

Explicit determinants of homogeneous polynomial evaluation matrices and applications

Research Article

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Abstract: In this work, the determinants of matrices constructed by evaluating homogeneous bivariate polynomials at pairs of vectors are investigated. For a polynomial $p(x, y) = \sum_{i=0}^k \alpha_i x^{k-i} y^i$, an explicit factorization of the determinant of the associated $n \times n$ evaluation matrix

$$A_{\mathbf{a}, \mathbf{b}}(p(x, y)) = (p(a_r, b_s))_{1 \leq r, s \leq n}$$

is presented for all $n \geq k + 1$ and for all pairs of vectors $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ of length n . In particular, it is proved that $\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) = 0$ when $n \geq k + 2$, while in the borderline case $n = k + 1$ a closed formula involving Vandermonde determinants is derived in terms of the vector sets and the coefficients of $p(x, y)$. Several well-known determinants, including those arising from $(x + y)^k$ and classical quotient forms $\frac{a^k - b^k}{a - b}$ and $\frac{a^k + b^k}{a + b}$, emerge as special cases. We also provide a discussion for $n \leq k$, connecting the problem to symmetric functions and generalized Vandermonde determinants. Finally, applications of such matrices and determinants are provided, including an explicit formula and equivariance law under linear change of variables for the sum-form $p(x, y) = f(x + y)$, and a non-vanishing bound over finite fields via the Schwartz-Zippel lemma.

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1. Introduction

Polynomial evaluation matrices arise naturally in many areas of mathematics, including interpolation theory, approximation theory, coding theory, and structured linear algebra (see, e.g., [2, 3, 6]). Given a polynomial $p(x, y)$ and two sets of evaluation points $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$, one may form the matrix

$$A_{\mathbf{a}, \mathbf{b}}(p(x, y)) = (p(a_r, b_s))_{1 \leq r, s \leq n},$$

whose algebraic and spectral properties reflect both the structure of the polynomial and the geometry of the evaluation points.

Classical examples include Vandermonde matrices and their variations, whose determinants encode polynomial interpolation and root separation:

$$V(x_1, \dots, x_n) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} \quad \text{and} \quad \det V(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

More recently, matrices arising from evaluating polynomials of the form $f(x + y)$ or $f(xy)$ have appeared in diverse contexts, ranging from additive convolution operators to coding theory and random matrix models. In particular, determinant formulas for matrices of the form $(f(a_r + b_s))$ have been obtained in special cases, often relying on ad-hoc arguments or strong structural assumptions on f and the evaluation sets. For example, in [8], a family of polynomial evaluation matrices of the form $((a_i + b_j)^k)_{1 \leq i, j \leq n}$ was analyzed, where the determinant was computed via a factorization through Vandermonde matrices. In particular, for the $n \times n$ matrix $C = ((a_i + b_j)^k)_{1 \leq i, j \leq n}$, it was shown that

$$\det(C) = (-1)^{\binom{k+1}{2}} \left(\prod_{i=0}^k \binom{k}{i} \right) \left(\prod_{1 \leq i < j \leq n} (a_j - a_i) \right) \left(\prod_{1 \leq i < j \leq n} (b_j - b_i) \right)$$

when $n = k + 1$, whereas $\det(C) = 0$ for all $n \geq k + 2$. Further extensions and related classes of polynomial evaluation matrices were discussed in [7]. However, existing results are typically subject to one or more limitations: They are restricted to specific polynomial forms, they address only the case $n = k + 1$, or they do not provide a unified explanation for the determinant when the matrix size exceeds the polynomial degree.

The aim of this paper is to develop a systematic and unified framework for analyzing determinants of polynomial evaluation matrices arising from homogeneous bivariate polynomials. Homogeneity is a crucial structural assumption: it confines the polynomial to a finite-dimensional coefficient space and enables the associated evaluation matrix to admit a factorization through structured Vandermonde-type matrices. This factorization serves as the fundamental mechanism driving all of our main results. Rather than focusing on isolated examples, we treat the general homogeneous polynomial

$$p(x, y) = \sum_{i=0}^k \alpha_i x^{k-i} y^i$$

and analyze the corresponding evaluation matrix for all values of n relative to k . This perspective clarifies when determinants are identically zero, when nontrivial closed forms exist, and how these phenomena depend on the support of the coefficient vector (α_i) . Let $\mathbb{C}$ denote the complex field and let $\mathbb{C}[x, y]$ denote the ring of polynomials in two variables over $\mathbb{C}$. For $k \in \mathbb{N} \cup \{0\}$ set

$$\widehat{\mathbb{C}}_k[x, y] := \left\{ p(x, y) = \sum_{i=0}^k \alpha_i x^{k-i} y^i : \alpha_i \in \mathbb{C} \right\},$$

the vector space of homogeneous polynomials of total degree k . We also write

$$\widehat{\mathbb{C}}[x, y] := \bigcup_{k \geq 0} \widehat{\mathbb{C}}_k[x, y].$$

Given vectors $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ in $\mathbb{C}^n$ and a polynomial $p(x, y) \in \widehat{\mathbb{C}}[x, y]$, we focus on the corresponding *polynomial evaluation matrix*

$$A_{\mathbf{a}, \mathbf{b}}(p(x, y)) := (p(a_r, b_s))_{1 \leq r, s \leq n}$$

and its determinant. The study of the determinant of the $n \times n$ matrices $C = ((a_i + b_j)^k)_{1 \leq i, j \leq n}$ in [7] and [8] can be viewed as evaluation matrices in the special case where the polynomial $p(x, y) = (x + y)^k$. The concept of polynomial evaluation matrices can be illustrated in the following example.

Example 1.1. Let $\mathbf{a} = (a_1, a_2, a_3, a_4)$ and $\mathbf{b} = (b_1, b_2, b_3, b_4)$ be vectors in $\mathbb{C}^4$. Let $p(x, y) = (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \in \widehat{\mathbb{C}}[x, y]$. Then

$$A_{\mathbf{a}, \mathbf{b}}(p(x, y)) = \begin{pmatrix} (a_1 + b_1)^3 & (a_1 + b_2)^3 & (a_1 + b_3)^3 & (a_1 + b_4)^3 \\ (a_2 + b_1)^3 & (a_2 + b_2)^3 & (a_2 + b_3)^3 & (a_2 + b_4)^3 \\ (a_3 + b_1)^3 & (a_3 + b_2)^3 & (a_3 + b_3)^3 & (a_3 + b_4)^3 \\ (a_4 + b_1)^3 & (a_4 + b_2)^3 & (a_4 + b_3)^3 & (a_4 + b_4)^3 \end{pmatrix}.$$

Similarly, for $q(x, y) = x^3 + xy^2 + y^3 \in \widehat{\mathbb{C}}[x, y]$, we have

$$A_{\mathbf{a}, \mathbf{b}}(q(x, y)) = \begin{pmatrix} a_1^3 + a_1b_1^2 + b_1^3 & a_1^3 + a_1b_2^2 + b_2^3 & a_1^3 + a_1b_3^2 + b_3^3 & a_1^3 + a_1b_4^2 + b_4^3 \\ a_2^3 + a_2b_1^2 + b_1^3 & a_2^3 + a_2b_2^2 + b_2^3 & a_2^3 + a_2b_3^2 + b_3^3 & a_2^3 + a_2b_4^2 + b_4^3 \\ a_3^3 + a_3b_1^2 + b_1^3 & a_3^3 + a_3b_2^2 + b_2^3 & a_3^3 + a_3b_3^2 + b_3^3 & a_3^3 + a_3b_4^2 + b_4^3 \\ a_4^3 + a_4b_1^2 + b_1^3 & a_4^3 + a_4b_2^2 + b_2^3 & a_4^3 + a_4b_3^2 + b_3^3 & a_4^3 + a_4b_4^2 + b_4^3 \end{pmatrix}.$$

In this paper, a systematic and unified study of polynomial evaluation matrices associated with homogeneous bivariate polynomials is undertaken, encompassing and extending classical Vandermonde and sum-form constructions. A general factorization of the evaluation matrix $A_{\mathbf{a}, \mathbf{b}}(p(x, y))$ into structured Vandermonde-type matrices and a coefficient-dependent core matrix is established for all homogeneous polynomials of fixed degree k . This factorization allows us to prove that the determinant is identically zero whenever $n \geq k + 2$. In the borderline square case $n = k + 1$, explicit determinant formulas are obtained, including a closed-form expression for sum-form polynomials of the type $p(x, y) = f(x + y)$. Equivariance properties under linear change of variables are also analyzed. Further applications are developed over finite fields, where probabilistic nonvanishing bounds are derived via the Schwartz–Zippel lemma. A discussion for $n \leq k$, connecting the problem to symmetric functions and generalized Vandermonde determinants is given as well.

The paper is organized as follows: In Section 2, the basic factorization of evaluation matrices is presented together with explicit determinant formulas in the square case $n = k + 1$ and the rectangular case $n \leq k$. In Section 3, various applications of such matrices and determinants are discussed, including the sum-form $p(x, y) = f(x + y)$, equivariance under linear change of variables, and behavior over finite fields. Finally, a summary is given in Section 4.

2. Determinants of polynomial evaluation matrices

In this section, a factorization of the polynomial evaluation matrix $A_{\mathbf{a}, \mathbf{b}}(p(x, y))$ through Vandermonde-type matrices is established. This factorization reveals the behavior of the matrix according

to the size parameter n relative to the degree k : the cases $n < k + 1$, $n = k + 1$, and $n \geq k + 2$ exhibit fundamentally different rank and determinant properties. As immediate consequences, explicit rank bounds are obtained and closed-form determinant formulas are derived in the square case $n = k + 1$, while several special configurations are examined in greater detail. When $n \geq k + 2$, the matrix $A_{\mathbf{a}, \mathbf{b}}(p(x, y))$ necessarily has rank at most $k + 1$ and is therefore singular. For the general regime $n < k + 1$, the Cauchy–Binet formula together with generalized Vandermonde identities is applied to express $\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y)))$ in a structured and computable form.

2.1. Factorization of polynomial evaluation matrices

Fix $k \geq 0$ and $p(x, y) = \sum_{i=0}^k \alpha_i x^{k-i} y^i \in \widehat{\mathbb{C}}_k[x, y]$. Given vectors $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ in $\mathbb{C}^n$, define the $n \times (k + 1)$ Vandermonde-type matrices

$$V_{\mathbf{a}}^{(k)} := (a_r^{k-j})_{1 \leq r \leq n, 0 \leq j \leq k} = \begin{pmatrix} a_1^k & a_1^{k-1} & \cdots & a_1^0 \\ a_2^k & a_2^{k-1} & \cdots & a_2^0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n^k & a_n^{k-1} & \cdots & a_n^0 \end{pmatrix},$$

and

$$W_{\mathbf{b}}^{(k)} := (b_s^j)_{1 \leq s \leq n, 0 \leq j \leq k} = \begin{pmatrix} b_1^0 & b_1^1 & \cdots & b_1^k \\ b_2^0 & b_2^1 & \cdots & b_2^k \\ \vdots & \vdots & \ddots & \vdots \\ b_n^0 & b_n^1 & \cdots & b_n^k \end{pmatrix}.$$

By expanding $p(a_r, b_s)$ we obtain the matrix identity

$$A_{\mathbf{a}, \mathbf{b}}(p(x, y)) = V_{\mathbf{a}}^{(k)} D_{\alpha} \left(W_{\mathbf{b}}^{(k)} \right)^T, \quad (1)$$

where $D_{\alpha} = \text{diag}(\alpha_0, \alpha_1, \dots, \alpha_k)$ is given from the coefficients of $p(x, y)$.

Example 2.1. For $k = 3$ and $n = 3$, let $p(x, y) = (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ and let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in $\mathbb{C}^3$. Then $D_{\alpha} = \text{diag}(1, 3, 3, 1)$,

$$V_{\mathbf{a}}^{(3)} = \begin{pmatrix} a_1^3 & a_1^2 & a_1^1 & a_1^0 \\ a_2^3 & a_2^2 & a_2^1 & a_2^0 \\ a_3^3 & a_3^2 & a_3^1 & a_3^0 \end{pmatrix}, \quad \text{and} \quad W_{\mathbf{b}}^{(3)} = \begin{pmatrix} b_1^0 & b_1^1 & b_1^2 & b_1^3 \\ b_2^0 & b_2^1 & b_2^2 & b_2^3 \\ b_3^0 & b_3^1 & b_3^2 & b_3^3 \end{pmatrix}.$$

Hence,

$$A_{\mathbf{a}, \mathbf{b}}(p(x, y)) = V_{\mathbf{a}}^{(3)} \text{diag}(1, 3, 3, 1) \left(W_{\mathbf{b}}^{(3)} \right)^T.$$

In the special case where $\alpha_i = \binom{k}{i}$, it was shown in [8, Proposition 1] that $\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) = 0$ for all $n \geq k + 2$. We now present the following proposition, which gives a more general result for an arbitrary polynomial $p(x, y) \in \widehat{\mathbb{C}}_k[x, y]$.

Proposition 2.2. Let $k \geq 0$ and $n \geq 1$ be integers and let $p(x, y) \in \widehat{\mathbb{C}}_k[x, y]$. Then

$$\text{rank}(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) \leq k + 1.$$

In particular, if $n \geq k + 2$, then $\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) = 0$.

Proof. The factorization (1) expresses A as a product

$$A_{\mathbf{a},\mathbf{b}}(p(x,y)) = V_{\mathbf{a}}^{(k)} D_{\alpha} \left(W_{\mathbf{b}}^{(k)}\right)^T,$$

where $V_{\mathbf{a}}^{(k)}$, D_{α} , and $\left(W_{\mathbf{b}}^{(k)}\right)^T$ are an $n \times (k+1)$ matrix, a $(k+1) \times (k+1)$ matrix, and a $(k+1) \times n$ matrix, respectively. By the rank inequality for matrix products, it follows that

$$\text{rank}(A_{\mathbf{a},\mathbf{b}}(p(x,y))) \leq \min\{\text{rank}(V_{\mathbf{a}}^{(k)}), \text{rank}(D_{\alpha}), \text{rank}\left(\left(W_{\mathbf{b}}^{(k)}\right)^T\right)\} \leq k+1.$$

If $n \geq k+2$, an $n \times n$ matrix of rank less than or equal to $k+1$ is always singular. $\square$

2.2. The borderline case $n = k+1$

In this subsection, we focus on the case where $n = k+1$. From (1),

$$A_{\mathbf{a},\mathbf{b}}(p(x,y)) = V_{\mathbf{a}}^{(k)} D_{\alpha} \left(W_{\mathbf{b}}^{(k)}\right)^T,$$

which becomes a product of three square matrices. The determinants can then be computed explicitly in terms of Vandermonde products.

In the special case $n = k+1$ and $\alpha_i = \binom{k}{i}$, an explicit formula for $\det(A_{\mathbf{a},\mathbf{b}}(p(x,y)))$ was given in [8, Proposition 2]. The following theorem extends this result to all polynomials $p(x,y) \in \widehat{\mathbb{C}}_k[x,y]$.

Theorem 2.3. *Let $k \geq 0$ and $n = k+1$ be integers and let $p(x,y) = \sum_{i=0}^k \alpha_i x^{k-i} y^i \in \widehat{\mathbb{C}}_k[x,y]$. Then*

$$\det(A_{\mathbf{a},\mathbf{b}}(p(x,y))) = (-1)^{\binom{k+1}{2}} \left(\prod_{i=0}^k \alpha_i\right) \left(\prod_{1 \leq i < j \leq n} (a_j - a_i)\right) \left(\prod_{1 \leq i < j \leq n} (b_j - b_i)\right),$$

where $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ in $\mathbb{C}^n$. In particular, $\det(A_{\mathbf{a},\mathbf{b}}(p(x,y))) \neq 0$ if and only if $\alpha_i \neq 0$ for all i and the vector sets $\{a_i\}$ and $\{b_i\}$ are pairwise distinct.

Proof. From (1), we have

$$\det(A) = \det\left(V_{\mathbf{a}}^{(k)}\right) \cdot \det(D_{\alpha}) \cdot \det\left(W_{\mathbf{b}}^{(k)}\right)^T,$$

where $D_{\alpha} = \text{diag}(\alpha_0, \alpha_1, \dots, \alpha_k)$. We note that $W_{\mathbf{b}}^{(k)}$ is the standard $(k+1) \times (k+1)$ Vandermonde matrix which implies that

$$\det(W_{\mathbf{b}}^{(k)}) = \prod_{1 \leq i < j \leq n} (b_j - b_i).$$

The matrix $V_{\mathbf{a}}^{(k)}$ has columns in reverse order of the standard $(k+1) \times (k+1)$ Vandermonde matrix which requires $\binom{k+1}{2}$ adjacent transpositions of the standard order, and hence

$$\det\left(V_{\mathbf{a}}^{(k)}\right) = (-1)^{\binom{k+1}{2}} \prod_{1 \leq i < j \leq n} (a_j - a_i).$$

Finally, $\det(D_{\alpha}) = \prod_{i=0}^k \alpha_i$. Multiplying the three determinants gives the claim. $\square$

Remark 2.4. For $c \in \mathbb{C}$ and $p(x, y) = c \cdot q(x, y)$ with $q(x, y) \in \widehat{\mathbb{C}}_k[x, y]$, then $D_\alpha = cD_\beta$, where α and β are the vector coefficients of $p(x, y)$ and $q(x, y)$, respectively. Hence, $\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) = c^n \det(A_{\mathbf{a}, \mathbf{b}}(q(x, y)))$. In particular, the determinant is multi-linear in the columns of D_α but not linear in $p(x, y)$ as a polynomial.

In each of the following special cases, we use Theorem 2.3 when $n = k + 1$, and apply Proposition 2.2 for the vanishing when $n \geq k + 2$.

Corollary 2.5 ([8, Propositions 1 and 2]). Let $k \geq 0$ and $n \geq 1$ be integers and let $p(x, y) = (x + y)^k = \sum_{i=0}^k \binom{k}{i} x^{k-i} y^i$. Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ in $\mathbb{C}^n$. Then the following statements hold.

1. If $n \geq k + 2$, then $\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) = 0$.
2. If $n = k + 1$, then

$$\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) = (-1)^{\binom{k+1}{2}} \left(\prod_{i=0}^k \binom{k}{i} \right) \left(\prod_{1 \leq i < j \leq n} (a_j - a_i) \right) \left(\prod_{1 \leq i < j \leq n} (b_j - b_i) \right).$$

In particular, $\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) \neq 0$ if and only if the a_i 's are pairwise distinct and the b_j 's are pairwise distinct.

Proof. We note that for each $i \in \{0, 1, \dots, k\}$, we have $\alpha_i = \binom{k}{i}$ which implies that $\det(D_\alpha) = \prod_{i=0}^k \binom{k}{i}$. The result follows from Theorem 2.3. $\square$

Corollary 2.6. Let $k \geq 0$ and $n \geq 1$ be integers and let $p(x, y) = \sum_{i=0}^k x^{k-i} y^i$. Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ in $\mathbb{C}^n$. Then the following statements hold.

1. If $n \geq k + 2$, then $\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) = 0$.
2. If $n = k + 1$, then

$$\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) = (-1)^{\binom{k+1}{2}} \left(\prod_{1 \leq i < j \leq n} (a_j - a_i) \right) \left(\prod_{1 \leq i < j \leq n} (b_j - b_i) \right).$$

Moreover, we have the identity

$$p(a_r, b_s) = \frac{a_r^{k+1} - b_s^{k+1}}{a_r - b_s} \quad (a_r \neq b_s),$$

which implies that $A_{\mathbf{a}, \mathbf{b}}(p(x, y))$ agrees with $p(a_r, b_s) = \left(\frac{a_r^{k+1} - b_s^{k+1}}{a_r - b_s} \right)$ on the domain $a_r \neq b_s$.

Proof. We note that for each $i \in \{0, 1, \dots, k\}$, we have $\alpha_i = 1$ which implies that $\det(D_\alpha) = 1$. The result follows from Theorem 2.3. $\square$

Corollary 2.7. Let $k \geq 0$ and $n \geq 1$ be integers and let $p(x, y) = \sum_{i=0}^k (-1)^i x^{k-i} y^i$. Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ in $\mathbb{C}^n$. Then the following statements hold.

1. If $n \geq k + 2$, then $\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) = 0$.

2. If $n = k + 1$, then

$$\det(A_{\mathbf{a},\mathbf{b}}(p(x, y))) = \left(\prod_{1 \leq i < j \leq n} (a_j - a_i) \right) \left(\prod_{1 \leq i < j \leq n} (b_j - b_i) \right).$$

Moreover, we have the identity

$$p(a_r, b_s) = \frac{a_r^{k+1} - (-b_s)^{k+1}}{a_r + b_s} \quad (a_r \neq -b_s),$$

which implies that $A_{\mathbf{a},\mathbf{b}}(p(x, y))$ agrees with $\left(\frac{a_r^{k+1} - (-b_s)^{k+1}}{a_r + b_s} \right)$ on the domain $a_r \neq -b_s$.

Proof. We note that for each $i \in \{0, 1, \dots, k\}$, we have $\alpha_i = (-1)^i$ which implies that $\det(D_\alpha) = \prod_{i=0}^k (-1)^i = (-1)^{\binom{k+1}{2}}$ which cancels with the same power from $\det(V_{\mathbf{a}}^{(k)})$ to give the stated sign. The result follows from Theorem 2.3. $\square$

2.3. The case $n \leq k$

In this subsection, we focus on the case where $n \leq k$. However, the results hold true for $n = k + 1$ as well. Therefore, the results are written in terms of $n \leq k + 1$ if there is no ambiguity.

For an $m \times n$ matrix M and an index set $I \subseteq \{1, 2, \dots, n\}$ of increasing order, let $M(:, I)$ denote the submatrix of M formed by the columns with indices in I .

Theorem 2.8 ([4, Theorem 3.A.5 (Cauchy-Binet Formula)]). *Let $m \leq n$ be positive integers. Let A and B be $m \times n$ and $n \times m$ matrices, respectively. Then*

$$\det(AB) = \sum_{I=\{1 \leq i_1 < i_2 < \dots < i_m \leq n\}} \det(A(:, I)) \det(B(:, I)).$$

From (1), we recall the factorization $A_{\mathbf{a},\mathbf{b}}(p(x, y)) = V_{\mathbf{a}}^{(k)} D_\alpha (W_{\mathbf{b}}^{(k)})^T$ with

$$V_{\mathbf{a}}^{(k)} = (a_r^{k-j})_{1 \leq r \leq n, 0 \leq j \leq k}, \quad W_{\mathbf{b}}^{(k)} = (b_s^j)_{1 \leq s \leq n, 0 \leq j \leq k}, \quad \text{and } D_\alpha = \text{diag}(\alpha_0, \dots, \alpha_k).$$

Assume that $n \leq k + 1$. It follows that $V_{\mathbf{a}}^{(k)}$ and $W_{\mathbf{b}}^{(k)}$ are rectangular with the number of columns is greater than or equal to the number of rows. Applying the Cauchy-Binet formula in Theorem 2.8 to the product $V_{\mathbf{a}}^{(k)} D_\alpha (W_{\mathbf{b}}^{(k)})^T$, we have

$$\det(A_{\mathbf{a},\mathbf{b}}(p(x, y))) = \sum_{I=\{0 \leq i_1 < i_2 < \dots < i_n \leq k\}} \det(V_{\mathbf{a}}^{(k)}(:, I)) \left(\prod_{i \in I} \alpha_i \right) \det(W_{\mathbf{b}}^{(k)}(:, I)). \quad (2)$$

For $m \in \mathbb{Z}$ and variables $\mathbf{x} = (x_1, \dots, x_n)$ define

$$H_0(\mathbf{x}) = 1, \quad H_m(\mathbf{x}) = 0 \quad (m < 0),$$

and for $m \geq 1$ recursively

$$H_m(x_1, \dots, x_n) = H_m(x_1, \dots, x_{n-1}) + x_n H_{m-1}(x_1, \dots, x_n). \quad (3)$$

Proposition 2.9 ([5, Page 41 (3.6)]). For $e_1 > \cdots > e_n \geq 0$,

$$\det \left((a_r^{e_j})_{1 \leq r, j \leq n} \right) = \prod_{1 \leq r < r' \leq n} (a_{r'} - a_r) \det \left((H_{e_j - (n-j)}(\mathbf{a}))_{1 \leq j \leq n} \right).$$

Theorem 2.10. Let $1 \leq n \leq k+1$ be integers and let $p(x, y) = \sum_{i=0}^k \alpha_i x^{k-i} y^i$. Then

$$\begin{aligned} \det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) &= \prod_{1 \leq r < r' \leq n} (a_{r'} - a_r) \prod_{1 \leq r < r' \leq n} (b_{r'} - b_r) \\ &\times \sum_{I=\{0 \leq i_1 < i_2 < \cdots < i_n \leq k\}} \left(\prod_{i \in I} \alpha_i \right) \det \left((H_{k-i_j - (n-j)}(\mathbf{a}))_{1 \leq j \leq n} \right) \det \left((H_{i_j - (n-j)}(\mathbf{b}))_{1 \leq j \leq n} \right), \end{aligned}$$

where H_m are as in (3).

Proof. By (2), it follows that

$$\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) = \sum_{I=\{0 \leq i_1 < i_2 < \cdots < i_n \leq k\}} \det \left(V_{\mathbf{a}}^{(k)}(:, I) \right) \left(\prod_{i \in I} \alpha_i \right) \det \left(W_{\mathbf{b}}^{(k)}(:, I) \right),$$

where the sum runs over n -subsets $I = \{0 \leq i_1 < i_2 < \cdots < i_n \leq k\}$. For each $j \in \{1, 2, \dots, n\}$, let $e_j = k - i_j$. Applying Proposition 2.9 to the exponent e_j in $V_{\mathbf{a}}^{(k)}(:, I)$ yields

$$\begin{aligned} \det(V_{\mathbf{a}}^{(k)}(:, I)) &= \prod_{1 \leq r < r' \leq n} (a_{r'} - a_r) \det \left((H_{e_j - (n-j)}(\mathbf{a}))_{1 \leq j \leq n} \right) \\ &= \prod_{1 \leq r < r' \leq n} (a_{r'} - a_r) \det \left((H_{k-i_j - (n-j)}(\mathbf{a}))_{1 \leq j \leq n} \right). \end{aligned}$$

Similarly, applying Proposition 2.9 to the exponents i_j in $W_{\mathbf{b}}^{(k)}(:, I)$ gives

$$\det(W_{\mathbf{b}}^{(k)}(:, I)) = \prod_{1 \leq r < r' \leq n} (b_{r'} - b_r) \det \left((H_{i_j - (n-j)}(\mathbf{b}))_{j=1}^n \right).$$

Combining the results, the theorem follows. $\square$

Corollary 2.11. Let $1 \leq n \leq k+1$ be integers and let $p(x, y) = \sum_{i=0}^k \alpha_i x^{k-i} y^i$. Let $S = \{i \in \{0, 1, \dots, k\} : \alpha_i \neq 0\}$. Then

$$\text{rank}(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) \leq \min\{n, |S|\}.$$

If $|S| < n$, then $\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) = 0$.

Proof. Since $\alpha_i = 0$ for $i \notin S$, we may delete the zero columns and obtain the reduced factorization

$$A_{\mathbf{a}, \mathbf{b}}(p(x, y)) = V_{\mathbf{a}}^{(k)} D_{\alpha} (W_{\mathbf{b}}^{(k)})^T = V_{\mathbf{a}}^{(k)}(:, S) D_{\alpha}(S) \left(W_{\mathbf{b}}^{(k)}(:, S) \right)^T,$$

where $D_{\alpha} = \text{diag}(\alpha_0, \dots, \alpha_k)$, $V_{\mathbf{a}}^{(k)}(:, S)$, $W_{\mathbf{b}}^{(k)}(:, S)$ are $n \times |S|$, and $D_{\alpha}(S)$ is $|S| \times |S|$ diagonal with nonzero diagonal entries. Hence,

$$\text{rank}(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) \leq \min\{\text{rank}(V_{\mathbf{a}}^{(k)}(:, S)), \text{rank}(D_{\alpha}(S)), \text{rank}(W_{\mathbf{b}}^{(k)}(:, S))\} \leq |S|.$$

Clearly, $\text{rank}(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) \leq n$ which implies that $\text{rank}(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) \leq \min\{n, |S|\}$. If $|S| < n$, then $\text{rank}(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) < n$, and hence $\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) = 0$. $\square$

3. Applications

In this section, we explore several consequences and applications of the determinant factorization. We begin with a closed-form formula for the determinant of polynomial evaluation matrices in the special case $p(x, y) = f(x + y)$, followed by equivariance under linear change of variables. We then analyze the behavior of the determinant over finite fields, providing probabilistic bounds on when it is identically zero.

3.1. Determinant formula for $p(x, y) = f(x + y)$

We begin with the special case where the bivariate polynomial is of *sum-form*,

$$p(x, y) = f(x + y),$$

with $f(t)$ a univariate polynomial of degree k and the matrix size $n = k + 1$. In this setting, the determinant of the polynomial evaluation matrix admits a clean factorization in terms of the Vandermonde factors of the evaluation points. This gives an explicit and computationally convenient closed-form formula for $\det(A_{\mathbf{a}, \mathbf{b}}(f(x + y)))$, revealing its full algebraic structure.

Proposition 3.1. *Let $n = k + 1$ be integers and let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ be in $\mathbb{C}^n$. If $f(t) = \sum_{i=0}^k \alpha_i t^i$ with leading coefficient $\alpha_k \neq 0$, then*

$$\det(A_{\mathbf{a}, \mathbf{b}}(f(x + y))) = \alpha_k^n (-1)^{\binom{n}{2}} \prod_{i=0}^k \binom{k}{i} \prod_{1 \leq r < r' \leq n} (a_{r'} - a_r) \prod_{1 \leq s < s' \leq n} (b_{s'} - b_s).$$

Proof. Let $V_{\mathbf{a}}^{(k)} = (a_r^{k-j})_{1 \leq r \leq n, 0 \leq j \leq k}$ and $W_{\mathbf{b}}^{(k)} = (b_s^j)_{1 \leq s \leq n, 0 \leq j \leq k}$. Write also the “ascending-power” evaluation matrices

$$E_{\mathbf{a}} = (a_r^{i-1})_{1 \leq r, i \leq n}, \quad E_{\mathbf{b}} = (b_s^{j-1})_{1 \leq s, j \leq n}.$$

Let R be the $(k + 1) \times (k + 1)$ column-reversal permutation matrix (ones on the anti-diagonal, zeros elsewhere). Then

$$E_{\mathbf{a}} = V_{\mathbf{a}}^{(k)} R, \quad E_{\mathbf{b}} = W_{\mathbf{b}}^{(k)}.$$

By definition of $f(t)$ and the binomial theorem, we have

$$f(a_r + b_s) = \sum_{m=0}^k \alpha_m (a_r + b_s)^m = \sum_{m=0}^k \alpha_m \sum_{i=0}^m \binom{m}{i} a_r^i b_s^{m-i}.$$

Set $j = m - i$, so $i + j = m$. The coefficient of $a_r^i b_s^j$ is $\alpha_{i+j} \binom{i+j}{i}$, so

$$f(a_r + b_s) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \alpha_{i+j} \binom{i+j}{i} a_r^i b_s^j.$$

Restricting to $i, j \leq n - 1$ ensures all terms fit into $n \times n$ matrices (and if $n = k + 1$ all needed terms appear).

Now write this in matrix form: the i th power a_r^i gives the $(r, i + 1)$ entry of $E_{\mathbf{a}}$, similarly b_s^j gives $(s, j + 1)$ entry of $E_{\mathbf{b}}$, and the coefficient in the middle is exactly

$$(C_f)_{i+1, j+1} = \alpha_{i+j} \binom{i+j}{i}.$$

The desired factorization

$$A_{\mathbf{a},\mathbf{b}}(f(x+y)) = E_{\mathbf{a}} C_f E_{\mathbf{b}}^T = V_{\mathbf{a}}^{(k)} (RC_f) \left(W_{\mathbf{b}}^{(k)}\right)^T$$

is concluded. It follows that

$$\det(A_{\mathbf{a},\mathbf{b}}(f(x+y))) = \det(V_{\mathbf{a}}^{(k)}) \det(RC_f) \det(W_{\mathbf{b}}^{(k)}). \quad (4)$$

We now compute each factor: Since $V_{\mathbf{a}}^{(k)}$ lists descending powers, reversing columns to the standard Vandermonde order needs $\binom{n}{2}$ adjacent swaps:

$$\det(V_{\mathbf{a}}^{(k)}) = (-1)^{\binom{n}{2}} \prod_{1 \leq r < r' \leq n} (a_{r'} - a_r).$$

This is the usual Vandermonde determinant:

$$\det(W_{\mathbf{b}}^{(k)}) = \prod_{1 \leq s < s' \leq n} (b_{s'} - b_s).$$

We have $\det(R) = (-1)^{\binom{n}{2}}$. A standard Pascal-basis argument gives $\det(C_f) = \alpha_k^n (-1)^{\binom{n}{2}} \prod_{i=0}^k \binom{k}{i}$. Thus $\det(RC_f) = \det(R) \det(C_f) = \alpha_k^n \prod_{i=0}^k \binom{k}{i}$.

Multiplying the three determinants,

$$\begin{aligned} \det(A_{\mathbf{a},\mathbf{b}}(f(x+y))) &= \left((-1)^{\binom{n}{2}} \prod_{r < r'} (a_{r'} - a_r) \right) \left(\alpha_k^n \prod_{i=0}^k \binom{k}{i} \right) \left(\prod_{s < s'} (b_{s'} - b_s) \right) \\ &= \alpha_k^n (-1)^{\binom{n}{2}} \prod_{i=0}^k \binom{k}{i} \prod_{1 \leq r < r' \leq n} (a_{r'} - a_r) \prod_{1 \leq s < s' \leq n} (b_{s'} - b_s), \end{aligned}$$

as desired. $\square$

From Proposition 3.1, for $f(t) = \sum_{i=0}^k \alpha_i t^i$ with leading coefficient $\alpha_k \neq 0$ and $n = k + 1$, it is not difficult to see that $\det(A_{\mathbf{a},\mathbf{b}}(f(x+y)))$ is independent from α_i for all $i \in \{0, 1, \dots, k-1\}$. Hence, we have the following corollary.

Corollary 3.2. *Let $n = k + 1$ be integers and let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ be in $\mathbb{C}^n$. If $f(t) = \sum_{i=0}^k \alpha_i t^i$ with leading coefficient $\alpha_k \neq 0$ and $g(t) = \alpha_k t^k$, then*

$$\det(A_{\mathbf{a},\mathbf{b}}(f(x+y))) = \det(A_{\mathbf{a},\mathbf{b}}(g(x+y))).$$

Remark 3.3. *We note that $p(x,y) = f(x+y)$ is homogeneous if and only if $f(t) = ct^k$ for some integer $k \geq 0$ and scalar c . In this case, the result in Proposition 3.1 agrees with the results in Corollary 2.5 and Remark 2.4.*

3.2. Equivariance under linear change of variables

Let $\mathrm{GL}_2(\mathbb{C})$ denote the set of 2×2 nonsingular matrices over $\mathbb{C}$. Next, we study how $A_{\mathbf{a},\mathbf{b}}(p(x,y))$ transforms when we apply a linear change of variables $(x,y) \mapsto (x',y') = B \cdot (x,y)$ with $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$. For a homogeneous $p(x,y)$ of degree k put $p_B(x,y) := p(x',y')$. For the *sum-form* $p(x,y) = f(x+y)$ we obtain a clean and explicit rule.

Proposition 3.4. Let $f(t)$ be a polynomial of degree k and let $p(x, y) = f(x + y)$. Let $n \leq k + 1$ and let

$$c := \alpha + \gamma \quad \text{and} \quad d := \beta + \delta.$$

Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ be vectors in $\mathbb{C}^n$. Then

$$A_{\mathbf{a}, \mathbf{b}}(p_B(x, y)) = (f(ca_r + db_s))_{1 \leq r, s \leq n} = A_{c\mathbf{a}, d\mathbf{b}}(f(x + y)).$$

In particular, if $n = k + 1$, then

$$\det(A_{\mathbf{a}, \mathbf{b}}(p_B(x, y))) = c^{\binom{n}{2}} d^{\binom{n}{2}} \det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))).$$

Proof. Since $p_B(x, y) = f(x' + y') = f((\alpha + \gamma)x + (\beta + \delta)y) = f(cx + dy)$, we have

$$(A_{\mathbf{a}, \mathbf{b}}(p_B(x, y)))_{rs} = f(ca_r + db_s) = (A_{c\mathbf{a}, d\mathbf{b}}(f(x + y)))_{rs}.$$

Precisely, $A_{\mathbf{a}, \mathbf{b}}(p_B(x, y)) = A_{c\mathbf{a}, d\mathbf{b}}(f(x + y))$.

For the determinant scaling at $n = k + 1$, we have

$$\det(A_{\mathbf{a}, \mathbf{b}}(p_B(x, y))) = \det(A_{c\mathbf{a}, d\mathbf{b}}(f(x + y))) = \det(V_{c\mathbf{a}}^{(k)}) \det(RC_f) \det(W_{d\mathbf{b}}^{(k)})$$

by (4) in Proposition 3.1. Since

$$\det(V_{c\mathbf{a}}^{(k)}) = c^{\binom{n}{2}} \det(V_{\mathbf{a}}^{(k)}) \quad \text{and} \quad \det(W_{d\mathbf{b}}^{(k)}) = d^{\binom{n}{2}} \det(W_{\mathbf{b}}^{(k)}),$$

it can be deduced that

$$\det(A_{\mathbf{a}, \mathbf{b}}(p_B(x, y))) = c^{\binom{n}{2}} d^{\binom{n}{2}} \det(A_{\mathbf{a}, \mathbf{b}}(f(x + y))).$$

□

Combining Proposition 3.1 and Proposition 3.4, an explicit formula for $\det(A_{\mathbf{a}, \mathbf{b}}(f(x + y)))$ follows immediately.

3.3. Behavior over finite fields

Next, we consider the behavior of $\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y)))$ in the case where the entries a_i, b_j are chosen randomly from a finite field $\mathbb{F}_q$. We focus on the case where $n \leq k + 1$ and give an upper bound for the case where the determinant is identically zero. This is obtained via an application of the Schwartz–Zippel lemma.

Lemma 3.5 ([1, Lemma A.36], Schwartz–Zippel). Let $\mathbb{F}_q$ be a field of order q and let $p(x_1, \dots, x_n) \in \mathbb{F}_q[x_1, \dots, x_n]$ be a nonzero polynomial of total degree $d \geq 0$. Let $S \subseteq \mathbb{F}_q$ be a non-empty set and choose $(r_1, \dots, r_n) \in S^n$ uniformly at random. Then

$$\Pr(p(r_1, \dots, r_n) = 0) \leq \frac{d}{|S|}.$$

Proposition 3.6. Let $\mathbb{F}_q$ be a finite field of order q and characteristic ℓ . Let $n = k + 1 \leq \ell$ be integers. Let $\alpha_i \in \mathbb{F}_q \setminus \{0\}$ for $0 \leq i \leq k$ and let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ be chosen independently and uniformly from $\mathbb{F}_q^n$. If $p(x, y) = \sum_{i=0}^k \alpha_i (x + y)^i$, then the probability that $\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) = 0$ is

$$\Pr(\det(A_{\mathbf{a}, \mathbf{b}}(p(x, y))) = 0) \leq \frac{nk}{q}.$$

Proof. From (4), we have

$$\det(A_{\mathbf{a},\mathbf{b}}(p(x,y))) = \det(V_{\mathbf{a}}^{(k)}) \det(RC_p) \det(W_{\mathbf{b}}^{(k)}) \text{ and } \det(RC_p) = \alpha_k^n \prod_{i=0}^k \binom{k}{i}.$$

Since $\alpha_i \neq 0$ for all $1 \leq i \leq k$, $\det(RC_p) \neq 0$ which implies that $\det(A_{\mathbf{a},\mathbf{b}}(p(x,y)))$ is a nonzero polynomial.

For each $1 \leq r \leq n$ and $1 \leq s \leq n$, we note that for each entry $p(a_r, b_s) = \sum_{i=0}^k \alpha_i (a_r + b_s)^i$ has degree at most k in the $2n$ variables a_r and b_s . Using the Leibniz expansion, $\det(A_{\mathbf{a},\mathbf{b}}(p(x,y)))$ is a product of n entries and each depending on a distinct pair (a_r, b_s) which implies that the total degree is at most nk . Hence, $\det(A_{\mathbf{a},\mathbf{b}}(p(x,y)))$ is a nonzero polynomial of total degree at most nk in $2n$ independent uniform variables over $\mathbb{F}_q$. By Lemma 3.5, it follows that

$$\Pr(\det(A_{\mathbf{a},\mathbf{b}}(p(x,y))) = 0) \leq \frac{nk}{q}.$$

The proof is complete. □

4. Conclusion

Using the simple yet powerful Vandermonde factorization $A_{\mathbf{a},\mathbf{b}}(p(x,y)) = V_{\mathbf{a}}^{(k)} D_{\alpha} (W_{\mathbf{b}}^{(k)})^T$, we have developed a unified framework for understanding determinants of polynomial evaluation matrices associated with homogeneous bivariate polynomials. Our analysis covers the full range of matrix sizes, from $n \leq k+1$ to the square borderline case $n = k+1$ and beyond, where rank constraints force the determinant to be zero for $n \geq k+2$.

In the square case $n = k+1$, we obtained a closed-form expression for $\det(A_{\mathbf{a},\mathbf{b}}(p(x,y)))$ in terms of Vandermonde products and coefficient factors, recovering and extending classical formulas for matrices such as $((a_i + b_j)^k)_{1 \leq i,j \leq n}$. For $n \leq k+1$, applying the Cauchy–Binet formula together with an elementary evaluation of generalized Vandermonde minors via complete homogeneous polynomials H_m yielded a symmetric expansion for the determinant, leading to immediate rank bounds and vanishing criteria.

Several applications illustrate the versatility of our determinant factorization. We derived a closed-form formula for the square sum-form case where $p(x,y) = f(x+y)$ with $n = k+1$. Beyond this special case, we established an explicit equivariance law under linear change of variables for sum-form polynomials, and proved probabilistic non-vanishing bounds over finite fields via the Schwartz–Zippel lemma.

More generally, it would be of interest to analyze determinants of polynomial evaluation matrices associated with nonhomogeneous bivariate polynomials.

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