



Partitions with unique largest part and their generating functions

Research Article

Harman Kaur • Meenakshi Rana

Abstract: The paper introduces a new type of partition where the largest part appears only once, and the remaining parts together form a partition of that largest part. We also derive the generating function for these partitions, a variation of it.

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1. Introduction

Partitions have been the object of interest to mathematicians for many years. A *partition* of a positive integer n is a way of expressing it as a sum of positive integer parts; generally, the parts are listed in descending order, from the largest (ℓ) to the smallest (s). The generating function to count the number of partitions $p(n)$ of n is

$$\sum_{n \geq 0} p(n)q^n = \frac{1}{(q; q)_{\infty}},$$

Harman Kaur (Corresponding Author); Department of Mathematics, Chandigarh University, Mohali-140413, Punjab, India, (kaur.harman196@gmail.com).

Meenakshi Rana; Department of Mathematics, Thapar Institute of Engineering and Technology, Patiala-147004, Punjab, India, (mrana@thapar.edu).

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where

$$(t; q)_n = \begin{cases} 1 & \text{for } n = 0, \\ (1-t)(1-tq)(1-tq^2)(1-tq^3) \cdots (1-tq^{n-1}) & \text{for } n > 0 \end{cases}$$

is the q -shifted factorial and

$$(t; q)_\infty = \lim_{n \rightarrow \infty} (t; q)_n, \quad \text{where } |q| < 1.$$

For a detailed understanding of integer partitions, see [1, 2]. The study of restricted partitions dates back to Euler, who established a remarkable result: the number of partitions of any integer into odd parts, $o(n)$, is equal to the number of partitions into distinct parts or strict partitions, $d(n)$. The corresponding sequence of partitions is listed in the On-Line Encyclopedia of Integer Sequences (OEIS) under the number A000009. OEIS is a vast database of well-defined sequences of integers, many of which originate from partition theory. Analogous to Euler's results, many appealing results on partitions can be found in [3–6].

In this paper, we discuss the partition Λ , in which the largest part occurs only once, while the remaining parts collectively form a partition of the largest part.

For example,

$$5 + 4 + 1, \quad 5 + 3 + 2, \quad 5 + 2 + 2 + 1.$$

However, some of the other partitions do not satisfy this condition, such as $9 + 1$, $8 + 2$, $7 + 3$, and so on. Since the sum of all parts other than the largest part always equals the largest part (ℓ), the total sum is always an even number (2ℓ). We provide the generating function corresponding to the partition and to some associated restricted partitions analogously. The partition structures that are discussed in this paper are based on a common partition-theoretic principle: imposing specific restrictions or structural conditions on the parts of an integer partition creates new classes of partitions whose enumerative aspects can be studied using generating functions. The integer sequences corresponding to partitions described in Theorem 1.1–Theorem 1.4 are listed in OEIS. However, the sequences in Theorem 1.5–Theorem 1.6 are new. The notations are defined wherever necessary; otherwise, it is defined before the theorem in which it is used. The generating functions for $\Lambda_d, \Lambda_o, \Lambda_{od}$, respectively, are given in the following theorems.

Theorem 1.1. *The generating function to count the number of partitions Λ is*

$$\sum_{n=0}^{\infty} \rho(n)q^n = \frac{1}{(q^2; q^2)_\infty} - \frac{1}{(1 - q^2)}. \quad (1)$$

Motivated from Euler's theorem on partitions of n into distinct/odd parts, we now propose the similar restricted partitions for Λ . We use the notations as follows:

Λ_d : partition Λ where the summands are distinct,

Λ_o : partition Λ where the summands are odd,

Λ_{od} : partition Λ where the summands are odd and distinct.

The generating functions for $\Lambda_d, \Lambda_o, \Lambda_{od}$, respectively, are given in the following theorems.

Theorem 1.2. *The generating function to count the number of partitions Λ_d is*

$$\sum_{n=0}^{\infty} \rho_d(n)q^n = (-q^2; q^2)_\infty - \frac{1}{(1 - q^2)}. \quad (2)$$

Theorem 1.3. *The generating function to count the number of partitions Λ_o is*

$$\sum_{n=0}^{\infty} \rho_o(n)q^n = \frac{1}{(q^2; q^4)_\infty} - \frac{q^2}{(1 - q^4)} - 1. \quad (3)$$

Theorem 1.4. *The generating function to count the number of partitions Λ_{od} is*

$$\sum_{n=0}^{\infty} \rho_{od}(n)q^n = (-q^2; q^4)_{\infty} - \frac{q^2}{(1-q^4)} - 1. \quad (4)$$

Inspired by Schur's theorem, which states that the number of partitions into distinct parts $\equiv \pm 1 \pmod{3}$ is the same as the number of partitions into parts $\equiv \pm 1 \pmod{6}$, we consider the partition Λ with distinct summands and $\equiv \pm 1 \pmod{3}$ except the largest part, denoted by Λ_3 .

Theorem 1.5. *The generating function to count the number of partitions Λ_3 is*

$$\sum_{n=0}^{\infty} \rho_3(n)q^n = (-q^2, -q^4; q^6)_{\infty} - \frac{q^2}{1-q^2} + \frac{q^6}{1-q^6}. \quad (5)$$

Analogously, the generating function for the partitions Λ with summands $\equiv \pm 1 \pmod{6}$ except the largest part, denoted by Λ_6 , is given in the next theorem.

Theorem 1.6. *The generating function to count the number of partitions Λ_6 is*

$$\sum_{n=0}^{\infty} \rho_6(n)q^n = \frac{1}{(q^2, q^{10}; q^{12})_{\infty}} - \frac{(q^2 + q^{10})}{1 - q^{12}}. \quad (6)$$

2. Proofs of the theorems

Proof of Theorem 1.1. The sequence of the partition Λ can be generated as:

$$\begin{aligned} \sum_{n=0}^{\infty} \rho(n)q^n &= q^2 \cdot q^{1+1} + q^3 \cdot (q^{2+1} + q^{1+1+1}) + q^4 \cdot (q^{3+1} + q^{2+2} + q^{2+1+1} + q^{1+1+1+1}) \\ &\quad + q^5 \cdot (q^{4+1} + q^{3+2} + q^{3+1+1} + q^{2+2+1} + q^{2+1+1+1} + q^{1+1+1+1+1}) + \dots \\ &= \sum_{n=0}^{\infty} q^n (p(n) - 1)q^n \\ &= \sum_{n=0}^{\infty} (p(n) - 1)q^{2n} \\ &= \sum_{n=0}^{\infty} p(n)q^{2n} - \sum_{n=0}^{\infty} q^{2n} \\ &= \frac{1}{(q^2; q^2)_{\infty}} - \frac{1}{(1 - q^2)}. \end{aligned} \quad (7)$$

□

Proof of Theorem 1.2. Consider partition Λ_d with the largest part of multiplicity 1, and the rest of the summands are distinct that represent the partition of the corresponding largest part, it can be generated as:

$$\begin{aligned} \sum_{n=0}^{\infty} \rho_d(n)q^n &= q^3 \cdot (q^{2+1}) + q^4 \cdot (q^{3+1}) + q^5 \cdot (q^{4+1} + q^{3+2}) + q^6 \cdot (q^{5+1} + q^{4+2} + q^{3+2+1}) \\ &\quad + q^7 \cdot (q^{6+1} + q^{5+2} + q^{4+3} + q^{4+2+1}) + \dots \end{aligned}$$

Let $d(n)$ represent the number of distinct partitions of n , then

$$\begin{aligned}
 \sum_{n=0}^{\infty} \rho_d(n) q^n &= \sum_{n \geq 3} q^n (d(n) - 1) q^n \\
 &= \sum_{n \geq 3} (d(n) - 1) q^{2n} \\
 &= \sum_{n \geq 3} d(n) q^{2n} - \sum_{n \geq 3} q^{2n} \\
 &= \sum_{n \geq 0} d(n) q^{2n} - \sum_{n \geq 0} q^{2n} \\
 &= (-q^2; q^2)_{\infty} - \frac{1}{(1 - q^2)}.
 \end{aligned} \tag{8}$$

□

Proof of Theorem 1.3. The generating function for partitions with the largest part and the rest of the summands are odd, which constitute the partition of the corresponding largest part, leading to

$$\begin{aligned}
 \sum_{n=0}^{\infty} \rho_o(n) q^n &= q^2 \cdot (q^{1+1}) + q^3 \cdot (q^{1+1+1}) + q^4 \cdot (q^{3+1} + q^{1+1+1+1}) \\
 &\quad + q^5 \cdot (q^{3+1+1} + q^{1+1+1+1+1}) + q^6 \cdot (q^{5+1} + q^{3+3} + q^{3+1+1+1}) \\
 &\quad + q^7 \cdot (q^{5+1+1} + q^{3+3+1} + q^{3+1+1+1+1} + q^{1+1+1+1+1+1+1}) + \dots
 \end{aligned}$$

Let $o(n)$ represent the number of odd partitions of n , then

$$\begin{aligned}
 \sum_{n=0}^{\infty} \rho_o(n) q^n &= \sum_{\substack{n \geq 2 \\ n \text{ even}}} o(n) q^{2n} + \sum_{\substack{n \geq 3 \\ n \text{ odd}}} (o(n) - 1) q^{2n} \\
 &= \sum_{n \geq 2} o(n) q^{2n} - \sum_{\substack{n \geq 3 \\ n \text{ odd}}} q^{2n} \\
 &= \sum_{n \geq 0} o(n) q^{2n} - \sum_{\substack{n \geq 1 \\ n \text{ odd}}} q^{2n} - (o(0) q^0 + o(1) q^2) + q^2 \\
 &= \frac{1}{(q^2; q^4)_{\infty}} - \frac{q^2}{(1 - q^4)} - 1.
 \end{aligned} \tag{9}$$

□

Proof of Theorem 1.4. The generating function for partition Λ with the summands, apart from the largest part, are distinct and odd, which constitute the largest part, leading to

$$\begin{aligned}
 \sum_{n=0}^{\infty} \rho_{od}(n) q^n &= q^4 \cdot q^{3+1} + q^6 \cdot q^{5+1} + q^8 \cdot (q^{7+1} + q^{5+3}) + q^9 \cdot (q^{5+3+1}) \\
 &\quad + q^{10} \cdot (q^{9+1} + q^{7+3}) + q^{11} \cdot (q^{7+3+1}) + q^{12} \cdot (q^{11+1} + q^{9+3} + q^{7+5}) \\
 &\quad + q^{13} \cdot (q^{9+3+1} + q^{7+5+1}) + \dots
 \end{aligned}$$

Let $od(n)$ represent the number of odd and distinct partitions of n , then

$$\sum_{n=0}^{\infty} \rho_{od}(n) q^n = \sum_{\substack{n \geq 1 \\ n \text{ even}}} q^n od(n) q^n + \sum_{\substack{n \geq 1 \\ n \text{ odd}}} q^n (od(n) - 1) q^n \tag{10}$$

$$\begin{aligned}
&= \sum_{n \geq 1} \text{od}(n) q^{2n} - \sum_{\substack{n \geq 1 \\ n \text{ odd}}} q^{2n} \\
&= \sum_{n \geq 0} \text{od}(n) q^{2n} - \frac{q^2}{1 - q^4} - 1 \\
&= (-q^2; q^4)_\infty - \frac{q^2}{1 - q^4} - 1.
\end{aligned}$$

□

In Table 1, the corresponding sequences listed in OEIS with relevant examples are presented:

Table 1. Corresponding OEIS sequences and their relevant examples.

$\rho_i(n)$	Relevant example as per interpretation	OEIS sequence no.	OEIS interpretation
$\rho(8) = 4$	$4 + 3 + 1, 4 + 2 + 2,$ $4 + 2 + 1 + 1, 4 + 1 + 1 + 1 + 1$	A000065	$-1 + \#$ partitions of n
$\rho_d(8) = 1$	$4 + 3 + 1$	A111133	$\#$ partitions of n into at least two distinct parts
$\rho_o(8) = 2$	$4 + 3 + 1, 4 + 1 + 1 + 1 + 1$	A357456	$\#$ partitions of n into two or more odd parts
$\rho_{od}(10) = 1$	$4 + 3 + 1$	A357457	$\#$ partitions of n into two or more distinct odd parts

Proof of Theorem 1.5. The generating function for partitions Λ with the summands (apart from the largest part) are distinct and $\equiv \pm 1 \pmod{3}$ which constitute the largest part, lead to

$$\begin{aligned}
\sum_{n=0}^{\infty} \rho_3(n) q^n &= q^3 \cdot q^{1+2} + q^5 \cdot q^{4+1} + q^6 \cdot (q^{5+1} + q^{4+2}) + q^7 \cdot (q^{5+2} + q^{4+2+1}) \\
&\quad + q^8 \cdot (q^{7+1} + q^{5+2+1}) + q^9 \cdot (q^{8+1} + q^{7+2} + q^{5+4}) \\
&\quad + q^{10} \cdot (q^{8+2} + q^{7+2+1} + q^{5+4+1}) + q^{11} \cdot (q^{10+1} + q^{8+2+1} + q^{7+1} + q^{5+4+2}) + \dots
\end{aligned}$$

Let $dp_3(n)$ represent the number of distinct partitions of n with parts $\equiv \pm 1 \pmod{3}$, then

$$\begin{aligned}
\sum_{n=0}^{\infty} \rho_3(n) q^n &= \sum_{\substack{n \geq 3 \\ 3 \nmid n}} q^n dp_3(n) q^n + \sum_{\substack{n \geq 1 \\ 3 \nmid n}} q^n (dp_3(n) - 1) q^n \\
&= \sum_{n \geq 1} dp_3(n) q^{2n} - \sum_{\substack{n \geq 1 \\ 3 \nmid n}} q^{2n} \\
&= \sum_{n \geq 1} dp_3(n) q^{2n} - \frac{q^2}{1 - q^2} + \frac{q^6}{1 - q^6} \\
&= (-q^2, -q^4; q^6)_\infty - \frac{q^2}{1 - q^2} + \frac{q^6}{1 - q^6}.
\end{aligned} \tag{11}$$

□

Proof of Theorem 1.6. The generating function for partitions Λ with the summands (apart from the largest part) are $\equiv \pm 1 \pmod{6}$ which constitute the largest part, leading to

$$\begin{aligned} \sum_{n=0}^{\infty} \rho_6(n) q^n &= q^2 \cdot q^{1+1} + q^3 \cdot q^{1+1+1} + q^4 \cdot (q^{1+1+1+1}) + q^5 \cdot (q^{1+1+1+1+1}) \\ &\quad + q^6 \cdot (q^{5+1} + q^{1+1+1+1+1+1}) + q^7 \cdot (q^{5+1+1} + q^{1+1+1+1+1+1+1}) \\ &\quad + q^8 \cdot (q^{7+1} + q^{5+1+1+1} + q^{1+1+1+1+1+1+1+1}) \\ &\quad + q^9 \cdot (q^{7+1+1} + q^{5+1+1+1+1} + q^{1+1+1+1+1+1+1+1}) + \dots \end{aligned}$$

Let $p_6(n)$ represent the number of partitions of n with parts $\equiv \pm 1 \pmod{6}$, then

$$\begin{aligned} \sum_{n=0}^{\infty} \rho_6(n) q^n &= \sum_{\substack{n \geq 2 \\ n \not\equiv \pm 1 \pmod{6}}} q^n p_6(n) q^n + \sum_{\substack{n \geq 1 \\ n \equiv \pm 1 \pmod{6}}} q^n (p_6(n) - 1) q^n \\ &= \sum_{n \geq 1} p_6(n) q^{2n} - \sum_{\substack{n \geq 1 \\ n \equiv \pm 1 \pmod{6}}} q^{2n} \\ &= \frac{1}{(q^2, q^{10}; q^{12})_{\infty}} - \frac{(q^2 + q^{10})}{1 - q^{12}}. \end{aligned} \tag{12}$$

□

The partition sequences obtained in Theorem 1.5–Theorem 1.6 are new, given as:

Sequence 1: $\langle 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 2, 0, 2, 0, 2, 0, 3, 0, 3, 0, 4, 0, 6, 0, 6, 0, 7, 0, 9, 0, 9, 0, 11, 0, \dots \rangle$

Sequence 2: $\langle 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 2, 0, 2, 0, 3, 0, 3, 0, 4, 0, 4, 0, 6, 0, 6, 0, 8, 0, 9, 0, 10, 0, 11, 0, \dots \rangle$

Here, Sequence 1 represents the coefficients $\rho_3(n)$ due to Theorem 1.5, while Sequence 2 represents the coefficients $\rho_6(n)$ due to Theorem 1.6.

3. Conclusions

The mathematical significance of the partition structures discussed in this paper lies in extending the existing theory of integer partitions. In particular, we introduce new families with distinct structural properties and establish their enumerative generating functions. The findings offer more illustrations of how partition restrictions can result in significant generating-function identities and could be used as a foundation for additional research on partition identities, congruence properties, recurrence relations, and potential relationships with other classes of restricted partitions. In this way, the partition families presented in this work add to the larger investigation of the relationship between generating functions and combinatorial structures.

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The authors declare no conflict of interest.

Author contributions

All authors contributed to the manuscript and approved the final version.

Conflict of interest

The authors declare that they have no conflict of interest.

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The authors declare that no generative AI or AI-assisted technologies were used in the preparation of this manuscript, except for tools used solely for spelling, grammar, language editing, reference formatting, or similar routine editorial corrections.

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