

M^* -matrices: Tridiagonal structure and signature similarity to M -matrices

Research Article

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Abstract: This paper introduces and studies a new class of matrices, called M^* -matrices. These matrices have a prescribed tridiagonal sign structure together with the positivity of all principal minors. Specifically, M^* -matrices are matrices with positive diagonal entries, positive first sub- and super-diagonal entries, non-positive entries outside the tridiagonal band, and positive principal minors. The specified sign structure and characteristics of M^* -matrices extend both classical tridiagonal M -matrices and totally positive tridiagonal matrices. We identify the subclass, denoted by M_T^* , consisting of all tridiagonal M^* -matrices, such that each matrix in this class is always diagonally signature-similar to a tridiagonal M -matrix. The classical M -matrix theory can be transferred directly to this subclass. In particular, M_T^* -matrices admit LU factorizations without pivoting with positive pivots, have eigenvalues with positive real parts, exhibit inverse-sign regularity, and possess invariance of induced norm condition numbers. The symmetric M_T^* -matrices coincide with Jacobi matrices and are therefore positive definite, admitting Cholesky factorizations and strict eigenvalue interlacing. We also investigate scaling invariance and numerical conditioning. Applications of this structured tridiagonal system arise in compartmental and population models with alternating nearest-neighbor transfer. This paper illustrates the natural occurrence of M^* -matrices in applied and computational settings.

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1. Introduction

Matrices with positivity properties and specific sign structure have important applications in linear algebra, matrix theory, and the analysis of differential and dynamical systems. Among them, the most influential examples are M -matrices, which arise in the discretization of elliptic partial differential equations, Markov chains, compartmental systems, and monotone dynamical system models and many other fields [1, 3, 5, 14]. Classical M -matrix theory provides powerful tools for analyzing non-singularity, spectral positivity, inverse-positivity, and numerical stability, and has been developed extensively over several decades [2, 13, 17]. Closely related matrix classes, including totally positive matrices, H -matrices and oscillatory matrices are also significant due to their strong algebraic and spectral properties [6–8].

Even though the M -matrix theory was well established, many structured matrices arising in applications fall outside its framework. In particular, models governed by nearest-neighbor interactions often result in tridiagonal linear systems with mixed but highly structured sign patterns. Such cases arise in discretizations of the one-dimensional convection-diffusion problem with sign-changing couplings, compartmental and population models exhibiting alternating cooperative-competitive effects, and in networked systems [11, 17]. In many such settings, positivity occurs along the main diagonal and the first sub- and super-diagonals, reflecting direct local influence, while interactions beyond nearest neighbors are inhibitory and therefore result in non-positive entries. In general, these types of matrices violate the classical requirement that all off-diagonal entries are non-positive, even though they have strong positivity and stability properties at the level of principal minors and spectra.

This gap between classical M -matrix theory and the practical situation motivates the introduction of a broader class of matrices that relaxes the off-diagonal sign restrictions of M -matrices while preserving positivity of principal minors. In this paper, we introduce a new class of matrices, called M^* -matrices. An M^* -matrix is defined by a prescribed sign structure: positive diagonal entries, positive first sub- and super-diagonal entries, and non-positive entries outside the tridiagonal band, together with positivity of all principal minors. The positivity of principal minors place M^* -matrices within the broader family of P -matrices [7], while the tridiagonal sign structure captures systems governed by nearest-neighbor interactions with alternating influences.

Although M^* -matrices possess positivity of principal minors as that of M -matrices, they are distinct in both sign structure and algebraic behavior. Classical M -matrices require all off-diagonal entries to be non-positive, satisfy inverse-positivity, and obey comparison principles. The defining positivity of the first sub- and super-diagonals in an M^* -matrix precludes membership in the M -matrix class, but M^* -matrices are not merely sign-perturbed M -matrices. The positivity of all principal minors places them closely with totally positive and oscillatory matrices. In this way, the class of M^* -matrices extends classical M -matrix theory. We identify an important subclass, denoted by M_T^* , consisting of tridiagonal M^* -matrices. These matrices are always diagonally signature-similar to tridiagonal M -matrices. For matrices in this subclass, classical M -matrix theory transfers directly through diagonal similarity. In particular, M_T^* -matrices inherit positive real spectra, inverse-sign regularity, and invariance of the induced norm condition numbers. These properties do not hold in general for M^* -matrices, and some additional structure is required to hold these properties.

The present work focuses primarily on the tridiagonal case. For tridiagonal M^* -matrices, we show that positivity of leading principal minors are equivalent to the positivity of all principal minors, derive explicit three-term determinant recursions, and establish the existence of LU factorization without pivoting. In the symmetric case, tridiagonal M^* -matrices become Jacobi matrices and are therefore positive definite, admitting a Cholesky factorization and strict eigenvalue interlacing. We also prove invariance under positive diagonal scaling and diagonal signature similarity, including preservation of induced norm condition numbers. Applications to structured tridiagonal systems arising in compartmental models with bidirectional flow, network systems, and convection-diffusion problems with flow reversal illustrate the natural occurrence of M^* -matrices in applied and computational settings.

The remainder of the paper is organized as follows. Section 2 recalls preliminary definitions and basic results from matrix theory. Section 3 introduces the definition of M^* -matrices and establishes their fundamental structural properties. Section 4 develops the structural and spectral theory of the

M_T^* -matrices, with emphasis on diagonal signature similarity. Section 5 examines the relationships between M^* -matrices and other related matrix classes. Section 6 addresses factorization properties of M_T^* -matrices. Section 7 presents illustrative examples and applications. Finally, Section 8 concludes the paper and outlines directions for future research.

2. Preliminaries

We begin by recalling some standard concepts from matrix theory that will be used throughout this paper.

Definition 2.1. [2] A real matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is called a Z -matrix if all its off-diagonal entries are nonpositive, that is, $a_{ij} \leq 0$ for all $i \neq j$.

Definition 2.2. [9] A Z -matrix A is called an M_0 -matrix if it can be written in the form $A = sI - B$, where $B \geq 0$ and $s \geq \rho(B)$, where $\rho(B)$ denotes the spectral radius of B . If $s > \rho(B)$, then A is called an M -matrix. Equivalently, a Z -matrix A is an M -matrix if and only if it is **non-singular** and $A^{-1} \geq 0$.

Theorem 2.3. [2, 16] Let $A = (a_{ij})$ be a real Z -matrix. Then the following statements are equivalent:

1. A is an M -matrix.
2. All principal minors of A are positive.
3. There exists a vector $x > 0$ such that $Ax > 0$.
4. A is positive stable, i.e., all its eigenvalues have positive real parts.

Definition 2.4. Two real matrices $A, B \in \mathbb{R}^{n \times n}$ are said to be diagonally signature-similar if there exists a diagonal matrix $S = \text{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, $\varepsilon_i \in \{-1, 1\}$, such that $B = SAS^{-1}$. Then the matrix S is called a signature matrix. Since $S^{-1} = S$, diagonal signature similarity preserves eigenvalues, principal minors, and induced matrix norms.

Definition 2.5. [2] A real matrix $A \in \mathbb{R}^{n \times n}$ is called a P -matrix if all of its principal minors are positive, that is, $|A[I, I]| > 0$, for every nonempty index set $I \subseteq \{1, 2, \dots, n\}$.

3. M^* -matrices and its structural properties

In this section, we introduce a new class of matrices, and we establish several elementary but fundamental structural properties of M^* -matrices, showing that under suitable structural assumptions, they share important characteristics with non-singular M -matrices [12]. Throughout this section, $A = (a_{ij})$ denotes an $n \times n$ M^* -matrix unless stated otherwise.

Definition 3.1. A real matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is called a M^* -matrix if

- (i) $a_{ii} > 0$ for all i ;
- (ii) $a_{i,i+1} > 0$ and $a_{i+1,i} > 0$ for all admissible i ;
- (iii) $a_{ij} \leq 0$ for all $|i - j| > 1$;
- (iv) all principal minors of A are positive (implies A is non-singular).

An M^* -matrix A is called a tridiagonal M^* -matrix if $a_{ij} = 0$ for all $|i - j| > 1$. The class M_T^* consists of all tridiagonal M^* -matrices. Each matrix in this class is called an M_T^* -matrix.

The following matrix is an example of an M^* -matrix.

Example 3.2. Consider the matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 2 \end{pmatrix}$, which is not an M -matrix, since some of its off-diagonal entries are positive. But A satisfies all defining conditions of an M^* -matrix.

The following proposition states the basic invariance of M^* -matrices under positive scaling.

Proposition 3.3. Let A be an M^* -matrix and let $\alpha > 0$. Then αA is also an M^* -matrix.

Proof. Multiplication by a positive scalar does not alter the sign pattern of the entries of a matrix. Therefore, αA has positive diagonal entries, positive first sub- and super-diagonal entries, and non-positive entries away from the tridiagonal band. Also, all principal minors of αA are equal to α^k times the corresponding principal minors of A , where k denotes the order of the principal minor. Since $\alpha > 0$ and all principal minors of A are positive, all principal minors of αA remain positive. Hence, αA is an M^* -matrix. $\square$

Next proposition shows that the class of M^* -matrices is not invariant under permutation similarity.

Proposition 3.4. The class of M^* -matrices is not closed under permutation similarity.

Proof. Let $A = [a_{ij}]$ be an M^* -matrix and P be a permutation matrix. Since the permutation similarity preserves all principal minors, $B = PAP^T$ is a P -matrix. M^* -matrices are required to have positive entries on the first sub- and super-diagonals, while all entries outside the tridiagonal band must be non-positive. If the matrix P corresponds to a non-trivial permutation, then (i, j) th entry of B is $b_{ij} = a_{\pi(i)\pi(j)}$, where π is the associated permutation of $\{1, \dots, n\}$.

There is no guarantee that the condition $|\pi(i) - \pi(j)| = 1$ is equivalent to $|i - j| = 1$, nor that $|\pi(i) - \pi(j)| > 1$ is preserved. Hence, a positive nearest-neighbor entry of A may be mapped to a position outside the tridiagonal band of B , or a non-positive entry of A may be mapped to a first sub- or super-diagonal position of B . In either case, the defining sign structure of an M^* -matrix is violated.

Therefore, although permutation similarity preserves positivity of principal minors, it does not preserve the positional sign constraints defining an M^* -matrix. Hence, the class of M^* -matrices is not closed under permutation similarity. $\square$

Like M -matrices, the class of M^* -matrices is invariant under positive diagonal similarity, a property that is central to matrix normalization and preconditioning [2, 17], and it is proved in the following theorem.

Theorem 3.5. Let A be an M^* -matrix and let $D = \text{diag}(d_1, \dots, d_n)$ be a diagonal matrix with $d_i > 0$ for all i . Then DAD^{-1} is also an M^* -matrix.

Proof. Diagonal similarity preserves the zero-nonzero structure of a matrix. Also,

$$(DAD^{-1})_{ii} = a_{ii} > 0 \text{ and } (DAD^{-1})_{ij} = \frac{d_i}{d_j} a_{ij}.$$

So the sign pattern of the off-diagonal entries is unchanged, and hence $(DAD^{-1})_{ij} \leq 0$ for all $|i - j| > 1$, and $(DAD^{-1})_{i,i\pm 1} > 0$ whenever $a_{i,i\pm 1} > 0$. Thus, the sign structure of M^* -matrix is satisfied.

For any index set I , the corresponding principal minor satisfies.

$$\det((DAD^{-1})[I, I]) = \det(D[I, I]) \det(A[I, I]) \det(D[I, I]^{-1}) = \det(A[I, I]),$$

since $\det(D[I, I]) \det(D[I, I]^{-1}) = 1$. Thus, the positivity of all principal minors is preserved.

Hence DAD^{-1} satisfies all defining properties of an M^* -matrix. $\square$

4. Structural and spectral properties of M_T^* -matrices

In this section, we investigate fundamental structural properties of M_T^* -matrices that follow from their tridiagonal form, prescribed sign structure, and positivity of principal minors.

Proposition 4.1. *Every M_T^* -matrix is diagonally signature-similar to a non-singular M -matrix.*

Proof. Let $A = [a_{ij}]$ be an M_T^* -matrix. By definition, A is a tridiagonal M^* -matrix, so $a_{ii} > 0$, $a_{i,i+1} > 0$, $a_{i+1,i} > 0$, and $a_{ij} = 0$ for $|i - j| > 1$, and all principal minors of A are positive.

Define $S = \text{diag}(s_1, \dots, s_n)$, $s_1 = 1$, $s_{i+1} = -s_i$ ($i = 1, \dots, n-1$). Let $B = SAS^{-1}$. Since S is diagonal, B preserves tridiagonality. Also, $b_{ii} = a_{ii} > 0$, and for $i = 1, \dots, n-1$, $b_{i,i+1} = s_i s_{i+1} a_{i,i+1} = -a_{i,i+1} < 0$, $b_{i+1,i} = s_{i+1} s_i a_{i+1,i} = -a_{i+1,i} < 0$. Hence B is a tridiagonal Z -matrix with positive diagonal entries.

Diagonal similarity preserves all principal minors. Therefore, all principal minors of B are positive, and B is nonsingular. By the standard characterization of M -matrices, a Z -matrix with positive principal minors is an M -matrix. Thus, A is diagonally signature-similar to an M -matrix. $\square$

While the determinant recursion is valid for all tridiagonal matrices, it provides a useful tool for studying the positivity of principal minors and the structural behavior of M_T^* -matrices.

Proposition 4.2. *Let*

$$A_n = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & c_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_{n-1} & b_{n-1} \\ 0 & \cdots & 0 & c_{n-1} & a_n \end{pmatrix}$$

be an M_T^ -matrix where $a_i, b_i, c_i > 0$ for all admissible i . Let A_k denote the leading $k \times k$ principal submatrix of A_n . Then the determinants satisfy the recurrence: $\det(A_1) = a_1$, $\det(A_2) = a_1 a_2 - b_1 c_1$, and for $n \geq 3$, $\det(A_n) = a_n \det(A_{n-1}) - b_{n-1} c_{n-1} \det(A_{n-2})$.*

Proof. The result follows from expanding the determinant along the last row (or column) of the tridiagonal matrix. Since A_n is tridiagonal, the Laplace expansion gives

$$\det(A_n) = a_n \det(A_{n-1}) - c_{n-1} b_{n-1} \det(A_{n-2}),$$

because all other minors involving non-adjacent entries vanish. The initial cases $\det(A_1) = a_1$ and $\det(A_2) = a_1 a_2 - b_1 c_1$ follow directly from the 1×1 and 2×2 determinants. $\square$

Lemma 4.3. *Let A_n be a tridiagonal M^* -matrix as in Proposition 4.2 with $a_i, b_i, c_i > 0$. Then A_n has all principal minors positive if and only if $D_k > 0$ for all $k = 1, \dots, n$, where $D_k := \det(A_k)$ and A_k is the leading principal submatrix of A_n of order k .*

Proof. If A_n is an M^* -matrix, then by definition all its principal minors are positive. In particular, all leading principal minors D_k , $k = 1, \dots, n$, are positive. Conversely, assume that A_k denote the leading principal submatrix of A_n of size $k \times k$, for $1 \leq k \leq n$ and $D_k > 0$ for all $k = 1, \dots, n$. For each $1 \leq i \leq j \leq n$, let $D_{i,j}$ denotes the determinant of $A[\{i, i+1, \dots, j\} \times \{i, i+1, \dots, j\}]$, that is the submatrix of $A := A_n$ corresponding to a contiguous index set $\{i, i+1, \dots, j\}$. Since A is a tridiagonal matrix, if the determinants of all principal submatrices corresponding to contiguous index sets are positive, then all principal minors of A are positive. So our aim is to show that $D_{i,j} > 0$. Suppose that $1 < i < j$. Expanding the determinant of A_j along the i^{th} row, we get,

$\det(A_j) = -b_{i-1}c_{i-1}D_{1,i-2}D_{i+1,j} + a_iD_{1,i-1}D_{i+1,j} - c_i b_i D_{1,i-1}D_{i+2,j}$, where $D_{1,i-2} = 1$ when $i = 2$, and $D_{i+2,j} = 1$ when $i = j - 1$.

Since $\det(A_j) > 0$, it follows that, $(a_i D_{1,i-1} - b_{i-1} c_{i-1} D_{1,i-2}) D_{i+1,j} - c_i b_i D_{1,i-1} D_{i+2,j} > 0$. By Proposition 4.2 we get, $\det(A_i) D_{i+1,j} - c_i b_i D_{1,i-1} D_{i+2,j} > 0$. Since $D_{k,k} = a_k > 0$, we may proceed by induction, with the hypothesis that $\det(A[I \times I]) > 0$ for every contiguous index set I whose cardinality is less than $j - i$. Then, by the induction hypothesis $D_{i+2,j} > 0$. Moreover, since $b_i > 0$ and $c_i > 0$, and since $D_{1,i-1} = \det(A_{i-1}) > 0$, we obtain $\det(A_i) D_{i+1,j} > 0$. Hence $D_{i+1,j} > 0$. In this way, one can prove that $D_{i,j} > 0$ whenever $i > 2$. To show that $D_{2,j} > 0$, we can use the formula: $\det(A_j) = a_1 D_{2,j} - b_1 c_1 D_{3,j}$. $\square$

Although M^* -matrices belong to the class of P -matrices, spectral positivity does not hold in general. However, for M_T^* -matrices this property is recovered due to their diagonal signature similarity to M -matrices. Since similarity preserves the spectrum [10], M_T^* -matrices inherit the spectral positivity of M -matrices. The following theorem makes this connection precise.

Theorem 4.4. *Let A be an M_T^* -matrix. Then all eigenvalues of A have positive real parts.*

Proof. Since A is an M_T^* -matrix, there exists a diagonal signature matrix $S = \text{diag}(\pm 1, \dots, \pm 1)$ such that $B = SAS^{-1}$ is a non-singular M -matrix by Proposition 4.1. But all eigenvalues of a non-singular M -matrix have positive real parts [2, 17]. Because A and B are similar, they have the same spectrum. Therefore, all eigenvalues of A have positive real parts. $\square$

The inverse-sign structure of M^* -matrices is not automatic and, in general, requires additional structural assumptions. But it can be guaranteed for M_T^* -matrices, as stated in the following corollary.

Corollary 4.5. *Let A be an M_T^* -matrix. Then A is inverse-sign regular.*

Proof. Since A is an M_T^* -matrix, there exists a diagonal signature matrix $S = \text{diag}(s_1, \dots, s_n)$, $s_i = \pm 1$, such that $B = SAS^{-1}$ is a non-singular M -matrix.

Since B is a non-singular M -matrix, its inverse B^{-1} is entry wise non-negative. Using the similarity relation $A^{-1} = S^{-1}B^{-1}S$, we obtain $(A^{-1})_{ij} = s_i s_j (B^{-1})_{ij}$. Because $(B^{-1})_{ij} \geq 0$ for all i, j , the sign of $(A^{-1})_{ij}$ is completely determined by the product $s_i s_j$. Hence A^{-1} exhibits a fixed sign pattern and A is inverse-sign regular. $\square$

Spectral localization. The eigenvalue localization for M_T^* -matrices is discussed using the Gershgorin disc theorem. This provides simple sufficient conditions for positivity of the real parts of eigenvalues and leads naturally to the symmetric case, for which a full spectral description is available.

Theorem 4.6. *Let A be a symmetric tridiagonal M^* -matrix. Then all eigenvalues of A are real, positive, and simple, and can be ordered as $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n$.*

Proof. Since A is symmetric and all its leading principal minors are positive, A is positive definite by Sylvester's criterion [10]. Hence, all eigenvalues of A are real and positive. Thus, A is symmetric and tridiagonal with strictly positive sub- and super-diagonal entries, and hence it is a Jacobi matrix. But Jacobi matrices have real, simple, and strictly interlacing eigenvalues [8, 10]. Hence the theorem. $\square$

Spectral bound: The Gershgorin disc theorem [10] can be applied to obtain explicit spectral localization regions for M_T^* -matrices.

Theorem 4.7. *Let $A = [a_{ij}]$ be an $n \times n$ M_T^* -matrix with $a_{kk} = a_k > 0$, $a_{k,k+1} = b_k > 0$, and $a_{k+1,k} = c_k > 0$. Then every eigenvalue λ of A satisfies*

$$\lambda \in \bigcup_{k=1}^n \{z \in \mathbb{C} : |z - a_k| \leq b_k + c_{k-1}\},$$

where $b_n = c_0 = 0$. If, in addition, A is strictly diagonally dominant, i.e., $a_k > b_k + c_{k-1}$ for all k , then all eigenvalues of A lie in the open right half-plane.

Proof. The only nonzero off-diagonal entries of A are $a_{k,k-1} = c_{k-1}$ and $a_{k,k+1} = b_k$, for each k . Hence, the sum of absolute values of the off-diagonal entries in the k^{th} row is $R_k = b_k + c_{k-1}$, with $b_n = c_0 = 0$. By the Gershgorin disc theorem, every eigenvalue λ of A lies in at least one disc $\mathcal{D}_k = \{z \in \mathbb{C} : |z - a_k| \leq R_k\}$, where $k = 1, \dots, n$. This proves the stated localization. If A is strictly diagonally dominant, then $a_k > R_k$ for all k , so all Gershgorin discs lie in the open right half-plane, and hence $\Re(\lambda) > 0$ for all eigenvalues λ . $\square$

Condition number estimates for M^* -matrices require additional assumptions such as strict diagonal dominance and symmetry, as stated in the following proposition.

Proposition 4.8. *Let A be a symmetric M_T^* -matrix that is strictly diagonally dominant, i.e., $a_k > b_k + c_{k-1}$ for all k . Then the 2-norm condition number $\kappa_2(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$ satisfies $1 \leq \kappa_2(A) \leq \frac{\max_k(a_k + b_k + c_{k-1})}{\min_k(a_k - b_k - c_{k-1})}$.*

Proof. Since A is symmetric and strictly diagonally dominant, it is positive definite, and all its eigenvalues are real and positive. By the Gershgorin disc theorem, each eigenvalue λ of A satisfies $a_k - (b_k + c_{k-1}) \leq \lambda \leq a_k + (b_k + c_{k-1})$ for some index k . Taking the maximum of the upper bounds and the minimum of the lower bounds over all k yields $\lambda_{\max}(A) \leq \max_k(a_k + b_k + c_{k-1})$ and $\lambda_{\min}(A) \geq \min_k(a_k - b_k - c_{k-1})$ and hence we get the stated bound for $\kappa_2(A)$. $\square$

We next show that diagonal signature similarity preserves the induced norm condition numbers for matrices in the class M_T^* .

Theorem 4.9. *Let $A \in M_T^*$ and $S = \text{diag}(\pm 1, \dots, \pm 1)$ be the diagonal signature matrix such that $B = SAS^{-1}$ is a non-singular M -matrix. Then for every induced matrix p -norm, $1 \leq p \leq \infty$, the corresponding condition numbers satisfy $\kappa_p(A) = \kappa_p(B)$.*

Proof. Since S is diagonal with entries ± 1 , it is orthogonal and $\|S\|_p = \|S^{-1}\|_p = 1$, for $1 \leq p \leq \infty$. Using submultiplicative property of induced matrix norms [10, 13], we obtain

$$\|B\|_p = \|SAS^{-1}\|_p \leq \|S\|_p \|A\|_p \|S^{-1}\|_p = \|A\|_p,$$

$$\|A\|_p = \|S^{-1}BS\|_p \leq \|S^{-1}\|_p \|B\|_p \|S\|_p = \|B\|_p,$$

and hence $\|A\|_p = \|SAS^{-1}\|_p$. Similarly, $\|A^{-1}\|_p = \|(SAS^{-1})^{-1}\|_p$.

Therefore,

$$\kappa_p(A) = \|A\|_p \|A^{-1}\|_p = \|SAS^{-1}\|_p \|(SAS^{-1})^{-1}\|_p = \kappa_p(B),$$

$\square$

We now discussed properties of M^* -matrices by means of a numerical example.

Example 4.10. *Consider the matrix*

$$A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 \\ 0 & 2 & 6 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

The diagonal entries and first sub- and super-diagonal entries of A are positive, and all remaining off-diagonal entries are zero. Also, the leading principal minors are $\Delta_1 = 4$, $\Delta_2 = 18$, $\Delta_3 =$

100, $\Delta_4 = 382$, and hence all principal minors of A are positive. Therefore, A is an M_T^* -matrix. Let $S = \text{diag}(1, -1, 1, -1)$. Then

$$SAS^{-1} = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -2 & 5 & -1 & 0 \\ 0 & -2 & 6 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

is an M -matrix. Thus, A is signature similar to an M -matrix. The eigenvalues of A have positive real parts and its spectrum is given by, $\sigma(A) \approx \{2.689, 3.646, 5.266, 7.399\}$. Also, the inverse of A exhibits the alternating sign pattern predicted by Corollary 4.5.

5. Relationship of M^* -matrices with other matrix classes

In this section, we compare M^* -matrices with other well-known matrix classes, namely, H -matrices, and totally positive tridiagonal matrices. The following results clarify the relationships and inclusions among these families.

Totally positive tridiagonal matrices form a well-studied class characterized by the positivity of all minors [6, 8, 15]. The following proposition clarifies their relationship with M^* -matrices.

Proposition 5.1. *Every totally positive tridiagonal matrix is an M^* -matrix, but the converse does not hold.*

Proof. Let A be a totally positive tridiagonal matrix. Since A is tridiagonal, we have $a_{ij} = 0$ for all $|i - j| > 1$. Total positivity implies that all minors of A are positive and hence, all 1×1 minors are positive. Therefore, all diagonal and first sub- and super-diagonal entries and all principal minors of A are positive. Therefore, A satisfies all defining properties of an M^* -matrix. The converse does not hold, since an M^* -matrix is required to have positive principal minors only, whereas total positivity requires positivity of all minors. $\square$

Definition 5.2. A matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is called an H -matrix if its comparison matrix $\mathcal{M}(A)$, defined by

$$\mathcal{M}(A)_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j, \end{cases}$$

is an M -matrix. i.e., if A is an H -matrix then its comparison matrix is inverse-positive.

Although M^* -matrices are not H -matrices in general, a meaningful connection arises under additional structural assumptions. In particular, every M_T^* -matrix belongs to the class of H -matrices, as stated below.

Proposition 5.3. *If A is an M_T^* -matrix, then A is an H -matrix.*

Proof. Since A is an M_T^* -matrix, there exists a diagonal signature matrix $S = \text{diag}(s_1, \dots, s_n)$ with $s_i = \pm 1$ such that $B = SAS^{-1}$ is an M -matrix. Hence, B is a Z -matrix with positive diagonal entries. The entries of A and B satisfy $a_{ij} = s_i b_{ij} s_j$, where $i, j = 1, \dots, n$. Consequently, for all i , $\mathcal{M}(A)_{ii} = |a_{ii}| = |b_{ii}| = \mathcal{M}(B)_{ii}$, and for $i \neq j$, $\mathcal{M}(A)_{ij} = -|a_{ij}| = -|b_{ij}| = \mathcal{M}(B)_{ij}$. Hence, $\mathcal{M}(A) = \mathcal{M}(B)$. Since B is an M -matrix, it is a Z -matrix, and its comparison matrix satisfies $\mathcal{M}(B) = B$. But B is itself an M -matrix. Therefore, $\mathcal{M}(A)$ is a non-singular M -matrix, which implies that A is an H -matrix. $\square$

6. Factorization properties of M^* -matrices

In this section, we discuss the existence and properties of various matrix factorizations for M^* -matrices. Such decompositions play a central role in numerical stability, inverse estimation, and structural interpretation of the M^* class.

6.1. LU factorization without pivoting

An important structural property of the M_T^* matrices is the existence of an LU factorization without pivoting, as shown in the following proposition.

Proposition 6.1. *Let A be an M_T^* -matrix. Then A admits an LU factorization without pivoting, $A = LU$, where L is unit lower bidiagonal and U is upper bidiagonal. Moreover, all diagonal entries of U (i.e., the pivots) are strictly positive.*

Proof. Since A is tridiagonal and all its leading principal minors are positive, Gaussian elimination can be performed without row exchanges. Therefore, an LU factorization without pivoting exists and is unique. For tridiagonal matrices, the pivots produced by Gaussian elimination satisfy $u_{kk} = \frac{D_k}{D_{k-1}}$, where $k = 1, \dots, n$, where D_k denotes the k th leading principal minor of A (with $D_0 = 1$). Because A is an M^* -matrix, all $D_k > 0$, and hence $u_{kk} > 0$ for all k . This shows that the LU factorization is well defined, proceeds without breakdown and has strictly positive pivots. $\square$

The following example illustrates the LU factorization result.

Example 6.2. The M_T^* -matrix $A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 \\ 0 & 3 & 6 & 1 \\ 0 & 0 & 2 & 7 \end{bmatrix}$ have the LU factorization $A = LU$ without pivoting,

where $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \ell_1 & 1 & 0 & 0 \\ 0 & \ell_2 & 1 & 0 \\ 0 & 0 & \ell_3 & 1 \end{bmatrix}$, $U = \begin{bmatrix} u_1 & 1 & 0 & 0 \\ 0 & u_2 & 1 & 0 \\ 0 & 0 & u_3 & 1 \\ 0 & 0 & 0 & u_4 \end{bmatrix}$. Then

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & \frac{3}{8} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & \frac{9}{2} & 1 & 0 \\ 0 & 0 & \frac{16}{3} & 1 \\ 0 & 0 & 0 & \frac{53}{8} \end{bmatrix},$$

by applying the recursion $u_1 = a_1$, $\ell_k = \frac{c_k}{u_k}$, $u_{k+1} = a_{k+1} - \ell_k b_k$. This confirms that tridiagonal M^* -matrices admit LU factorization without pivoting.

6.2. Cholesky and symmetric factorizations

If A is a symmetric and positive definite matrix, then it admits a Cholesky decomposition. The following theorem establishes the existence of a Cholesky factorization for symmetric M_T^* -matrices.

Theorem 6.3. *Let A be a symmetric M_T^* -matrix with $b_i = c_i > 0$. Then there exists a unique lower triangular matrix L such that $A = LL^T$, where the diagonal entries of L are strictly positive.*

Proof. Since A is symmetric and all its leading principal minors are positive, A is positive definite by Sylvester's criterion. Hence, the standard Cholesky algorithm can be applied. For a tridiagonal matrix, the nonzero entries of L satisfy,

$$\ell_{11} = \sqrt{a_1}, \quad \ell_{i,i-1} = \frac{b_{i-1}}{\ell_{i-1,i-1}}, \quad \ell_{ii} = \sqrt{a_i - \ell_{i,i-1}^2} > 0.$$

This existence and uniqueness of the Cholesky factorization for symmetric positive definite matrices is classical [10]. $\square$

The next example illustrates the Cholesky factorization of a symmetric M_T^* -matrix.

Example 6.4. Consider the M_T^* -matrix $A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 5 & 2 & 0 \\ 0 & 2 & 6 & 1 \\ 0 & 0 & 1 & 7 \end{bmatrix}$ which is symmetric and positive definite.

It admits a Cholesky factorization $A = LL^T$, where

$$L = \begin{bmatrix} \ell_{11} & 0 & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 & 0 \\ 0 & \ell_{32} & \ell_{33} & 0 \\ 0 & 0 & \ell_{43} & \ell_{44} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{19}}{2} & 0 & 0 \\ 0 & \frac{4}{\sqrt{19}} & \sqrt{\frac{98}{19}} & 0 \\ 0 & 0 & \sqrt{\frac{19}{98}} & \sqrt{\frac{667}{98}} \end{bmatrix},$$

All diagonal entries of L are strictly positive, and no pivoting is required.

7. Applications of M^* -matrices

The theoretical results developed above can be applied to linear systems arising in numerical analysis and modelling. In many such problems, sign-alternating tridiagonal structures arise from nearest-neighbor interactions or alternating flow directions. This may lead to matrices that fall outside the classical M -matrix framework yet naturally belong to the M^* class. The contribution of this work lies in identifying a unifying algebraic structure through diagonal signature similarity to M -matrices. In particular, sign changes in transport or interaction coefficients may destroy the M -matrix sign pattern even when diagonal dominance remains [4].

We illustrate the relevance of the M^* -matrix by considering a standard one-dimensional convection–diffusion model in which the transport direction changes within the domain.

7.1. Convection–diffusion problems with flow reversal

Consider the one-dimensional convection–diffusion interaction model

$$u''(x) + p(x)u'(x) + \gamma u(x) = f(x), \quad x \in (0, 1),$$

subject to the boundary conditions $u(0) = 0, u'(1) + u(1) = 0$, where $\gamma > 0$ is a reaction parameter.

Here $u(x)$ denotes the transported quantity and $f(x)$ is a prescribed source term. Let $p(x) = x - \frac{1}{2}$ and $f(x) = 1$, so that the convection coefficient changes sign at $x = \frac{1}{2}$, modelling flow reversal. Let $h = 1/(N+1)$ be a uniform mesh size and let $x_i = ih, i = 1, \dots, N$. Using standard central difference discretizations, $u''(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$, and $u'(x_i) \approx \frac{u_{i+1} - u_{i-1}}{2h}$, we obtain the linear system $A\mathbf{u} = \mathbf{f}$, where the tridiagonal matrix $A = (a_{ij})$ has entries

$$a_{i,i-1} = \frac{1}{h^2} - \frac{p(x_i)}{2h}, \quad a_{ii} = -\frac{2}{h^2} + \gamma, \quad a_{i,i+1} = \frac{1}{h^2} + \frac{p(x_i)}{2h}.$$

For $N = 4$, we have $h = \frac{1}{5}$, $\frac{1}{h^2} = 25$, $\frac{1}{2h} = 2.5$. Choosing $\gamma = 100$, we get $p(x_1) = -0.3$, $p(x_2) = -0.1$, $p(x_3) = 0.1$, $p(x_4) = 0.3$.

The resulting discretization matrix is

$$A = \begin{bmatrix} 50 & 24.25 & 0 & 0 \\ 25.25 & 50 & 24.75 & 0 \\ 0 & 24.75 & 50 & 25.25 \\ 0 & 0 & 24.25 & 50 \end{bmatrix}.$$

The off-diagonal entries do not satisfy the uniform sign condition required for classical M -matrices. But A is strictly diagonally dominant with positive diagonal entries. The matrix A has positive diagonal entries, positive first sub- and super-diagonal entries, and all principal minors are positive. Therefore, A is an M_T^* -matrix.

Consider a diagonal signature matrix $S = \text{diag}(1, -1, 1, -1)$, then the transformed matrix $B = SAS^{-1}$ is a tridiagonal Z -matrix with positive diagonal entries and positive principal minors, and hence B is an M -matrix. This example demonstrates that M^* -matrices arise naturally in the finite-difference discretization of convection–diffusion problems with flow reversal. The M^* framework provides a unified algebraic setting in which existence, stability, spectral properties, and efficient solution of the resulting linear systems can be analyzed, even when classical M -matrix theory does not apply directly.

7.2. Compartmental models with bidirectional flow

In epidemiological and compartmental dynamics, linearized models often take the form

$$\dot{x} = -Ax, \quad A = \begin{bmatrix} a_1 & b_1 & 0 \\ c_1 & a_2 & b_2 \\ 0 & c_2 & a_3 \end{bmatrix}, \quad a_i, b_i, c_i > 0,$$

where x_i denotes the population in the i^{th} compartment, a_i represent removal or decay rates, and b_i, c_i represent transfer rates between adjacent compartments.

This often occurs when transport between compartments is bidirectional or alternating (e.g., reversible migration or alternating infection–recovery flow), the nearest-neighbor interactions may induce sign-alternating coupling after linearization. In such cases, the system matrix A may fail to satisfy the global off-diagonal sign condition required for classical M -matrices, even though it retains a tridiagonal structure and positive diagonal entries. Consider the diagonal signature matrix $S = \text{diag}(1, -1, 1)$, the transformed matrix $B = SAS^{-1} = \begin{bmatrix} a_1 & -b_1 & 0 \\ -c_1 & a_2 & -b_2 \\ 0 & -c_2 & a_3 \end{bmatrix}$ has positive diagonal entries, non-positive off-diagonal entries, and positive principal minors or B is an M -matrix. Consequently, A belongs to the class M_T^* . A defining feature of this class is that it has spectral properties relevant to stability. In particular, M_T^* -matrices have spectra contained in the open right half-plane. So, the eigenvalues of $-A$ lie strictly in the left half-plane, and the equilibrium $x = 0$ of the compartmental system is asymptotically stable.

This example highlights the relevance of the M^* -matrix framework for compartmental systems with alternating bidirectional flow. Even though M -matrix sign conditions are violated, the M^* structure preserves stability through its algebraic properties.

7.3. Network systems

Structured network models with nearest-neighbor interactions often give rise to tridiagonal system matrices. Examples include linear chains of agents, traffic and supply networks. When interaction directions or feedback conventions alternate along the network, the resulting system matrix may violate the sign conditions required by classical M -matrix theory. Such matrices frequently belong to the class of M^* -matrices, which preserve stability and structural properties through their sign structure and principal

minor conditions. This shows the relevance of the M^* framework for stability analysis of networked systems with alternating interaction patterns.

8. Conclusion

In this paper, we have introduced and studied a new class of matrices, called M^* -matrices, defined by a prescribed sign structure together with positivity of all principal minors. This class relaxes the global off-diagonal sign restrictions of classical M -matrices while preserving many of their fundamental algebraic, spectral and numerical properties. In this sense, M^* -matrices provide a natural extension of both classical M -matrices and totally positive tridiagonal matrices. For tridiagonal systems, we established several core structural results. We showed that positivity of leading principal minors is equivalent to positivity of all principal minors, derived from the explicit determinant recursions, and proved the existence of LU factorizations without pivoting, with strictly positive pivots guaranteed by principal-minor positivity. We also identified an important subclass, denoted by M_T^* , consisting of tridiagonal M^* -matrices that are diagonally signature-similar to nonsingular M -matrices. For this subclass, classical M -matrix theory transfers directly, yielding positive real spectra, inverse-sign regularity, and invariance of induced norm condition numbers under signature similarity.

The relevance of the theory was illustrated through applications to convection-diffusion problems with flow reversal, compartmental models with bidirectional flow, and network systems. These examples demonstrate that M^* -matrices arise naturally in situations where classical M -matrix assumptions are violated by sign-alternating nearest-neighbor interactions, yet stability and numerical robustness are retained through diagonal signature similarity.

The study of M^* -matrices suggests several promising directions for further research. These include extending the theory to nonsymmetric and nontridiagonal M^* -matrix type systems arising in multidimensional PDE discretizations and network models; investigating connections with H -matrices, generalized inverse M -matrices, and monotone operator theory; developing sharper perturbation bounds, spectral radius estimates, and condition number analysis tailored to the M^* structure; and exploring applications in optimization, control, and large-scale numerical simulation, where the alternating-sign inverse structure may offer computational advantages. Overall, the M^* -matrix framework provides a unified algebraic perspective linking matrix theory, numerical analysis, and dynamical systems, and offers new tools for analyzing structured positivity, stability, and spectral regularity in tridiagonal linear systems.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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