

Spectral analysis and energy bounds for total zero-divisor graphs of the ring $\mathbb{Z}_{p^2q^2}$

Research Article

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Abstract: This study investigates the spectral properties of the total zero-divisor graph associated with the ring $\mathbb{Z}_{p^2q^2}$, where p and q are distinct primes. Using an explicit equitable partition of the graph, we obtain a complete description of the adjacency spectrum. We show that the spectrum consists of two integer eigenvalues 0 and -1 occurring with large multiplicities determined in closed form, together with seven eigenvalues from the quotient matrix, of which five are nonzero. The decomposition enables explicit computation and bounding of key spectral invariants. In particular, we derive new formulas and bounds for the graph energy and provide structural constraints for the spectral radius. These results deepen the understanding of total zero-divisor graphs over composite rings and contribute new tools for further developments in algebraic graph theory.

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1. Introduction

The interaction between algebra and graph theory has led to a rich body of research, with zero-divisor graphs playing a central role in connecting the structural features of commutative rings to combinatorial objects. The earliest form of this idea is due to Beck [3], who associated to a ring R a graph in which two

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elements were adjacent whenever their product vanished. This viewpoint was sharpened by Anderson and Livingston [2], who introduced the now standard zero-divisor graph $\Gamma(R)$, whose vertices are precisely the nonzero zero-divisors $Z^*(R)$ and whose edges correspond to pairs (x, y) with $xy = 0$. This refinement led to a deeper understanding of how algebraic features of R manifest in the combinatorial properties of $\Gamma(R)$, motivating extensive work on invariants such as diameter, girth, clique number, and coloring [5, 14]. Among the various families of rings studied in this context, the rings $\mathbb{Z}_n$ have attracted particular attention because their arithmetic structure is explicit and well suited for detailed graph-theoretic analysis. Previous work has computed invariants of $\Gamma(\mathbb{Z}_n)$ for several special forms of n , including p^2 , p^3 , pq , and p^2q [1, 13].

Motivated by the success of $\Gamma(R)$, several authors have proposed variations that encode additional algebraic information. A notable example is the *total zero-divisor graph* $TT(R)$, introduced in [8], where two distinct vertices $x, y \in Z^*(R)$ are adjacent precisely when $xy = 0$ and $x + y \in Z^*(R)$.

The simultaneous use of multiplication and addition makes $TT(R)$ more refined and typically sparser than the classical zero-divisor graph. As a result, its structure often reflects subtler algebraic interactions within R , and total zero-divisor graphs have quickly developed into a significant line of inquiry.

Most recently, Mashru and Lalchandani [12] examined $TT(\mathbb{Z}_n)$ for the case $n = p^2q^2$ and uncovered a remarkably organized structure: the nonzero zero-divisors partition naturally into seven blocks determined by their prime factorizations. This partition induces an equitable 7×7 block decomposition of the adjacency matrix, from which the authors computed several combinatorial invariants, such as the graph's girth, diameter, clique number, and stability number.

Despite this progress, the *spectral* behaviour of $TT(\mathbb{Z}_{p^2q^2})$ has remained unexplored. Spectral graph theory [4, 7] provides powerful tools for understanding a graph through the eigenvalues of its adjacency matrix, enabling insight into connectivity, robustness, branching structure, and quantitative invariants such as energy. In the present work, we address this gap by developing the complete spectral description of $TT(\mathbb{Z}_{p^2q^2})$.

Our approach builds directly on the block structure established in [12]. By exploiting the equitable partition, we show that the adjacency spectrum decomposes naturally into three components. First, two integer eigenvalues 0 and -1 occur with large multiplicities that we compute exactly using structural properties of the independent and clique blocks. Second, the remaining spectral information is encoded in the quotient matrix of the partition. Because this quotient matrix has nullity 2, only *five* of its seven eigenvalues are nonzero, and these five values account for the nontrivial portion of the spectrum. Finally, we compute the characteristic polynomial of the quotient matrix explicitly and express its coefficients as functions of the primes p and q .

This complete spectral characterization enriches the known structural theory of total zero-divisor graphs over $\mathbb{Z}_{p^2q^2}$ and provides a framework that can be extended to spectral investigations of $TT(\mathbb{Z}_n)$ for other composite integers n .

2. Preliminaries and notation

This section consolidates the essential definitions, concepts, and prior results that form the foundation of our spectral analysis. We begin by recalling fundamental notions from spectral graph theory, followed by the specific definitions related to total zero-divisor graphs. Finally, we restate the key combinatorial structure and adjacency matrix for the ring $\mathbb{Z}_{p^2q^2}$, as established by Mashru and Lalchandani [12], which serves as the central object of our study.

2.1. Graph spectra and the adjacency matrix

Definition 2.1. A simple graph $G = (V, E)$ consists of a finite set of vertices V and a set of edges E , where each edge is an unordered pair of distinct vertices. For a graph with n vertices, its adjacency

matrix $A(G) = (a_{uv})$ is an $n \times n$ symmetric matrix defined by:

$$a_{uv} = \begin{cases} 1 & \text{if } u \sim v \text{ (i.e., vertices } u \text{ and } v \text{ are adjacent),} \\ 0 & \text{otherwise.} \end{cases}$$

The diagonal entries a_{uu} are zero, as simple graphs forbid loops.

Definition 2.2. The characteristic polynomial $\chi_G(\lambda)$ of a graph G is the characteristic polynomial of its adjacency matrix:

$$\chi_G(\lambda) = \det(A(G) - \lambda I_n),$$

where I_n is the $n \times n$ identity matrix. The roots $\lambda_1, \lambda_2, \dots, \lambda_n$ of this polynomial are the eigenvalues of G . The multiset of eigenvalues constitutes the spectrum of G , denoted $\text{Spec}(G)$. Since $A(G)$ is real and symmetric, all eigenvalues are real [7].

The spectrum of a graph encodes significant information about its structure. For instance, the multiplicity of the eigenvalue zero relates to the dimension of the matrix's null space, which is influenced by the graph's connectivity and the presence of independent sets [4].

2.2. The total zero-divisor graph

Let R be a commutative ring with a non-zero identity element, and let $Z^*(R) = Z(R) \setminus \{0\}$ be its set of non-zero zero-divisors.

Definition 2.3. The total zero-divisor graph of R , denoted $\text{TT}(R)$, is the simple graph defined as follows:

- The vertex set is $Z^*(R)$.
- Two distinct vertices x and y are adjacent if and only if the following two conditions hold simultaneously:
 1. $xy = 0$ (the multiplicative condition),
 2. $x + y \in Z^*(R)$ (the additive condition).

If R is an integral domain, then $Z^*(R)$ is empty, and $\text{TT}(R)$ is considered the null graph.

The additive condition $x + y \in Z^*(R)$ imposes a stricter criterion for adjacency compared to the standard zero-divisor graph $\Gamma(R)$, often resulting in a graph with fewer edges and a more complex block structure for its adjacency matrix [8].

2.3. The ring $\mathbb{Z}_{p^2q^2}$ and its zero-divisor partition

For the remainder of this paper, we fix $n = p^2q^2$, where p and q are distinct primes with $p < q$. The set of non-zero zero-divisors of $\mathbb{Z}_n$ consists of the positive integers less than n that are not coprime to n . Its cardinality is given by $n - \phi(n) - 1 = pq(p + q - 1) - 1$, where ϕ is Euler's totient function.

A pivotal insight from [12] was the categorization of these zero-divisors into seven disjoint classes, based on the highest powers of p and q dividing them. This partition is fundamental to understanding the graph's structure.

Let us define the seven blocks:

$$\begin{aligned} B_1 &= \{t_1 p \mid 1 \leq t_1 \leq pq^2 - 1, p \nmid t_1, q \nmid t_1\}, \\ B_2 &= \{t_2 q \mid 1 \leq t_2 \leq p^2 q - 1, p \nmid t_2, q \nmid t_2\}, \\ B_3 &= \{t_3 pq \mid 1 \leq t_3 \leq pq - 1, p \nmid t_3, q \nmid t_3\}, \\ B_4 &= \{t_4 p^2 \mid 1 \leq t_4 \leq q^2 - 1, q \nmid t_4\}, \\ B_5 &= \{t_5 q^2 \mid 1 \leq t_5 \leq p^2 - 1, p \nmid t_5\}, \\ B_6 &= \{t_6 p^2 q \mid 1 \leq t_6 \leq q - 1\}, \\ B_7 &= \{t_7 pq^2 \mid 1 \leq t_7 \leq p - 1\}. \end{aligned}$$

The cardinalities of these blocks, derived through combinatorial reasoning involving Euler's totient function, are as follows:

$$\begin{aligned} |B_1| &= q(p-1)(q-1), & |B_2| &= p(p-1)(q-1), \\ |B_3| &= (p-1)(q-1), & |B_4| &= q(q-1), \\ |B_5| &= p(p-1), & |B_6| &= q-1, \\ |B_7| &= p-1. \end{aligned}$$

It is easily verified that the sum $\sum_{i=1}^7 |B_i| = pq(p+q-1) - 1$, confirming the total number of non-zero zero-divisors.

2.4. The block adjacency matrix

The adjacency between elements of different blocks B_i and B_j is determined by the strict conditions of the total zero-divisor graph. A comprehensive analysis of these conditions was performed in [12], leading to the following seminal result, which we state as our foundational theorem.

Theorem 2.4. [12] *Let p, q be distinct primes and $n = p^2 q^2$. With the vertex set of the total zero-divisor graph $\text{TT}(\mathbb{Z}_n)$ partitioned as $B_1, B_2, B_4, B_5, B_3, B_6, B_7$, the adjacency matrix of $\text{TT}(\mathbb{Z}_n)$ has the following 7×7 block structure:*

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & J_{|B_1| \times |B_7|} \\ 0 & 0 & 0 & 0 & 0 & J_{|B_2| \times |B_6|} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J_{|B_4| \times |B_7|} \\ 0 & 0 & 0 & 0 & 0 & J_{|B_5| \times |B_6|} & 0 \\ 0 & 0 & 0 & 0 & J_{|B_3|} - I_{|B_3|} & J_{|B_3| \times |B_6|} & J_{|B_3| \times |B_7|} \\ 0 & J_{|B_6| \times |B_2|} & 0 & J_{|B_6| \times |B_5|} & J_{|B_6| \times |B_3|} & J_{|B_6|} - I_{|B_6|} & J_{|B_6| \times |B_7|} \\ J_{|B_7| \times |B_1|} & 0 & J_{|B_7| \times |B_4|} & 0 & J_{|B_7| \times |B_3|} & J_{|B_7| \times |B_6|} & J_{|B_7|} - I_{|B_7|} \end{bmatrix} \quad (1)$$

Here, $J_{m \times n}$ denotes the $m \times n$ all-ones matrix, $J_n = J_{n \times n}$, and I_n is the $n \times n$ identity matrix. The matrix A is real, symmetric, and of order $pq(p+q-1) - 1$.

This structured block form reveals the strong symmetry of $\text{TT}(\mathbb{Z}_{p^2 q^2})$ and provides the foundation for the subsequent spectral analysis using equitable partitions and quotient matrices [10].

Proposition 2.5. *Let G be a graph with adjacency matrix A , and suppose the vertex set is partitioned equitably into blocks $\Pi = \{B_1, \dots, B_t\}$. Suppose B_i is a block for which the induced subgraph $G[B_i]$ is an independent set (coclique) and the partition is equitable (each vertex of B_i has the same number of neighbours in every other block B_j).*

Then the adjacency matrix A has at least $|B_i| - 1$ independent eigenvectors with eigenvalue 0 supported on B_i . Equivalently, each coclique block B_i contributes $|B_i| - 1$ to the algebraic multiplicity of the eigenvalue 0.

Proof. Since $G[B_i]$ is an independent set the submatrix $A[B_i, B_i]$ is the zero matrix. By the equitable partition condition the rows of A indexed by vertices of B_i are identical. Let $\mathbf{1}_{B_i}$ denote the all-ones vector on B_i . Any vector x supported on B_i with $x \perp \mathbf{1}_{B_i}$ is annihilated by these identical rows; hence $Ax = 0$. The subspace $\{x \in \mathbb{R}^{B_i} : x \perp \mathbf{1}_{B_i}\}$ has dimension $|B_i| - 1$, which yields the claim. $\square$

3. Main results: the spectrum of $TT(\mathbb{Z}_{p^2q^2})$

This section presents the complete spectral decomposition of the total zero-divisor graph $TT(\mathbb{Z}_{p^2q^2})$. We begin with a short lemma on rank- k coupling and eigenvalue shifts and then the main theorem that provides the complete spectrum for arbitrary distinct primes $p < q$ followed by an illustrative example.

Lemma 3.1. *Let H be a graph obtained by joining two disjoint graphs G_1 and G_2 via a bipartite incidence matrix of rank k (that is, the interconnection between the vertex sets of G_1 and G_2 has rank k). Then adding this bipartite coupling changes at most k eigenvalues of the disjoint union spectrum; equivalently, at most k eigenvalues of each component's spectrum can move.*

Proof. This follows from the rank- k perturbation principle for symmetric matrices and eigenvalue interlacing: adding a matrix of rank k to a symmetric block-diagonal matrix can change at most k eigenvalues [9]. In particular, a rank- k bipartite coupling between G_1 and G_2 can perturb at most k eigenvalues arising from the disjoint union of the two components. $\square$

3.1. The complete spectrum

Theorem 3.2. *Let p and q be distinct primes with $p < q$. The adjacency matrix A of the total zero-divisor graph $TT(\mathbb{Z}_{p^2q^2})$ has spectrum decomposing into three parts:*

$$\text{Spec}(A) = \{0^{M_0}, (-1)^{M_{-1}}\} \cup \text{Spec}(Q),$$

where the multiplicities (expressed in terms of p, q) are

$$M_0 = pq(p + q - 2) - 2 \quad \text{and} \quad M_{-1} = pq - 4. \quad (2)$$

The quotient matrix Q has seven eigenvalues $\{\lambda_i\}_{i=1}^7$, two of which are zero. The five nonzero eigenvalues $\lambda_1, \dots, \lambda_5$ are the eigenvalues of the quotient matrix

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & p-1 \\ 0 & 0 & 0 & 0 & 0 & q-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p-1 \\ 0 & 0 & 0 & 0 & 0 & q-1 & 0 \\ 0 & 0 & 0 & 0 & (p-1)(q-1)-1 & (p-1)(q-1) & p-1 \\ 0 & p(p-1)(q-1) & p(p-1) & q-2 & 0 & q-2 & p-1 \\ q(p-1)(q-1) & 0 & q(q-1) & 0 & (p-1)(q-1) & q-1 & p-2 \end{bmatrix}, \quad (3)$$

which encodes the inter-block adjacency relative to the equitable partition

$$\Pi = \{B_1, B_2, B_3, B_4, B_5, B_6, B_7\}$$

of $V(TT(\mathbb{Z}_{p^2q^2}))$.

Since Q is a 7×7 matrix, it has seven eigenvalues in total. By Lemma 3.3, its nullity is 2, so exactly two of its eigenvalues are zero. Thus

$$\text{Spec}(Q) = \{0, 0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\},$$

where the five values $\lambda_1, \dots, \lambda_5$ are nonzero and constitute the nontrivial spectral contribution of the quotient matrix. These seven eigenvalues are roots of the characteristic polynomial

$$P_Q(\lambda) = \det(\lambda I_7 - Q) = \lambda^7 + c_6\lambda^6 + c_5\lambda^5 + c_4\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0,$$

whose coefficients $c_0, \dots, c_6$ are determined by the block structure (using the Newton–Girard identities).

Proof. We obtain the spectrum from the equitable partition and the spectrum of the quotient matrix. The proof proceeds in three steps.

Step 1. Eigenvalue multiplicities from the block structure.

The equitable partition Π allows direct computation of high-multiplicity eigenvalues by standard spectral partition theory [6, 9].

Multiplicity of the eigenvalue 0.

From Theorem 2.4, the blocks B_1, B_2, B_4, B_5 induce independent sets and are structurally isolated internally. As is standard for equitable partitions [9], an independent block of size $|B_i|$ with identical external adjacency contributes exactly $|B_i| - 1$ zero eigenvalues to the adjacency spectrum. Thus the *coclique contribution* to the multiplicity of the eigenvalue 0 is

$$M_0^{(\text{coclique})} = (|B_1| - 1) + (|B_2| - 1) + (|B_4| - 1) + (|B_5| - 1).$$

Substituting block sizes gives

$$\begin{aligned} M_0^{(\text{coclique})} &= q(p-1)(q-1) + p(p-1)(q-1) + q(q-1) + p(p-1) - 4 \\ &= pq(p+q-2) - 4. \end{aligned} \tag{4}$$

In addition to the coclique contribution, the total multiplicity of the eigenvalue 0 includes the multiplicity $m_Q(0)$ of 0 as an eigenvalue of the quotient matrix Q . By Lemma 3.3, we have $\text{nullity}(Q) = 2$, so $m_Q(0) = 2$. Consequently,

$$M_0 = M_0^{(\text{coclique})} + m_Q(0) = (pq(p+q-2) - 4) + 2 = pq(p+q-2) - 2.$$

Multiplicity of the eigenvalue -1 .

The union $B_3 \cup B_6 \cup B_7$ induces a clique of size

$$n = |B_3| + |B_6| + |B_7| = (p-1)(q-1) + (q-1) + (p-1) = pq - 1,$$

and an isolated K_n has eigenvalue -1 with multiplicity $n - 1 = pq - 2$ [6]. The clique is coupled to the independent blocks by exactly two distinct connection patterns: B_1 and B_4 connect exclusively to B_7 , while B_2 and B_5 connect exclusively to B_6 . The incidence matrix of this bipartite coupling has rank 2, so by Lemma 3.1 at most two of the -1 eigenvalues of the isolated clique can be shifted.

On the other hand, the total number of vertices is

$$|V| = pq(p+q-1) - 1,$$

and from the previous computation we already know that the multiplicity of the eigenvalue 0 is

$$M_0 = pq(p+q-2) - 2,$$

while the quotient matrix Q contributes five additional nonzero eigenvalues $\lambda_1, \dots, \lambda_5$. Therefore the remaining eigenvalues must equal -1 , and their multiplicity is

$$M_{-1} = |V| - M_0 - 5 = (pq(p+q-1) - 1) - (pq(p+q-2) - 2) - 5 = pq - 4.$$

Thus the eigenvalue -1 has multiplicity $M_{-1} = pq - 4$.

Step 2. Characteristic polynomial and eigenvalue structure of the quotient matrix.

The quotient matrix Q is given explicitly in (3). By construction each entry Q_{ij} equals the number of neighbors in B_j of any fixed vertex in B_i ; see 2.3 for the block adjacency rules. Although Q need not be symmetric, it satisfies the balance relation

$$|B_i|Q_{ij} = |B_j|Q_{ji} \quad (1 \leq i, j \leq 7),$$

hence Q is symmetrizable and therefore has real eigenvalues.

Since Q is a 7×7 matrix, it has seven eigenvalues. Lemma 3.3 shows that $\text{nullity}(Q) = 2$, so Q has exactly two zero eigenvalues. Consequently,

$$\text{Spec}(Q) = \{0, 0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\},$$

where the five remaining eigenvalues $\lambda_1, \dots, \lambda_5$ are nonzero.

These seven eigenvalues are the roots of the characteristic polynomial

$$P_Q(\lambda) = \det(\lambda I_7 - Q) = \lambda^7 + c_6\lambda^6 + c_5\lambda^5 + c_4\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0,$$

and the coefficients $c_0, \dots, c_6$ are determined via the Newton–Girard identities [6, Chapter 1].

Formulas for the coefficients.

Let $\lambda_1, \dots, \lambda_7$ be the eigenvalues of Q and set $p_k = \sum_{i=1}^7 \lambda_i^k = \text{tr}(Q^k)$. Then the coefficients c_j are given by the Newton–Girard relations. Writing the characteristic polynomial in terms of the elementary symmetric polynomials e_j , we have

$$\chi_Q(\lambda) = \lambda^7 - e_1\lambda^6 + e_2\lambda^5 - e_3\lambda^4 + e_4\lambda^3 - e_5\lambda^2 + e_6\lambda - e_7.$$

Consequently, the coefficients satisfy $c_k = (-1)^k e_k$, yielding:

$$\begin{aligned} c_6 &= -e_1 = -\text{tr}(Q), \\ c_5 &= e_2 = \frac{1}{2}(p_1^2 - p_2), \\ c_4 &= -e_3 = -\frac{1}{6}(p_1^3 - 3p_1p_2 + 2p_3), \\ c_3 &= e_4 = \frac{1}{24}(p_1^4 - 6p_1^2p_2 + 3p_2^2 + 8p_1p_3 - 6p_4), \\ c_2 &= -e_5 = -\frac{1}{120}(p_1^5 - 10p_1^3p_2 + 15p_1p_2^2 + 20p_1^2p_3 - 20p_2p_3 - 30p_1p_4 + 24p_5), \\ c_1 &= e_6 = \frac{1}{720}(p_1^6 - 15p_1^4p_2 + 40p_1^3p_3 + 45p_1^2p_2^2 - 120p_1p_2p_3 + 40p_2^3 - 90p_1^2p_4 + 144p_1p_5 - 15p_2^3 + 90p_2p_4 - 120p_6), \end{aligned}$$

where $p_k = \text{tr}(Q^k)$ for $1 \leq k \leq 6$.

A symbolic computation confirms the following closed forms of the traces and coefficients.

$$\begin{aligned} \text{tr}(Q) &= pq - 4, \\ \text{tr}(Q^2) &= 5p^2q^2 - 6p^2q + 2p^2 - 6pq^2 + 6pq - 4p + 4q^2 - 10q + 12, \\ \text{tr}(Q^3) &= p^3q^3 + 6p^3q^2 - 6p^3q - 33p^2q^2 + 39p^2q - 12p^2 + 36pq^2 - 51pq + 27p + 3q^3 - 30q^2 + 54q - 37, \\ \text{tr}(Q^4) &= 4p^5q^2 - 8p^5q + 4p^5 + 5p^4q^4 - 4p^4q^3 - 14p^4q^2 + 32p^4q - 18p^4 - 12p^3q^4 + 4p^3q^3 - 4p^3q^2 - 20p^3q \\ &\quad + 24p^3 + 18p^2q^4 - 36p^2q^3 + 150p^2q^2 - 156p^2q + 48p^2 - 12pq^4 + 40pq^3 - 200pq^2 + 268pq - 128p \\ &\quad + 8q^4 - 52q^3 + 186q^2 - 252q + 128, \\ \text{tr}(Q^5) &= 5p^6q^3 - 10p^6q^2 + 5p^6q + p^5q^5 + 25p^5q^4 - 70p^5q^3 + 55p^5q^2 + 10p^5q - 20p^5 - 135p^4q^4 + 295p^4q^3 \\ &\quad - 205p^4q^2 - 45p^4q + 80p^4 + 250p^3q^4 - 545p^3q^3 + 525p^3q^2 - 115p^3q - 75p^3 + 10p^2q^5 - 320p^2q^4 \\ &\quad + 825p^2q^3 - 1250p^2q^2 + 900p^2q - 245p^2 - 10pq^5 + 210pq^4 - 665pq^3 + 1355pq^2 - 1350pq + 540p \\ &\quad + 10q^5 - 125q^4 + 480q^3 - 1020q^2 + 1055q - 434, \end{aligned}$$

$$\begin{aligned} \text{tr}(Q^6) = & 18p^7q^4 - 54p^7q^3 + 60p^7q^2 - 30p^7q + 6p^7 + 5p^6q^6 + 12p^6q^5 - 114p^6q^4 + 264p^6q^3 - 282p^6q^2 + 156p^6q \\ & - 40p^6 - 18p^5q^6 - 60p^5q^5 + 138p^5q^4 - 222p^5q^3 + 384p^5q^2 - 414p^5q + 180p^5 + 42p^4q^6 + 24p^4q^5 + 567p^4q^4 \\ & - 1338p^4q^3 + 849p^4q^2 + 300p^4q - 384p^4 - 54p^3q^6 + 96p^3q^5 - 1686p^3q^4 + 3960p^3q^3 - 3840p^3q^2 + 1236p^3q \\ & + 128p^3 + 60p^2q^6 - 354p^2q^5 + 2730p^2q^4 - 6564p^2q^3 + 8103p^2q^2 - 4974p^2q + 1239p^2 - 36pq^6 + 312pq^5 \\ & - 2064pq^4 + 5460pq^3 - 8160pq^2 + 6474pq - 2178p + 19q^6 - 216q^5 + 1206q^4 - 3304q^3 + 5154q^2 - 4260q + 1467. \end{aligned}$$

Coefficients of the characteristic polynomial.

$$\begin{aligned} c_6 = e_1 &= 4 - pq, \\ c_5 = e_2 &= -2p^2q^2 + 3p^2q - p^2 + 3pq^2 - 7pq + 2p - 2q^2 + 5q + 2, \\ c_4 = e_3 &= -2p^3q^3 + 5p^3q^2 - 3p^3q + 3p^2q^3 - 6p^2q^2 + 3p^2q - 2pq^3 + 5pq^2 - 3pq + p + q^3 - 2q^2 - 2q + 1, \\ c_3 = e_4 &= -p^5q^2 + 2p^5q - p^5 + p^4q^4 - 3p^4q^3 + 8p^4q^2 - 11p^4q + 5p^4 - 3p^3q^4 + 11p^3q^3 - 22p^3q^2 + 22p^3q \\ &\quad - 8p^3 + 4p^2q^4 - 16p^2q^3 + 25p^2q^2 - 17p^2q + 4p^2 - 2pq^4 + 9pq^3 - 13pq^2 + 6pq - q^3 + 2q^2 + q - 2, \\ c_2 = e_5 &= p^5q^5 - 4p^5q^4 + 6p^5q^3 - 4p^5q^2 + p^5q - 3p^4q^5 + 11p^4q^4 - 15p^4q^3 + 9p^4q^2 - 2p^4q + 4p^3q^5 - 15p^3q^4 \\ &\quad + 21p^3q^3 - 13p^3q^2 + 3p^3q - 3p^2q^5 + 12p^2q^4 - 17p^2q^3 + 10p^2q^2 - 2p^2q + pq^5 - 4pq^4 + 5pq^3 - 2pq^2, \\ c_1 = e_6 &= 0. \end{aligned} \tag{5}$$

Coefficient c_0 (Constant term).

By the discussion above, the constant term c_0 of $\chi_Q(\lambda) = \det(\lambda I_7 - Q)$ satisfies $c_0 = (-1)^7 \det(Q)$ and vanishes precisely when 0 is an eigenvalue of Q . Lemma 3.3 shows $\det(Q) = 0$, hence $c_0 = 0$.

Step 3. Spectral completeness and verification.

By the fundamental theorem for equitable partitions, the spectrum of A is the multiset union of (i) the eigenvalues produced by the block structure of the equitable partition, and (ii) the eigenvalues of the quotient matrix Q [6, 9]. From Step 1 we have

$$M_0 = pq(p + q - 2) - 2, \quad M_{-1} = pq - 4,$$

corresponding to the high-multiplicity eigenvalues 0 and -1 .

From Step 2, the quotient matrix Q has the spectral decomposition

$$\text{Spec}(Q) = \{0, 0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\},$$

in which exactly two eigenvalues are zero (which are counted in M_0), and the five remaining eigenvalues $\lambda_1, \dots, \lambda_5$ are nonzero.

Thus the total number of eigenvalues of A , counted with multiplicities, is

$$\begin{aligned} \# \text{Spec}(A) &= M_0 + M_{-1} + 5 \\ &= [pq(p + q - 2) - 2] + (pq - 4) + 5 \\ &= pq(p + q - 1) - 1. \end{aligned}$$

Finally, the graph $TT(\mathbb{Z}_{p^2q^2})$ has

$$|V| = pq(p + q - 1) - 1$$

vertices (see 2.3), so the total number of eigenvalues agrees exactly with the vertex count. This confirms that no eigenvalue has been omitted and that the spectral decomposition stated in Theorem 3.2 is complete. $\square$

Lemma 3.3. For the quotient matrix Q defined in (3) we have

$$\text{nullity}(Q) = m_Q(0) = 2.$$

In particular, Q has exactly two linearly independent eigenvectors corresponding to the eigenvalue 0.

Proof. Inspection of Q in (3) shows that

$$\text{row}_1 = \text{row}_3 = (0, 0, 0, 0, 0, 0, p-1), \quad \text{row}_2 = \text{row}_4 = (0, 0, 0, 0, 0, q-1, 0).$$

Hence $\text{row}_1 - \text{row}_3 = 0$ and $\text{row}_2 - \text{row}_4 = 0$, which are two independent linear relations among the rows. In particular, this implies $\text{rank}(Q) \leq 5$, and hence $\text{nullity}(Q) \geq 2$.

On the other hand, the characteristic polynomial of Q has the form

$$\chi_Q(\lambda) = \det(\lambda I_7 - Q) = \lambda^2(\lambda^5 + c_6\lambda^4 + c_5\lambda^3 + c_4\lambda^2 + c_3\lambda + c_2),$$

with $c_2, \dots, c_6$ polynomials in p and q that do not force any additional factor of λ . In particular, 0 is an eigenvalue of Q with algebraic multiplicity exactly 2. Therefore $\text{nullity}(Q) = 2$ and $\text{rank}(Q) = 5$, which yields $m_Q(0) = 2$, as claimed. $\square$

3.2. Illustrative example: $TT(\mathbb{Z}_{36})$

We now work through the case $p = 2$ and $q = 3$, which corresponds to the ring $\mathbb{Z}_{36}$. The total zero-divisor graph of this ring has 23 non-zero zero-divisor vertices, partitioned into seven blocks whose cardinalities are:

$$\begin{aligned} |B_1| &= q(p-1)(q-1) = 3 \cdot 1 \cdot 2 = 6, \\ |B_2| &= p(p-1)(q-1) = 2 \cdot 1 \cdot 2 = 4, \\ |B_3| &= (p-1)(q-1) = 1 \cdot 2 = 2, \\ |B_4| &= q(q-1) = 3 \cdot 2 = 6, \\ |B_5| &= p(p-1) = 2 \cdot 1 = 2, \\ |B_6| &= q-1 = 2, \\ |B_7| &= p-1 = 1. \end{aligned}$$

The total number of vertices is

$$\sum_{i=1}^7 |B_i| = 6 + 4 + 2 + 6 + 2 + 2 + 1 = 23,$$

which matches the general formula

$$pq(p+q-1) - 1 = 2 \cdot 3 \cdot 4 - 1 = 23.$$

Eigenvalue multiplicities. From Theorem 3.2, we have

$$\begin{aligned} M_0 &= pq(p+q-2) - 2 = 2 \cdot 3 \cdot 3 - 2 = 16, \\ M_{-1} &= pq - 4 = 2 \cdot 3 - 4 = 2. \end{aligned}$$

Quotient matrix. The 7×7 quotient matrix Q is

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 4 & 2 & 1 & 0 & 1 & 1 \\ 6 & 0 & 6 & 0 & 2 & 2 & 0 \end{bmatrix}.$$

Spectral structure of Q . Since Q is a 7×7 matrix, it has seven eigenvalues. By Lemma 3.3, its nullity is 2, so Q has exactly two zero eigenvalues. Thus

$$\text{Spec}(Q) = \{0, 0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\},$$

where the five eigenvalues $\lambda_1, \dots, \lambda_5$ are non-zero.

Trace computations. Computing traces of successive powers of Q gives:

$$\begin{aligned}\text{tr}(Q) &= 2, \\ \text{tr}(Q^2) &= 54, \\ \text{tr}(Q^3) &= 68, \\ \text{tr}(Q^4) &= 930, \\ \text{tr}(Q^5) &= 2292, \\ \text{tr}(Q^6) &= 19338.\end{aligned}$$

These agree with the general formulas in Theorem 3.2.

Characteristic polynomial coefficients. Substituting these values into the general expressions (5) yields:

$$\begin{aligned}c_6 &= -2, \\ c_5 &= -25, \\ c_4 &= 30, \\ c_3 &= 124, \\ c_2 &= -120, \\ c_1 &= 0, \\ c_0 &= 0.\end{aligned}$$

Therefore, the characteristic polynomial of Q is

$$\chi_Q(\lambda) = \lambda^7 - 2\lambda^6 - 25\lambda^5 + 30\lambda^4 + 124\lambda^3 - 120\lambda^2.$$

4. Graph invariants and spectral properties

This section examines key spectral invariants of the adjacency matrix and their significance for understanding the structural properties of $T\Gamma(\mathbb{Z}_{p^2q^2})$. We focus on the graph energy, a fundamental measure in spectral graph theory, and show how the eigenvalue decomposition obtained in Theorem 3.2 constrains this important quantity.

4.1. Graph energy: definition and computation

The graph energy $E(G)$ is defined as the sum of the absolute values of all eigenvalues of the adjacency matrix:

$$E(G) = \sum_{i=1}^n |\lambda_i|,$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A (with multiplicities).

From Theorem 3.2 we have:

$$\text{Spec}(A) = \{0^{M_0}, (-1)^{M-1}, \lambda_1, \lambda_2, \dots, \lambda_5\},$$

where: $M_0 = pq(p+q-2) - 2$ zero eigenvalues, $M_{-1} = pq - 4$ eigenvalues equal to -1 , and the remaining five nonzero eigenvalues $\lambda_1, \dots, \lambda_5$ come from the spectral structure of the quotient matrix Q , which has

$$\text{Spec}(Q) = \{0, 0, \lambda_1, \dots, \lambda_5\}.$$

Therefore, the energy of $TT(\mathbb{Z}_{p^2q^2})$ is

$$E(G) = M_0 \cdot |0| + M_{-1} \cdot |-1| + \sum_{i=1}^5 |\lambda_i| = (pq - 4) + \sum_{i=1}^5 |\lambda_i|. \quad (6)$$

4.2. Energy bounds

The energy of the graph can be bounded using standard spectral inequalities. For connected graphs, these bounds reveal how energy grows with the parameters p and q .

Theorem 4.1. *If $TT(\mathbb{Z}_{p^2q^2})$ is connected, then its graph energy satisfies:*

$$\sqrt{2(pq(p+q-1)-2)} \leq E(G) \leq (pq-4) + \sqrt{5 \sum_{i=1}^5 \lambda_i^2}.$$

Proof. Hong's inequality [11] states that any connected graph on n vertices satisfies

$$E(G) \geq \sqrt{2n-2}.$$

For $TT(\mathbb{Z}_{p^2q^2})$ we have

$$n = |V| = pq(p+q-1) - 1,$$

and hence

$$E(G) \geq \sqrt{2(pq(p+q-1)-2)}.$$

For the upper bound, apply the Cauchy-Schwarz inequality to the five nonzero eigenvalues $\{\lambda_1, \dots, \lambda_5\}$ of Q :

$$\left(\sum_{i=1}^5 |\lambda_i| \right)^2 \leq \left(\sum_{i=1}^5 |\lambda_i|^2 \right) \left(\sum_{i=1}^5 1^2 \right) = 5 \sum_{i=1}^5 \lambda_i^2.$$

Taking square roots,

$$\sum_{i=1}^5 |\lambda_i| \leq \sqrt{5 \sum_{i=1}^5 \lambda_i^2}.$$

Substituting into the energy formula (6) gives:

$$E(G) = (pq-4) + \sum_{i=1}^5 |\lambda_i| \leq (pq-4) + \sqrt{5 \sum_{i=1}^5 \lambda_i^2}.$$

□

5. Conclusion

The results of this work show that the ring $\mathbb{Z}_{p^2q^2}$ produces a highly structured and interpretable spectral pattern for its total zero-divisor graph. Using the equitable partition established in earlier studies, we obtained the complete adjacency spectrum and uncovered its clear split into two dominant integer eigenvalues and five additional nonzero eigenvalues arising from the quotient matrix. This fully determined spectral picture illustrates how the ring's additive and multiplicative features influence the behaviour of the graph. Moreover, the approach developed here demonstrates the usefulness of block-structured adjacency matrices for examining broader classes of total zero-divisor graphs. The framework and findings provide a foundation for future spectral analysis of $TT(\mathbb{Z}_n)$ for other composite values of n and contribute to the wider study of spectra in algebraically defined graphs.

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