 + \alpha_{31})e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = \alpha_{33}e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\lambda e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -e_3 - \alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

By the basis change $e'_3 = e_3 + \alpha_{11}e_4$, we can obtain $\alpha_{11} = 0$. Considering identity (4) for the triple $\{e_i, e_1, e_1\}$, $i = 1, 2, 3$, and also considering identity (5) for the triples $\{e_1, e_1, e_i\}$, $i = 1, 2$, we obtain the following restrictions on structure constants:

$$\alpha_{13} = -1, \alpha_{23} = \alpha_{33} = \alpha_{31} = \alpha_{32} = 0.$$

Now we rewrite

$$\begin{cases} e_1 \triangleright e_1 = e_3, \\ e_1 \triangleright e_2 = \lambda e_3 + (1 + \alpha_{12})e_4, \\ e_2 \triangleright e_1 = (-1 + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = e_3 + (1 + \alpha_{22})e_4, \\ e_3 \triangleright e_1 = e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_3, \\ e_1 \triangleleft e_2 = -\lambda e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -e_3 - \alpha_{22}e_4. \end{cases}$$

Similarly, considering the general change of the basis in this algebra we can show the following relations:

$$\alpha'_{12} = \alpha_{12}, \alpha'_{21} = \alpha_{21}, \alpha'_{22} = \alpha_{22}.$$

Hence, we have the algebra $AD_4^{42}[\lambda](\alpha, \beta, \gamma)$ □

Theorem 3.8. Any four-dimensional anti-dendriform algebra over the field of complex numbers whose associated associative algebra is As_4^{15} is isomorphic to one of the following pairwise non-isomorphic algebras:

$$\begin{aligned}
 AD_4^{43}(\alpha) : & \begin{cases} e_1 \triangleright e_2 = e_4, \ e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = e_4, \ e_1 \triangleleft e_2 = (\alpha - 1)e_4, \ e_2 \triangleleft e_1 = -\alpha e_4, \\ e_2 \triangleleft e_2 = e_4, \ e_3 \triangleleft e_1 = -e_4, \ e_3 \triangleleft e_3 = e_4; \end{cases} \\
 AD_4^{44}(\alpha) : & \begin{cases} e_1 \triangleright e_2 = e_4, \ e_2 \triangleright e_1 = -e_4, \ e_3 \triangleright e_3 = e_4, \\ e_1 \triangleleft e_1 = e_4, \ e_1 \triangleleft e_2 = (\alpha - 1)e_4, \ e_2 \triangleleft e_1 = (1 - \alpha)e_4, \ e_2 \triangleleft e_2 = e_4; \end{cases} \\
 AD_4^{45}(\alpha) : & \begin{cases} e_1 \triangleright e_1 = e_4, \ e_1 \triangleright e_2 = e_4, \ e_2 \triangleright e_1 = -e_4, \ e_3 \triangleright e_3 = e_4, \\ e_1 \triangleleft e_2 = (\alpha - 1)e_4, \ e_2 \triangleleft e_1 = (1 - \alpha)e_4, \ e_2 \triangleleft e_2 = e_4; \end{cases} \\
 AD_4^{46}(\alpha) : & \begin{cases} e_1 \triangleright e_2 = e_4, \ e_1 \triangleright e_3 = e_4, \ e_2 \triangleright e_1 = -e_4, \ e_2 \triangleright e_2 = e_4, \ e_3 \triangleright e_1 = e_4, \\ e_1 \triangleleft e_1 = e_4, \ e_1 \triangleleft e_2 = (\alpha - 1)e_4, \ e_1 \triangleleft e_3 = -e_4, \\ e_2 \triangleleft e_1 = (1 - \alpha)e_4, \ e_3 \triangleleft e_1 = -e_4, \ e_3 \triangleleft e_3 = e_4; \end{cases} \\
 AD_4^{47}(\alpha) : & e_1 \triangleright e_1 = e_4, \ e_1 \triangleright e_2 = \alpha e_4, \ e_2 \triangleright e_1 = -\alpha e_4, \ e_2 \triangleright e_2 = e_4, \ e_3 \triangleright e_3 = e_4.
 \end{aligned}$$

Proof. Considering identity (6) for the following triples

$$\{e_1, e_3, e_3\}, \{e_3, e_3, e_1\}, \{e_3, e_3, e_2\}, \{e_2, e_3, e_3\}, \{e_2, e_2, e_3\}, \{e_3, e_2, e_2\}, \{e_3, e_3, e_4\},$$

we obtain $e_1 \triangleleft e_4 = 0$, $e_4 \triangleright e_1 = 0$, $e_4 \triangleright e_2 = 0$, $e_2 \triangleleft e_4 = 0$, $e_4 \triangleright e_3 = 0$, $e_3 \triangleleft e_4 = 0$, $e_4 \triangleright e_4 = 0$, respectively.

By $e_4 e_i = e_4 \triangleright e_i + e_4 \triangleleft e_i = 0$ and $e_i e_4 = e_i \triangleright e_4 + e_i \triangleleft e_4 = 0$ these imply $e_4 \triangleleft e_i = 0$ and $e_i \triangleright e_4 = 0$. Then it is easy to see that $\langle e_4 \rangle$ is the annihilator of the algebras $(As_4^{15}(\alpha), \cdot)$ and $(As_4^{15}(\alpha), \triangleright, \triangleleft)$. If we take the algebra $As_4^{15}(\alpha)/\langle e_4 \rangle$, this algebra is a three-dimensional associative abelian algebra, and we know that any three-dimensional complex anti-dendriform algebra associated with the abelian algebra is isomorphic to one of the pairwise non-isomorphic algebras $AD_3^3 - AD_3^7(\lambda)$.

Case $(As_4^{15}(\alpha), \triangleright, \triangleleft)/\langle e_4 \rangle \cong AD_3^3$

$$\begin{cases} e_1 \triangleright e_1 = (1 + \alpha_{11})e_4, \\ e_1 \triangleright e_2 = (\alpha + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = (-\alpha + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = \alpha_{31}e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = (1 + \alpha_{33})e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

It is not difficult to see that $(As_4^{15}, \triangleright, \triangleleft)$ are 2-nilpotent and have three generator elements. Thus we have $(x \triangleright y) \triangleright z = x \triangleright (y \triangleright z) = 0$ which implies that $(As_4^{15}, \triangleright)$ is also associative 2-nilpotent algebra and has three generator elements. According to Theorem 2.8 there are five non-isomorphic four-dimensional indecomposable associative 2-nilpotent and the three generated algebras. Hence, we obtain $AD_4^{43}(\alpha) - AD_4^{47}(\alpha)$ algebras.

It is possible to consider the multiplication $\triangleleft$ as above. However, it is not difficult to show that the constructed algebras are isomorphic.

Now if $(As_4^{15}, \triangleright, \triangleleft)/\langle e_4 \rangle$ anti-dendriform algebra is isomorphic to one of the algebras $AD_3^4 - AD_3^7(\alpha)$, then we will proof there is not a four-dimensional complex anti-dendriform algebra associated to the associative algebra As_4^{15} .

Case $(As_4^{15}(\alpha), \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^4$

$$\begin{cases} e_1 \triangleright e_1 = (1 + \alpha_{11})e_4, \\ e_1 \triangleright e_2 = e_3 + (\alpha + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = -e_3 + (-\alpha + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = \alpha_{31}e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = (1 + \alpha_{33})e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = e_3 - \alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (5) for the triples $\{e_2, e_1, e_3\}$ and $\{e_3, e_1, e_2\}$, we obtain the following restrictions on structure constants:

$$\alpha_{33} = 0, \quad \alpha_{33} = -1.$$

Thus, we have a contradiction. Hence, there is no anti-dendriform algebra associated to the algebra As_4^{15} and satisfying the condition $(As_4^{15}(\alpha), \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^4$.

Case $(As_4^{15}(\alpha), \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^5$

$$\begin{cases} e_1 \triangleright e_1 = e_3 + (1 + \alpha_{11})e_4, \\ e_1 \triangleright e_2 = (\alpha + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = (-\alpha + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = \alpha_{31}e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = (1 + \alpha_{33})e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (5) for the triples $\{e_1, e_1, e_3\}$ and $\{e_3, e_1, e_1\}$, we obtain

$$\alpha_{33} = 0, \quad \alpha_{33} = -1,$$

which yields a contradiction. Hence, this case cannot occur.

Case $(As_4^{15}(\alpha), \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^6$

$$\begin{cases} e_1 \triangleright e_1 = (1 + \alpha_{11})e_4, \\ e_1 \triangleright e_2 = e_3 + (\alpha + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = (-\alpha + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = (1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = \alpha_{31}e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = (1 + \alpha_{33})e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -\alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -\alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (5) for the triples $\{e_1, e_2, e_3\}$ and $\{e_3, e_1, e_2\}$, we obtain $\alpha_{33} = 0$ and $\alpha_{33} = -1$; a contradiction. Therefore, this case cannot occur.

Case $(As_4^{15}(\alpha), \triangleright, \triangleleft) / \langle e_4 \rangle \cong AD_3^7(\lambda)$

$$\begin{cases} e_1 \triangleright e_1 = e_3 + (1 + \alpha_{11})e_4, \\ e_1 \triangleright e_2 = \lambda e_3 + (\alpha + \alpha_{12})e_4, \\ e_1 \triangleright e_3 = \alpha_{13}e_4, \\ e_2 \triangleright e_1 = (-\alpha + \alpha_{21})e_4, \\ e_2 \triangleright e_2 = e_3 + (1 + \alpha_{22})e_4, \\ e_2 \triangleright e_3 = \alpha_{23}e_4, \\ e_3 \triangleright e_1 = \alpha_{31}e_4, \\ e_3 \triangleright e_2 = \alpha_{32}e_4, \\ e_3 \triangleright e_3 = (1 + \alpha_{33})e_4, \end{cases} \quad \begin{cases} e_1 \triangleleft e_1 = -e_3 - \alpha_{11}e_4, \\ e_1 \triangleleft e_2 = -\lambda e_3 - \alpha_{12}e_4, \\ e_1 \triangleleft e_3 = -\alpha_{13}e_4, \\ e_2 \triangleleft e_1 = -\alpha_{21}e_4, \\ e_2 \triangleleft e_2 = -e_3 - \alpha_{22}e_4, \\ e_2 \triangleleft e_3 = -\alpha_{23}e_4, \\ e_3 \triangleleft e_1 = -\alpha_{31}e_4, \\ e_3 \triangleleft e_2 = -\alpha_{32}e_4, \\ e_3 \triangleleft e_3 = -\alpha_{33}e_4. \end{cases}$$

Considering identity (5) for the triples $\{e_1, e_1, e_3\}$ and $\{e_3, e_1, e_1\}$, we obtain $\alpha_{33} = 0$, $\alpha_{33} = -1$, which yields a contradiction. Hence, this case cannot occur. $\square$

4. Conclusion

In this paper, we classified all four-dimensional anti-dendriform algebras whose associated associative algebra is indecomposable and has a one-dimensional annihilator. The classification was obtained through a detailed study of compatible anti-dendriform structures and automorphism group actions, resulting in a complete list of pairwise non-isomorphic algebras. A major difficulty was the determination of isomorphism classes, which required careful analysis of automorphisms and basis transformations. Since this work is restricted to associative algebras with a one-dimensional annihilator, the classification in the cases of higher-dimensional annihilators and decomposable associative algebras remains open. Future research will focus on completing the classification of four-dimensional anti-dendriform algebras, extending the study to higher dimensions, and investigating derivations, automorphisms, cohomological properties, and connections with related algebraic structures.

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