



On $(n - 1)$ -bicolorable Steiner triple systems of order $2^n - 7$

Research Article

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Abstract: A *Steiner triple system* STS of order v is a hypergraph of rank 3, with v vertices and such that every 2-subset of vertices has degree 1. In this paper we construct $STS(2v + 7)$ containing subsystems STS of order v . This construction allows us to obtain, for any $n \geq 5$, $(n - 1)$ -bicolorable systems $STS(2^n - 7)$ having a maximal independent set of largest cardinality.

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1. Introduction

A Steiner system $S(h, k, v)$ is a pair $\Sigma = (X, \mathcal{B})$, where X is a v -set and $\mathcal{B}$ is a family of k -subsets of X such that every h -subset of X is contained in exactly one member of $\mathcal{B}$ (see [6, 8, 9]). Using hypergraph theory terminology, a Steiner system is a hypergraph $\Sigma = (X, \mathcal{B})$ of order v , uniform of rank k , such that every h -subset Y of X has degree $d(Y) = 1$ (see [1, 8, 16]).

A Steiner triple system STS is a system $S(2, 3, v)$. It is known that *an $STS(v)$ exists if and only if $v \equiv 1$ or $3 \pmod{6}$* .

Given an $STS \Sigma = (X, \mathcal{B})$, an *independent set* T is a subset $T \subset X$ that does not contain any triple of $\mathcal{B}$. T is a *maximal independent set* for Σ if it is not contained in another independent set. By [13] a maximal independent set in an $STS(v)$, with $v \equiv 1, 9 \pmod{12}$, has cardinality at most $\frac{v-1}{2}$ (see also [4]). Note that $v = 2^n - 7 \equiv 1$ or $9 \pmod{12}$ for any $n \in \mathbb{N}$.

A *bicoloring* of an $STS \Sigma = (X, \mathcal{B})$ is a coloring of the vertices in such a way that all the blocks contain vertices of exactly two colors. Σ is called *k -bicolorable* if there exists a bicoloring of Σ using exactly k colors. For a bicolorable Steiner triple system Σ we denote $\chi(\Sigma)$ (resp. $\bar{\chi}(\Sigma)$) the *lower* (resp. *upper*) chromatic number, which is the smallest (resp. largest) integer k for which there exists a k -bicoloring of Σ using exactly k colors. Moreover, by a simple count it is immediate to see that no STS is 2-bicolorable (see [12]).

These types of colorings were introduced by Voloshin [14, 15] in the context of mixed hypergraphs and for Steiner systems the study was initiated by Milazzo and Tuza [10]. It is relevant to remark that in [10] it was proved that there exists a strong connection between bicolorable $STS(2^n - 1)$ with the highest possible upper chromatic number and the “doubling plus one construction” $v \mapsto 2v + 1$. Indeed, if Σ is an $STS(2^n - 1)$ with $\bar{\chi}(\Sigma) = n$, then Σ is obtained from the $STS(3)$ by repeated applications of “doubling plus one constructions”. Later, many other papers dealt with bicolorings of Steiner systems (see, for example, [2, 3, 5, 7, 11]).

In this paper we give in Theorem 4.1 a construction $v \longrightarrow 2v + 7$, in which $v = 2^n - 7$. We show that, through an iteration of this construction, it is possible, starting from the $STS(9)$, to obtain for any $n \geq 5$ an $STS(2^n - 7)$, Σ , with a maximal independent set of largest cardinality and with $\bar{\chi}(\Sigma) = n - 1$, where $n - 1$ is the largest possible upper chromatic number.

2. Construction $v \longrightarrow 2v + 1$

It is known in the literature the following construction $v \longrightarrow 2v + 1$, which allow us to construct an $STS(2v + 1)$ starting from an $STS(v)$. It is also known that the order $2v + 1$ is the minimum value for which a system $STS(v)$ can be embedded in a system (v') for $v' > v$.

Theorem 2.1. *If $\Sigma = (X, \mathcal{B})$ is an $STS(v)$, then there exists an $STS(2v + 1)$ embedding Σ .*

Proof. Let $\Sigma = (X, \mathcal{B})$ be an $STS(v)$ defined on $X = \{x_1, x_2, \dots, x_v\}$. Further, let:

- $Y = \{y_1, y_2, \dots, y_{v+1}\}$ be a set of cardinality $v + 1$ (even number) such that $X \cap Y = \emptyset$;
- $\mathcal{F} = \{F_1, F_2, \dots, F_v\}$ be a factorization of the complete graph K_{v+1} defined on Y ;
- $\varphi : X \longrightarrow \mathcal{F}$ be any bijection from X into $\mathcal{F}$.

Define the hypergraph $\Sigma' = (X', \mathcal{B}')$ as follows:

- $X' = X \cup Y$;
- $\mathcal{B}' = \mathcal{B} \cup \Gamma$, where:

$$\Gamma = \{\{x, y, z\} : x \in X, y, z \in \varphi(x)\}.$$

We call the triples of $\mathcal{B}$ of type 1, the triples of Γ of type 2. We say that Σ' is an $STS(2v+1)$. Indeed:

1. it is immediate that Σ' is a hypergraph of order $|X'| = 2v+1$, uniform of rank 3;
2. for every $x, y \in X', x \neq y$, there exists exactly one triple of $\mathcal{B}'$ containing $\{x, y\}$. There are the possible following three cases:
 - 2.1. $x, y \in X$;
 - 2.2. $x, y \in Y$;
 - 2.3. $x \in X, y \in Y$.

Case 2.1. Since Σ is an STS and Γ does not contain pairs of vertices of X , there exists exactly one block in $\mathcal{B}$ containing $\{x, y\}$.

Case 2.2. There exists exactly one factor $F_j \in \mathcal{F}$ containing x, y . If $x_i \in X$ is such that $\varphi(x_i) = F_j$, the triple $\{x_i, x, y\}$ is of type 2 and is it the unique, triple of $\mathcal{B}'$ containing $\{x, y\}$.

Case 2.3. Consider the factor $\varphi(x) = F_j \in \mathcal{F}$. There exists exactly one pair of F_j containing y . Let $\{y, z\} \in F_j$. The triple $\{x, y, z\}$ is a triple of type 2 and it is the unique triple of $\mathcal{B}'$ containing the pair $\{x, y\}$. $\square$

3. Factorization on $\mathbb{Z}_{2^n}$ by differences

Let $r \in \mathbb{N}$ even and let $\mathbb{Z}_{2^n} = \{0, 1, 2, \dots, 2^n - 1\}$, $D_{2^n} = \{1, 2, \dots, 2^{n-1}\}$. In this section we define a factorization of the complete graph K_{2^n} defined in $\mathbb{Z}_{2^n}$, briefly a factorization on $\mathbb{Z}_{2^n}$, where every factor contains pair having all the same difference. Note that all the elements belong to $\mathbb{Z}_{2^n}$.

Theorem 3.1. 1) If $a \in D_{2^n} - \{2^{n-1}\}$, then there exist two disjoint factors of $\mathbb{Z}_{2^n}$, $G_{a,1}, G_{a,2}$, containing all the pairs $\{x, y\} \subset \mathbb{Z}_{2^n}$ having difference a . 2) If $a = 2^{n-1}$, then there exists an unique factor H containing all the pairs $\{x, y\} \subset \mathbb{Z}_{2^n}$ having difference a .

Proof. 1) Consider the complete graph K_{2^n} defined on $\mathbb{Z}_{2^n}$. If $a \in D_{2^n} - \{2^{n-1}\}$ is an odd number, then $pa \equiv 0 \pmod{2^n}$ if and only if $p \equiv 0 \pmod{2^n}$, hence there exists in K_{2^n} a hamiltonian cycle of length $\mathbb{Z}_{2^n}$ described as follows:

$$(0, a, 2a, 3a, \dots, 2^n - a, 0),$$

which can be decomposed in the following two factors:

$$F_{a,1} : \{0, a\}, \{2a, 3a\}, \dots, \{2^n - 2a, 2^n - a\}$$

$$F_{a,2} : \{a, 2a\}, \{3a, 4a\}, \{5a, 6a\}, \dots, \{2^n - a, 2^n = 0\}.$$

If $a \in D_{2^n} - \{2^{n-1}\}$ is an even number and $k = GCD(a, 2^n)$, since $pa \equiv 0 \pmod{2^n}$ if and only if $p \equiv 0 \pmod{\frac{2^n}{k}}$, in the complete graph K_{2^n} defined in $\mathbb{Z}_{2^n}$ there exist k disjoint cycles of length $\frac{2^n}{k}$, described as follows:

$$C(a, 1) = (0, a, 2a, 3a, \dots, 2^n - a, 0),$$

$$C(a, 2) = (1, 1 + a, 1 + 2a, 1 + 3a, \dots, 1 + 2^n - a, 1 + 2^n = 1),$$

...

$$C(a, k) = (k - 1, k - 1 + a, k - 1 + 2a, \dots, 2^n + k - 1 - a, 2^n + k - 1 = k - 1),$$

which can be decomposed in the following factors:

$$G_{a,1}: \{i, a+i\}, \{2a+i, 3a+i\}, \{4a+i, 5a+i\}, \dots \\ \{2^n - 2a + i, 2^n - a + i\}, \quad \text{for } i = 0, \dots, k-1,$$

$$G_{a,2}: \{a+i, 2a+i\}, \{3a+i, 4a+i\}, \{5a+i, 6a+i\}, \dots, \\ \{2^n - a + i, i\} \quad \text{for } i = 0, \dots, k-1.$$

2) If $a = 2^{n-1}$, it is immediate to verify that in the complete graph K_{2^n} , defined in $\mathbb{Z}_{2^n}$, there exist the following unique factor:

$$H: \{0, 2^{n-1}\}, \{1, 2^{n-1} + 1\}, \{2, 2^{n-1} + 2\}, \dots, \{2^{n-1} - 1, 2^n - 1\}.$$

□

Note that in what follows we will consider the following factorization $\mathcal{F}$, defined in $\mathbb{Z}_{2^n}$:

$$\mathcal{F} = \{H\} \cup \{F_{a,1}, F_{a,2} : a \in D_{2^n}, a \text{ odd}\} \cup \\ \cup \{G_{a,1}, G_{a,2} : a \in D_{2^n}, a \neq 2^{n-1}, a \text{ even}\}, \quad (1)$$

which we call *difference factorization* of $\mathbb{Z}_{2^n}$. Note that a is an odd number in $F_{a,1}, F_{a,2}$ and it is an even number in $G_{a,1}, G_{a,2}$.

4. A construction $v \longrightarrow 2v + 7$

Let $n \in \mathbb{N}$, $n \geq 3$, and let $v = 2^n - 7$. Note that $2^n - 7$ is a positive integer number such that for n odd $2^n - 7 \equiv 1 \pmod{6}$ and for n even $2^n - 7 \equiv 3 \pmod{6}$. Note also that $2v + 7 = 2^{n+1} - 7$.

Theorem 4.1. *Let $\Sigma = (X, \mathcal{B})$ be an STS(v), where $v = 2^n - 7$, $n \in \mathbb{N}$, $n \geq 3$. It is possible to define an STS($2v + 7$) $\Sigma' = (X', \mathcal{B}')$ embedding Σ .*

Proof. Let $\Sigma = (X, \mathcal{B})$ be an STS(v) defined on $X = \{x_1, x_2, \dots, x_v\}$. Further, let:

- $Y = \mathbb{Z}_{2^n}$ be a set such that $X \cap Y = \emptyset$;
- $\mathcal{F}$ be the difference factorization of $\mathbb{Z}_{2^n}$ given in (1);
- $\{a, b, c\}$ be a difference triple contained in D_{2^n} , for $a \neq 2^{n-1}, b \neq 2^{n-1}, c \neq 2^{n-1}$, with either $c = a + b$ or $a + b + c = 2^n$;
- $\mathcal{F}^*$ be the family of factors obtained by $\mathcal{F}$ by excluding the factors containing the pairs having differences either a or b or c .

Further, since $|\mathcal{F}^*| = |\mathcal{F}| - 6 = 2^n - 7 = v$, it is possible to define any bijection $\varphi : X \longrightarrow \mathcal{F}^*$ from X into $\mathcal{F}^*$.

In what follows we fix $a = 1, b = 2, c = 3$. Note that, from Theorem 3.1, this causes no loss of generality. At this point, define the hypergraph $\Sigma' = (X', \mathcal{B}')$ as follows:

- $X' = X \cup Y$;

- $\mathcal{B}' = \mathcal{B} \cup \Gamma \cup \Delta$, where:

$$\Gamma = \{\{x, y, z\} : x \in X, \{y, z\} \in \varphi(x)\},$$

$$\Delta = \{\{i, i+1, i+3\} : i = 0, 1, 2, \dots, 2^n - 1\}.$$

We call the triples of $\mathcal{B}$ of *type 1*, the triples of Γ of *type 2*, the triples of Δ of *type 3*. Observe that the triples of Δ are all the translates generated by the base block $\{0, 1, 3\}$, defined by the difference triple $\{1, 2, 3\}$. We are going to prove that Σ' is an $STS(2v+7)$. Indeed:

1. it is immediate that Σ' is a hypergraph of order $|X'| = 2v+7$, uniform of rank 3;
2. for every $x, y \in X', x \neq y$, there exists exactly one triple of $\mathcal{B}'$ containing the pair $\{x, y\}$. Consider the following three possible cases:
 - 2.1. $x, y \in X$;
 - 2.2.1. $x, y \in Y$ and the difference between x, y is either 1 or 2 or 3;
 - 2.2.2. $x, y \in Y$ and the difference between x, y is in $\{4, \dots, 2^{n-1}\}$;
 - 2.3. $x \in X, y \in Y$.

Case 2.1. Since Σ is an STS and $\Gamma \cup \Delta$ do not contain vertices of X , there exists exactly one block in $\mathcal{B}$, of type 1, containing x, y .

Case 2.2.1. In Σ there are not vertices of Y and in the factors of $\mathcal{F}^*$ there are not pairs of vertices having differences 1, 2, or 3. In Δ there are triples containing all the pairs of vertices of Y with differences either 1 or 2 or 3, therefore there exists in Δ a block, of type 3, containing $\{x, y\}$.

Case 2.2.2. In the factors of $\mathcal{F}^*$ there are all the pairs of Y having difference $a \in D_{2^n} - \{1, 2, 3\}$, therefore there exists in $\mathcal{F}^*$ a factor F containing $\{x, y\}$. If $z = \varphi^{-1}(F)$, then the triple $\{x, y, z\}$ is a block of type 2 of Σ' .

Case 2.3. In this case, consider the factor $\varphi(x) = F \in \mathcal{F}^*$. There exists exactly one pair of the factor F containing y . Let $\{y, z\} \in F$. The triple $\{x, y, z\}$ is a triple of type 2 containing the pair $\{x, y\}$.

So, we have proved that, in general, for every pair $\{x, y\} \subset X' = X \cup Y$ it is $d(x, y) \geq 1$. To prove that it is exactly $d(x, y) = 1$ it is sufficient to prove that $|\mathcal{B}'| = \frac{(2v+7)(2v+6)}{6}$, which is the exact number of blocks in any $STS(2v+7)$. Since $\mathcal{B}' = \mathcal{B} \cup \Gamma \cup \Delta$ and $\mathcal{B}, \Gamma, \Delta$ are pairwise disjoint, it follows that:

$$|\mathcal{B}'| = |\mathcal{B}| + |\Gamma| + |\Delta| = \frac{v(v-1)}{6} + v \cdot \frac{v+7}{2} + (v+7) = \frac{(2v+7)(2v+6)}{6}.$$

□

Corollary 4.2. Let $n \in \mathbb{N}$, $n \geq 4$, and let $\Sigma = (X, \mathcal{B})$ be an $STS(v)$, with $v = 2^n - 7$, having a maximal independent set T of largest cardinality $\frac{v-1}{2}$. Then there exists an $STS \Sigma'$ of order $v' = 2^{n+1} - 7$ embedding Σ with a maximal independent set $T' \supset T$ of largest cardinality $\frac{v'-1}{2}$.

Proof. In the proof of Theorem 4.1 if T is a maximal independent set for Σ , then $T \cup \{2i \mid i = 0, 1, \dots, 2^{n-1}\}$ is a maximal independent set for Σ' of cardinality $\frac{v'-1}{2}$, with $v' = 2^{n+1} - 7$, under the following conditions:

- the restriction $\varphi|_T : T \rightarrow \mathcal{F}^*$ induces a bijection with the subset of $\mathcal{F}^*$ determined by the differences $a \in D_{2^n}$, a odd;

- the restriction $\varphi|_{X-T}: X - T \rightarrow \mathcal{F}^*$ induces a bijection with the subset of $\mathcal{F}^*$ determined by the differences $a \in D_{2^n}$, a even.

□

Corollary 4.3. *There exists for any $n \in \mathbb{N}$, $n \geq 4$, an $STS(v)$, with $v = 2^n - 7$, having a maximal independent set of largest cardinality $\frac{v-1}{2}$.*

Proof. It is sufficient to apply iteratively the previous corollary, considering that the base case is $v = 9$, for which the statement is easily proved in the Appendix. □

Corollary 4.4. *Let $n \in \mathbb{N}$, $n \geq 4$. Then there exists an $STS(v)$ Σ , with $v = 2^n - 7$, such that $\bar{\chi}(\Sigma) = n - 1$.*

Proof. Note that by [10, Theorem 1 and Corollary 1] for any bicolored $STS(2^n - 7)$ Σ it must be $\bar{\chi}(\Sigma) < n$, which implies that we simply need to prove the existence of an $STS(2^n - 7)$ which is $(n - 1)$ -bicolored.

Let $v = 9$. In this case, given the system Σ in the Appendix, it is 3-bicolored with the color classes $\{x_1, x_2, x_4, x_5\}$, $\{x_3, x_6, x_7, x_8\}$ and $\{x_9\}$. Now, let $T = \{x_1, x_2, x_4, x_5\}$ a maximal independent set for Σ . When we apply the construction given in Corollary 4.2 to Σ , the system Σ' of order 25 that we get is 4-bicolored, since we can take as color classes $C_1 = \{x_1, x_2, x_4, x_5\} \cup \{2i \mid i = 0, \dots, 7\}$, $C_2 = \{x_3, x_6, x_7, x_8\}$, $C_3 = \{x_9\}$ and $C_4 = \{2i + 1 \mid i = 0, \dots, 7\}$. Moreover, a maximal independent set is $T \cup \{2i \mid i = 0, \dots, 7\} = C_1$.

For $n \geq 5$ we apply iteratively the construction given in Corollary 4.2, where the case $n = 5$ has been previously explained. So, we have an $STS(2^{n-1} - 7)$ which is $(n - 2)$ -bicolored and such that there exists a $(n - 2)$ -bicoloring having as color class, say C_1 , a maximal independent set T . In this case, proceeding as in Corollary 4.2 it is sufficient to give the color of 1 to the vertices in $\{2i \mid i = 0, \dots, 2^{n-1} - 1\}$ and the color $n - 1$ to the vertices in $\{2i + 1 \mid i = 0, \dots, 2^{n-1} - 1\}$. Note that in this way a color class in an $(n - 1)$ -bicoloring of Σ' coincides with a maximal independent set of largest cardinality. □

5. Appendix

In this section, we give an application of the construction described in this paper in the cases $v = 9$ and $2v + 7 = 25$. Let:

- $Y = \mathbb{Z}_{16} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$,
- $D_{16} = \{1, 2, 3, 4, 5, 6, 7, 8\}$,
- $\Sigma = (X, \mathcal{B}), STS(9)$,
- $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$,
- $\mathcal{B}$:

$$\{x_1, x_2, x_3\}, \{x_1, x_4, x_7\}, \{x_1, x_5, x_9\}, \{x_1, x_6, x_8\},$$

$$\{x_4, x_5, x_6\}, \{x_2, x_5, x_8\}, \{x_2, x_6, x_7\}, \{x_2, x_4, x_9\},$$

$$\{x_7, x_8, x_9\}, \{x_3, x_6, x_9\}, \{x_3, x_4, x_8\}, \{x_3, x_5, x_7\}.$$

Note that $T = \{x_1, x_2, x_4, x_5\}$ is a maximal independent set for Σ and that the sets $\{x_1, x_2, x_4, x_5\}$, $\{x_3, x_6, x_7, x_8\}$ and $\{x_9\}$ are the color classes of a 3-bicoloring of Σ . Let Γ be the family of triples containing an $x_i \in X$ and a pair indicated in the following columns:

x_1	x_2	x_3	x_4	
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
$F_{5,1} :$	$F_{5,2} :$	$G_{4,1} :$	$F_{7,2} :$	
$0 \cdot 5$	$5 \cdot 10$	$0 \cdot 4$	$7 \cdot 14$	
$10 \cdot 15$	$15 \cdot 4$	$8 \cdot 12$	$5 \cdot 12$	
$4 \cdot 9$	$9 \cdot 14$	$1 \cdot 5$	$3 \cdot 10$	
$14 \cdot 3$	$3 \cdot 8$	$9 \cdot 13$	$1 \cdot 8$	
$8 \cdot 13$	$13 \cdot 2$	$2 \cdot 6$	$15 \cdot 6$	
$2 \cdot 7$	$7 \cdot 12$	$10 \cdot 14$	$13 \cdot 4$	
$12 \cdot 1$	$1 \cdot 6$	$3 \cdot 7$	$11 \cdot 2$	
$6 \cdot 11$	$11 \cdot 0$	$11 \cdot 15$	$9 \cdot 0$	
x_5	x_6	x_7	x_8	x_9
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$F_{7,1} :$	$G_{4,2} :$	$F_{6,1} :$	$F_{6,2} :$	$H :$
$0 \cdot 7$	$4 \cdot 8$	$0 \cdot 6$	$6 \cdot 12$	$0 \cdot 8$
$14 \cdot 5$	$12 \cdot 0$	$12 \cdot 2$	$2 \cdot 8$	$1 \cdot 9$
$12 \cdot 3$	$5 \cdot 9$	$8 \cdot 14$	$14 \cdot 4$	$2 \cdot 10$
$10 \cdot 1$	$13 \cdot 1$	$4 \cdot 10$	$10 \cdot 0$	$3 \cdot 11$
$8 \cdot 15$	$6 \cdot 10$	$1 \cdot 7$	$7 \cdot 13$	$4 \cdot 12$
$6 \cdot 13$	$14 \cdot 2$	$13 \cdot 3$	$3 \cdot 9$	$5 \cdot 13$
$4 \cdot 11$	$7 \cdot 11$	$9 \cdot 15$	$15 \cdot 5$	$6 \cdot 14$
$2 \cdot 9$	$15 \cdot 3$	$5 \cdot 11$	$11 \cdot 1$	$7 \cdot 15$

Let $X' = X \cup Y$ and $\mathcal{B}' = \mathcal{B} \cup \Gamma \cup \Delta$, where $\mathcal{B}$ and Γ are already defined and

$$\Delta = \{\{i, i+1, i+3\} : i = 0, 1, 2, \dots, 15\}.$$

It is immediate to verify that $\Sigma'(X', \mathcal{B}')$ is an $STS(25)$.

6. Conclusion

In this paper, we propose a construction of a Steiner triple system of order $2v+7$ embedding a Steiner triple system of order v , where $v = 2^n - 7$ and $n \geq 3$. We show that, by iterating this construction, it is possible to obtain, starting from an $STS(9)$, a Steiner triple system Σ of order $2^n - 7$ for every $n \geq 5$, with a maximum independent set of largest possible cardinality and upper chromatic number $\chi(\Sigma) = n - 1$, where $n - 1$ is the maximum possible value. It is the authors' intention to further investigate this type of construction for Steiner triple systems.

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Declarations

Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare that they have no conflict of interest.

Data availability

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