



Zagreb indices of the generalized zero-divisor graph of the ring $\mathbb{Z}_n$

Research Article

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Abstract: In this paper, we compute the first and second Zagreb indices and the first and second multiplicative Zagreb indices of the generalized zero-divisor graph $\Gamma'(\mathbb{Z}_n)$, where $n = pq$, $p^\alpha q$ and $p^\alpha q^\beta$, with p and q distinct primes and α, β positive integers. For each case, we construct an explicit partition of the vertex set, determine the degree of each vertex class, and derive closed-form expressions for all four Zagreb indices. These indices are important in chemical graph theory and network analysis, providing quantitative measures of the structural characteristics of the graph.

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1. Introduction

The zero-divisor graph is a visual representation of an algebraic structure and has become an effective means for the exploration of structural features of rings. The concept was originally introduced by Beck [2]

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in 1988 in the context of studying colorings of commutative rings, wherein all elements of the ring were taken as vertices and two vertices were connected by an edge whenever their product is zero. This definition was subsequently refined by Anderson and Livingston [1], who restricted the vertex set to the collection of nonzero zero-divisors of the ring, giving rise to the graph now standardly denoted $\Gamma(R)$. Since this reformulation, zero-divisor graphs have been studied extensively in both commutative and non-commutative settings, with investigations spanning spectral as well as combinatorial and structural properties, including work on adjacency matrices and eigenvalues of such graphs [7, 8, 10–13]. The definition of the zero-divisor graph was modified to generalized zero-divisor graph, denoted by $\Gamma'(R)$, by Lande and Khairnar [9]. Here $\Gamma'(R)$ is defined as a simple undirected graph with vertex set $Z^*(R)$, i.e., the set of all nonzero zero-divisors of R . Two distinct vertices x, y in $\Gamma'(R)$ are said to be adjacent if and only if there exists a positive integer n such that $x^n y = 0$ or $y^n x = 0$.

The generalized zero-divisor graph is not a routine relabelling of the classical construction: the two adjacency rules are genuinely different conditions. In the classical zero-divisor graph $\Gamma(R)$, two elements x, y are adjacent only if $xy = 0$. In $\Gamma'(R)$, adjacency instead requires only $x^n y = 0$ or $y^n x = 0$ for some positive integer n , a strictly weaker condition that allows x and y to be adjacent in $\Gamma'(R)$ while satisfying $xy \neq 0$ in R . This is precisely why the nonzero nilpotent elements of $\mathbb{Z}_n$ form a distinguished vertex class – adjacent to every other vertex – with no counterpart in the classical graph. Capturing this additional adjacency, and hence the extra topological complexity it introduces, is the reason $\Gamma'(\mathbb{Z}_n)$ must be studied on its own terms rather than inferred from results on $\Gamma(\mathbb{Z}_n)$.

The method used to obtain these closed-form expressions – partitioning the vertex set, computing the degree of each class, substituting into the definitions of M_1, M_2, Π_1, Π_2 , and simplifying – is standard in the literature on degree-based topological indices. The contribution of the present paper, however, does not rest on the method itself but on two further points. First, the closed-form expressions for M_1, M_2, Π_1, Π_2 on $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ and $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ have not appeared in the literature before, including in earlier work on this graph family [8, 9, 11]. Second, and more importantly, having these explicit formulas in hand is what makes it possible to establish several structural consequences about $\Gamma'(\mathbb{Z}_n)$: that the graph is never regular in any of the three families considered, that M_1 and Π_1 satisfy a strict inequality as a direct consequence of this non-regularity, that all four indices increase strictly and monotonically as the nilpotency parameters α, β increase, that the case $\alpha = 1$ is an extremal (minimizing) boundary case recovering the complete bipartite structure of $\Gamma'(\mathbb{Z}_{pq})$ exactly, and that the relative degree structure between vertex classes converges to fixed rational ratios as $\alpha \rightarrow \infty$ (see Subsection 3.1). These five results follow directly from the closed-form degree expressions derived here, and their derivation, rather than the computation alone.

This also clarifies the relationship between the present paper and earlier work on $\Gamma'(\mathbb{Z}_n)$ [8, 9, 11]. Those papers establish the vertex partition $\{X_1, \dots, X_5\}$ and the adjacency relations between these classes using spectral methods, but neither computes any topological index from this structure, nor derives any of the five results listed above. Knowing that the nilpotent class is a clique adjacent to every other vertex – a structural fact already established in that earlier work – does not by itself imply, for instance, that M_1 increases strictly with α , nor does it identify where the extremal case lies, nor determine how the degree ratios behave asymptotically; establishing each of these requires the explicit closed-form degree formulas derived in the present paper. The results of Subsection 3.1 therefore go beyond what can be inferred from the previously known graph structure alone.

Graph invariants are characteristics of a graph that provide quantitative measures of its structural features. Among these, topological indices have proven particularly effective in mathematical chemistry, as they encode combinatorial information about molecular graphs into numerical values that correlate with physical and chemical properties of molecules [3]. Zagreb indices are one of the most well-studied families of such topological indices. They were first introduced by Gutman et al. [6] in 1972, with subsequent theoretical development by Gutman and Das [5] and further analysis by Nikolić et al. [15]. These indices capture degree-based information about the vertices of a graph and have found sustained application in the study of molecular structure. Study of Zagreb indices is explored in [13, 14].

Definition 1.1. Let G be a simple graph. The first Zagreb index M_1 was introduced by Gutman et al. [6]

in 1972 and is the sum of the squares of the degrees of all vertices in the graph:

$$M_1(G) = \sum_{v \in V(G)} \deg(v)^2.$$

The second Zagreb index $M_2(G)$ is the sum of the product of the degrees of pairs of adjacent vertices in the graph:

$$M_2(G) = \sum_{(u,v) \in E(G)} \deg(u) \cdot \deg(v).$$

Eliasi et al. [4] and Todeschini et al. [16] have introduced modified versions of conventional Zagreb indices using multiplication operations:

$$\Pi_1(G) = \prod_{v \in V(G)} \deg(v)^2, \quad \Pi_2(G) = \prod_{(u,v) \in E(G)} \deg(u) \cdot \deg(v).$$

These multiplicative indices, based on products of vertex degrees and edge degrees respectively, have been shown to correlate well with the total π -electrons energy of conjugated hydrocarbons and contribute meaningfully to Quantitative Structure-Activity Relationship (QSAR) studies.

Calculations involving topological index invariants of the zero-divisor graph represent an intersection between two separately interesting fields – namely, algebraic graph theory and chemical graph theory. The ring of integers modulo n ($\mathbb{Z}_n$), as a mathematical object with its arithmetic properties well-characterized through prime factorization, serves as an ideal setting for these calculations.

In [13], Mahale and Pawar determined Zagreb indices of Beck's zero-divisor graph $\Gamma_0(\mathbb{Z}_n)$, where $n = p^k, pq, pqr$ and p, q, r are distinct primes. In this article, we compute the first Zagreb index, second Zagreb index, first multiplicative Zagreb index, and second multiplicative Zagreb index of the generalized zero-divisor graph $\Gamma'(\mathbb{Z}_n)$ for $n = pq, p^\alpha q$ and $p^\alpha q^\beta$, where p, q are distinct primes and α, β are positive integers. For each case, we construct an explicit partition of the vertex set in terms of the zero-divisor structure of $\mathbb{Z}_n$, determine the degree of each vertex class, and derive closed-form expressions for all four Zagreb indices.

The remainder of this paper is organized as follows. Section 2 recalls necessary preliminaries. Section 3 contains the main results for Zagreb indices of $\Gamma'(\mathbb{Z}_n)$ where $n = pq, p^\alpha q$ and $p^\alpha q^\beta$.

2. Preliminaries

In this section, we recall some basic definitions and results from graph theory and ring theory that will be used throughout this paper.

A graph $G = (V, E)$ consists of a nonempty set $V(G)$ of vertices and a set $E(G)$ of edges, where each edge is an unordered pair of distinct vertices. A graph is called *simple* if it has no loops or multiple edges. A graph is called *complete* if every two distinct vertices are adjacent. The complete graph on n vertices is denoted by K_n . A graph is called *bipartite* if its vertex set can be partitioned into two disjoint subsets U and V such that every edge connects a vertex in U to a vertex in V . A *complete bipartite graph*, denoted $K_{m,n}$, is a bipartite graph in which every vertex of U is adjacent to every vertex of V , where $|U| = m$ and $|V| = n$. The *degree* of a vertex v in a graph G , denoted $\deg(v)$, is the number of edges incident to v , that is, the number of vertices adjacent to v . The set of all nonzero zero-divisors of R is denoted by $Z^*(R)$. An element $x \in R$ is called *nilpotent* if there exists a positive integer n such that $x^n = 0$.

Definition 2.1. [9] The generalized zero-divisor graph of a commutative ring R , denoted $\Gamma'(R)$, is a simple undirected graph with vertex set $Z^*(R)$ and two distinct vertices x and y are adjacent if and only if $x^n y = 0$ or $y^n x = 0$ for some positive integer n .

Recall the following notions:

Let $\varphi(n)$ denote *Euler's totient function*, that is, the number of positive integers less than n and relatively prime to n . The canonical decomposition of an integer $n > 1$ is given by $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where $p_1, p_2, \dots, p_r$ are distinct primes and $k_1, k_2, \dots, k_r$ are positive integers. We know that

$$\varphi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right) = (p_1^{k_1} - p_1^{k_1-1}) \cdots (p_r^{k_r} - p_r^{k_r-1}).$$

Let $T_d = \{a \in \mathbb{Z}_n : (a, n) = d\}$, where (a, n) denotes the greatest common divisor of a and n . Then the cardinality of T_d is $\varphi\left(\frac{n}{d}\right)$ [17].

Note that the number of nonzero zero-divisors of $\mathbb{Z}_n$ is $n - \varphi(n) - 1$, that is, $|Z^*(\mathbb{Z}_n)| = n - \varphi(n) - 1$.

3. Main results

We first consider the Zagreb indices of the generalized zero-divisor graph $\Gamma'(\mathbb{Z}_{pq})$.

Remark 3.1. For distinct primes p and q , the graph $\Gamma'(\mathbb{Z}_{pq})$ is isomorphic to the complete bipartite graph $K_{p-1, q-1}$; see [9]. Using this graph structure, the Zagreb indices of $\Gamma'(\mathbb{Z}_{pq})$ are:

$$M_1(\Gamma'(\mathbb{Z}_{pq})) = (p-1)(q-1)(p+q-2); \quad M_2(\Gamma'(\mathbb{Z}_{pq})) = (p-1)^2(q-1)^2;$$

$$\Pi_1(\Gamma'(\mathbb{Z}_{pq})) = (q-1)^{2(p-1)} \cdot (p-1)^{2(q-1)}; \quad \Pi_2(\Gamma'(\mathbb{Z}_{pq})) = [(p-1)(q-1)]^{(p-1)(q-1)}.$$

In the following theorem, we determine the Zagreb indices of $\Gamma'(\mathbb{Z}_n)$ for $n = p^\alpha q$.

Theorem 3.2. Let p and q be distinct primes and $\alpha > 1$ be a positive integer. Then the Zagreb indices of $\Gamma'(\mathbb{Z}_{p^\alpha q})$ are given by:

1.

$$M_1(\Gamma'(\mathbb{Z}_{p^\alpha q})) = p^{\alpha-1}(q-1)(p^\alpha-1)^2 + p^{\alpha-1}(p-1)(p^{\alpha-1}q-1)^2 \\ + (p^{\alpha-1}-1)[p^{\alpha-1}(p+q-1)-2]^2;$$

2.

$$M_2(\Gamma'(\mathbb{Z}_{p^\alpha q})) = p^{2(\alpha-1)}(p-1)(q-1)(p^\alpha-1)(p^{\alpha-1}q-1) \\ + p^{\alpha-1}(p^{\alpha-1}-1)[p^{\alpha-1}(p+q-1)-2][(q-1)(p^\alpha-1) + (p-1)(p^{\alpha-1}q-1)] \\ + \frac{1}{2}(p^{\alpha-1}-1)(p^{\alpha-1}-2)[p^{\alpha-1}(p+q-1)-2]^2;$$

3.

$$\Pi_1(\Gamma'(\mathbb{Z}_{p^\alpha q})) = (p^\alpha-1)^{2p^{\alpha-1}(q-1)} \cdot (p^{\alpha-1}q-1)^{2p^{\alpha-1}(p-1)} \cdot [p^{\alpha-1}(p+q-1)-2]^{2(p^{\alpha-1}-1)};$$

4.

$$\Pi_2(\Gamma'(\mathbb{Z}_{p^\alpha q})) = \\ [(p^\alpha-1)(p^{\alpha-1}q-1)]^{p^{2(\alpha-1)}(p-1)(q-1)} \cdot [(p^\alpha-1)(p^{\alpha-1}(p+q-1)-2)]^{p^{\alpha-1}(q-1)(p^{\alpha-1}-1)} \\ \cdot [(p^{\alpha-1}q-1)(p^{\alpha-1}(p+q-1)-2)]^{p^{\alpha-1}(p-1)(p^{\alpha-1}-1)} \cdot [p^{\alpha-1}(p+q-1)-2]^{(p^{\alpha-1}-1)(p^{\alpha-1}-2)}.$$

Proof. Let p, q be distinct primes, $\alpha > 1$ be a positive integer, and $n = p^\alpha q$. Consider a partition of vertex set of $\Gamma'(\mathbb{Z}_{p^\alpha q})$ into three sets in terms of zero-divisors of $\mathbb{Z}_n$:

$$X_1 = \{x \in \mathbb{Z}_n : \gcd(x, n) = p^i, i = 1, 2, \dots, \alpha\}, \quad X_2 = \{x \in \mathbb{Z}_n : \gcd(x, n) = q\}, \\ X_3 = \{x \in \mathbb{Z}_n \setminus \{0\} : x = kp^i q, i = 1, 2, \dots, \alpha\}.$$

Note that X_3 is the set of all nonzero nilpotent elements of $\mathbb{Z}_n$. All the sets X_1, X_2, X_3 are mutually disjoint. Thus, $\mathcal{P} = \{X_1, X_2, X_3\}$ forms a partition of the vertex set of $\Gamma'(\mathbb{Z}_{p^\alpha q})$.

Next, we determine the cardinalities of each set. Let $x \in X_1$. Then $\gcd(x, n) = p^i$ for some $i \in \{1, 2, \dots, \alpha\}$. Therefore the number of elements in X_1 with $\gcd(x, n) = p^i$ is

$$\varphi\left(\frac{p^\alpha q}{p^i}\right) = \varphi(p^{\alpha-i} q) = p^{\alpha-i-1}(p-1)(q-1), \quad i = 1, 2, \dots, \alpha-1,$$

and for $i = \alpha$:

$$\varphi\left(\frac{p^\alpha q}{p^\alpha}\right) = \varphi(q) = q-1.$$

This gives,

$$n_1 = |X_1| = \varphi\left(\frac{p^\alpha q}{p^1}\right) + \varphi\left(\frac{p^\alpha q}{p^2}\right) + \dots + \varphi\left(\frac{p^\alpha q}{p^\alpha}\right) \\ = p^{\alpha-2}(p-1)(q-1) + p^{\alpha-3}(p-1)(q-1) + \dots + (p-1)(q-1) + (q-1) \\ = (q-1)(p^{\alpha-1} - 1) + (q-1) = (q-1)p^{\alpha-1}.$$

Hence, $n_1 = |X_1| = p^{\alpha-1}(q-1)$.

If $x \in X_2$, then $\gcd(x, n) = q$, so

$$n_2 = |X_2| = \varphi\left(\frac{p^\alpha q}{q}\right) = \varphi(p^\alpha) = p^{\alpha-1}(p-1).$$

Clearly, the number of multiples of pq in $\mathbb{Z}_{p^\alpha q}$ is $p^{\alpha-1}$. Therefore,

$$n_3 = |X_3| = p^{\alpha-1} - 1.$$

Since $\varphi(p^\alpha q) = p^{\alpha-1}(p-1)(q-1)$, the total number of nonzero zero-divisors in $\mathbb{Z}_{p^\alpha q}$ is

$$n - \varphi(n) - 1 = p^\alpha q - p^{\alpha-1}(p-1)(q-1) - 1 = p^{\alpha-1}(p+q-1) - 1.$$

Next, we determine the adjacency structure. If every element of X is adjacent to every element of Y , we write $X \sim Y$; if no element of X is adjacent to any element of Y , we write $X \not\sim Y$. We have the following adjacency relations [9]:

1. Let $a \in X_1, b \in X_2$. Then there exists a positive integer m such that $a^m b = 0$. Also, for any $b \in X_1$ and for any positive integer k , $a^k b \neq 0$ and $b^k a \neq 0$. Similarly, let $b \in X_3$. Then there exists a positive integer m such that $a^m b = 0$. Therefore, every element in X_1 is adjacent to every element in X_2 and X_3 , and no element of X_1 is adjacent to any element of X_1 . Thus, $X_1 \sim X_2, X_1 \sim X_3, X_1 \not\sim X_1$.

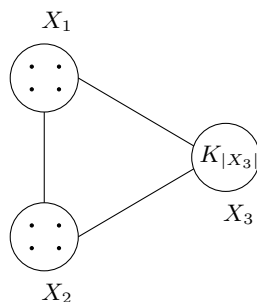


Figure 1. $\Gamma'(\mathbb{Z}_{p^\alpha q})$

2. Similarly we have, $X_2 \sim X_1$, $X_2 \sim X_3$, $X_2 \not\sim X_2$.
3. $X_3 \sim X_1$, $X_3 \sim X_2$, and every element of X_3 is adjacent to every other element of X_3 .

The adjacency structure of $\Gamma'(\mathbb{Z}_{p^\alpha q})$ is depicted in Fig. 1.

Next, we determine the vertex degrees. Since all vertices within each partition set have the same degree, we compute the degree of one representative vertex from each set. Denoting the degrees as d_1, d_2, d_3 respectively:

For any $u \in X_1$: u is adjacent to all vertices in X_2 and X_3 . Therefore,

$$d_1 = \deg(u \in X_1) = n_2 + n_3 = p^{\alpha-1}(p-1) + (p^{\alpha-1} - 1) = p^\alpha - 1,$$

For any $u \in X_2$: u is adjacent to all vertices in X_1 and X_3 . Therefore,

$$d_2 = \deg(u \in X_2) = n_1 + n_3 = p^{\alpha-1}(q-1) + (p^{\alpha-1} - 1) = p^{\alpha-1}q - 1,$$

For any $u \in X_3$: u is adjacent to all vertices in X_1 , X_2 , and all remaining vertices in X_3 . Therefore, $d_3 = \deg(u \in X_3) = n_1 + n_2 + (n_3 - 1) = p^{\alpha-1}(q-1) + p^{\alpha-1}(p-1) + (p^{\alpha-1} - 2) = p^{\alpha-1}(p+q-1) - 2$.

We now compute the Zagreb indices.

(1) *First Zagreb Index:*

$$\begin{aligned} M_1(\Gamma'(\mathbb{Z}_{p^\alpha q})) &= \sum_{v \in V(G)} \deg(v)^2 = \sum_{v \in X_1} \deg(v)^2 + \sum_{v \in X_2} \deg(v)^2 + \sum_{v \in X_3} \deg(v)^2 \\ &= \sum_{v \in X_1} d_1^2 + \sum_{v \in X_2} d_2^2 + \sum_{v \in X_3} d_3^2 = n_1 d_1^2 + n_2 d_2^2 + n_3 d_3^2 \\ &= p^{\alpha-1}(q-1)(p^\alpha - 1)^2 + p^{\alpha-1}(p-1)(p^{\alpha-1}q - 1)^2 + (p^{\alpha-1} - 1) \\ &\quad [p^{\alpha-1}(p+q-1) - 2]^2. \end{aligned}$$

(2) *Second Zagreb Index:* The edge classes correspond to adjacent pairs X_1 – X_2 , X_1 – X_3 , X_2 – X_3 , and

X_3-X_3 :

$$\begin{aligned} M_2(\Gamma'(\mathbb{Z}_{p^\alpha q})) &= \sum_{(u,v) \in E(G)} \deg(u) \cdot \deg(v) = \sum_{\substack{u \in X_1 \\ v \in X_2}} d_1 d_2 + \sum_{\substack{u \in X_1 \\ v \in X_3}} d_1 d_3 + \sum_{\substack{u \in X_2 \\ v \in X_3}} d_2 d_3 + \frac{1}{2} \sum_{\substack{u,v \in X_3 \\ u \neq v}} d_3^2 \\ &= n_1 n_2 d_1 d_2 + n_1 n_3 d_1 d_3 + n_2 n_3 d_2 d_3 + \frac{1}{2} n_3 (n_3 - 1) d_3^2 \\ &= p^{2(\alpha-1)} (p-1)(q-1)(p^\alpha-1)(p^{\alpha-1}q-1) \\ &\quad + p^{\alpha-1}(q-1)(p^{\alpha-1}-1)(p^\alpha-1)[p^{\alpha-1}(p+q-1)-2] \\ &\quad + p^{\alpha-1}(p-1)(p^{\alpha-1}-1)(p^{\alpha-1}q-1)[p^{\alpha-1}(p+q-1)-2] \\ &\quad + \frac{1}{2}(p^{\alpha-1}-1)(p^{\alpha-1}-2)[p^{\alpha-1}(p+q-1)-2]^2. \end{aligned}$$

Factoring $p^{\alpha-1}(p^{\alpha-1}-1)[p^{\alpha-1}(p+q-1)-2]$ from the middle two terms:

$$\begin{aligned} M_2(\Gamma'(\mathbb{Z}_{p^\alpha q})) &= p^{2(\alpha-1)} (p-1)(q-1)(p^\alpha-1)(p^{\alpha-1}q-1) \\ &\quad + p^{\alpha-1}(p^{\alpha-1}-1)[p^{\alpha-1}(p+q-1)-2][(q-1)(p^\alpha-1) + (p-1)(p^{\alpha-1}q-1)] \\ &\quad + \frac{1}{2}(p^{\alpha-1}-1)(p^{\alpha-1}-2)[p^{\alpha-1}(p+q-1)-2]^2. \end{aligned}$$

(3) *First Multiplicative Zagreb Index:*

$$\begin{aligned} \Pi_1(\Gamma'(\mathbb{Z}_{p^\alpha q})) &= \prod_{v \in V(G)} \deg(v)^2 = \prod_{v \in X_1} d_1^2 \cdot \prod_{v \in X_2} d_2^2 \cdot \prod_{v \in X_3} d_3^2 = d_1^{2n_1} \cdot d_2^{2n_2} \cdot d_3^{2n_3} \\ &= (p^\alpha-1)^{2p^{\alpha-1}(q-1)} \cdot (p^{\alpha-1}q-1)^{2p^{\alpha-1}(p-1)} \cdot [p^{\alpha-1}(p+q-1)-2]^{2(p^{\alpha-1}-1)}. \end{aligned}$$

(4) *Second Multiplicative Zagreb Index:*

$$\begin{aligned} \Pi_2(\Gamma'(\mathbb{Z}_{p^\alpha q})) &= \prod_{(u,v) \in E(G)} \deg(u) \cdot \deg(v) = \prod_{\substack{u \in X_1 \\ v \in X_2}} d_1 d_2 \cdot \prod_{\substack{u \in X_1 \\ v \in X_3}} d_1 d_3 \cdot \prod_{\substack{u \in X_2 \\ v \in X_3}} d_2 d_3 \cdot \prod_{\substack{u \neq v \\ u,v \in X_3}} (d_3^2) \\ &= (d_1 d_2)^{n_1 n_2} \cdot (d_1 d_3)^{n_1 n_3} \cdot (d_2 d_3)^{n_2 n_3} \cdot (d_3^2)^{\frac{n_3(n_3-1)}{2}} \\ &= [(p^\alpha-1)(p^{\alpha-1}q-1)]^{p^{2(\alpha-1)}(p-1)(q-1)} \\ &\quad \cdot [(p^\alpha-1)(p^{\alpha-1}(p+q-1)-2)]^{p^{\alpha-1}(q-1)(p^{\alpha-1}-1)} \\ &\quad \cdot [(p^{\alpha-1}q-1)(p^{\alpha-1}(p+q-1)-2)]^{p^{\alpha-1}(p-1)(p^{\alpha-1}-1)} \\ &\quad \cdot [p^{\alpha-1}(p+q-1)-2]^{(p^{\alpha-1}-1)(p^{\alpha-1}-2)}. \end{aligned}$$

□

3.1. Illustrative examples

Example 3.3. Let $n = pq = 2 \cdot 3 = 6$. By Remark 3.1, $\Gamma'(\mathbb{Z}_6) \cong K_{1,2}$ as shown in Fig. 2. Explicitly, the nonzero zero-divisors of $\mathbb{Z}_6$ are $\{2, 3, 4\}$, with edge set $\{(2, 3), (3, 4)\}$ and degree sequence $\deg(2) = 1$, $\deg(3) = 2$, $\deg(4) = 1$. Direct computation gives

$$M_1 = 1^2 + 2^2 + 1^2 = 6, \quad M_2 = (1)(2) + (2)(1) = 4,$$

$$\Pi_1 = 1^2 \cdot 2^2 \cdot 1^2 = 4, \quad \Pi_2 = (1 \cdot 2)(2 \cdot 1) = 4,$$

which agrees exactly with [Remark 3.1](#):

$$\begin{aligned} M_1 &= (p-1)(q-1)(p+q-2) = 6, & M_2 &= (p-1)^2(q-1)^2 = 4, \\ \Pi_1 &= (q-1)^{2(p-1)}(p-1)^{2(q-1)} = 4, & \Pi_2 &= [(p-1)(q-1)]^{(p-1)(q-1)} = 4. \end{aligned}$$

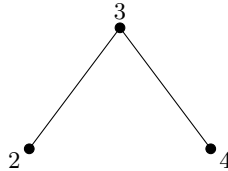


Figure 2. The graph $\Gamma'(\mathbb{Z}_6)$

Example 3.4. Let $n = p^\alpha q = 2^2 \cdot 3 = 12$. The nonzero zero-divisors of $\mathbb{Z}_{12}$ are $\{2, 3, 4, 6, 8, 9, 10\}$, and direct computation of the adjacency relation in [Definition 2.1](#) gives the fourteen edges (see [Fig. 3](#))

$$(2, 3), (2, 6), (2, 9), (3, 4), (3, 6), (3, 8), (3, 10), (4, 6), (4, 9), (6, 8), (6, 9), (6, 10), (8, 9), (9, 10)$$

with degree sequence

$$\deg(2) = \deg(4) = \deg(8) = \deg(10) = 3, \quad \deg(3) = \deg(9) = 5, \quad \deg(6) = 6.$$

This confirms the partition of [Theorem 3.2](#) for $p = 2$, $\alpha = 2$, $q = 3$: the four vertices of degree 3 form X_1 (so $n_1 = 4$, $d_1 = 3$), the two vertices of degree 5 form X_2 (so $n_2 = 2$, $d_2 = 5$), and the single vertex of degree 6 forms X_3 (so $n_3 = 1$, $d_3 = 6$).

Substituting into [Theorem 3.2](#) gives

$$\begin{aligned} M_1 &= 4(3)^2 + 2(5)^2 + 1(6)^2 = 122, \\ M_2 &= (4)(2)(3)(5) + (4)(1)(3)(6) + (2)(1)(5)(6) = 252, \\ \Pi_1 &= 3^8 \cdot 5^4 \cdot 6^2 = 147,622,500, \\ \Pi_2 &= 15^8 \cdot 18^4 \cdot 30^2 = 242,137,805,625,000,000. \end{aligned}$$

Each value was independently verified by direct computation on the graph constructed from [Definition 2.1](#), confirming the closed-form expressions of [Theorem 3.2](#). The disparity in magnitude between Π_2 and M_2 already for this small example illustrates the growth-rate contrast between the additive and multiplicative indices discussed in [Section 4](#).

In the following theorem, we determine the Zagreb indices of $\Gamma'(\mathbb{Z}_n)$ for $n = p^\alpha q^\beta$.

Theorem 3.5. Let p and q be distinct primes and α, β be positive integers. Then the Zagreb indices of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ are given by:

1.

$$\begin{aligned} M_1(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})) &= q^{\beta-1}(q-1)(p^{\alpha-1}-1)[p^{\alpha-1}(p+q^{\beta-1}-1)-1]^2 \\ &\quad + p^{\alpha-1}(p-1)(q^{\beta-1}-1)[q^{\beta-1}(p^{\alpha-1}+q-1)-1]^2 \\ &\quad + q^{\beta-1}(q-1)(p^\alpha q^{\beta-1}-1)^2 + p^{\alpha-1}(p-1)(p^{\alpha-1}q^\beta-1)^2 \\ &\quad + (p^{\alpha-1}q^{\beta-1}-1)[p^{\alpha-1}q^{\beta-1}(p+q-1)-2]^2; \end{aligned}$$

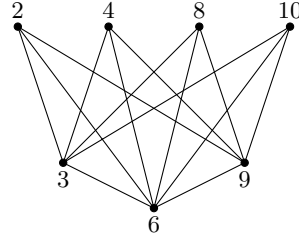


Figure 3. The graph $\Gamma'(\mathbb{Z}_{12})$

2.

$$\begin{aligned} M_2(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})) = & p^{\alpha-1} q^{\beta-1} (p-1)(q-1) \left[(p^{\alpha-1} - 1)(p^{\alpha-1}(p + q^{\beta-1} - 1) - 1)(p^{\alpha-1} q^{\beta-1} - 1) \right. \\ & + (q^{\beta-1} - 1)(q^{\beta-1}(p^{\alpha-1} + q - 1) - 1)(p^\alpha q^{\beta-1} - 1) \\ & \left. + (p^\alpha q^{\beta-1} - 1)(p^{\alpha-1} q^{\beta-1} - 1) \right] + (p^{\alpha-1} q^{\beta-1} - 1) [p^{\alpha-1} q^{\beta-1} (p + q - 1) - 2] \\ & \left[q^{\beta-1} (q-1)(p^{\alpha-1} - 1)(p^{\alpha-1}(p + q^{\beta-1} - 1) - 1) + p^{\alpha-1} (p-1)(q^{\beta-1} - 1) \right. \\ & (q^{\beta-1}(p^{\alpha-1} + q - 1) - 1) + q^{\beta-1} (q-1)(p^\alpha q^{\beta-1} - 1) + p^{\alpha-1} (p-1) \\ & \left. (p^{\alpha-1} q^{\beta-1} - 1) \right] + \frac{1}{2} (p^{\alpha-1} q^{\beta-1} - 1)(p^{\alpha-1} q^{\beta-1} - 2) [p^{\alpha-1} q^{\beta-1} (p + q - 1) - 2]^2; \end{aligned}$$

3.

$$\begin{aligned} \Pi_1(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})) = & [p^{\alpha-1}(p + q^{\beta-1} - 1) - 1]^{2q^{\beta-1}(q-1)(p^{\alpha-1}-1)} \\ & \cdot [q^{\beta-1}(p^{\alpha-1} + q - 1) - 1]^{2p^{\alpha-1}(p-1)(q^{\beta-1}-1)} \\ & \cdot (p^\alpha q^{\beta-1} - 1)^{2q^{\beta-1}(q-1)} \cdot (p^{\alpha-1} q^{\beta-1} - 1)^{2p^{\alpha-1}(p-1)} \\ & \cdot [p^{\alpha-1} q^{\beta-1} (p + q - 1) - 2]^{2(p^{\alpha-1} q^{\beta-1} - 1)}; \end{aligned}$$

4.

$$\begin{aligned} \Pi_2(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})) = & \left[(p^{\alpha-1}(p + q^{\beta-1} - 1) - 1)(p^{\alpha-1} q^{\beta-1} - 1) \right]^{p^{\alpha-1} q^{\beta-1} (p-1)(q-1)(p^{\alpha-1}-1)} \\ & \cdot \left[(q^{\beta-1}(p^{\alpha-1} + q - 1) - 1)(p^\alpha q^{\beta-1} - 1) \right]^{p^{\alpha-1} q^{\beta-1} (p-1)(q-1)(q^{\beta-1}-1)} \\ & \cdot \left[(p^\alpha q^{\beta-1} - 1)(p^{\alpha-1} q^{\beta-1} - 1) \right]^{p^{\alpha-1} q^{\beta-1} (p-1)(q-1)} \\ & \cdot \left[(p^{\alpha-1}(p + q^{\beta-1} - 1) - 1) \right]^{q^{\beta-1}(q-1)(p^{\alpha-1}-1)(p^{\alpha-1} q^{\beta-1}-1)} \\ & \cdot \left[(p^{\alpha-1} q^{\beta-1} (p + q - 1) - 2) \right]^{q^{\beta-1}(q-1)(p^{\alpha-1}-1)(p^{\alpha-1} q^{\beta-1}-1)} \\ & \cdot \left[(q^{\beta-1}(p^{\alpha-1} + q - 1) - 1) \right]^{p^{\alpha-1}(p-1)(q^{\beta-1}-1)(p^{\alpha-1} q^{\beta-1}-1)} \\ & \cdot \left[(p^{\alpha-1} q^{\beta-1} (p + q - 1) - 2) \right]^{p^{\alpha-1}(p-1)(q^{\beta-1}-1)(p^{\alpha-1} q^{\beta-1}-1)} \\ & \cdot \left[(p^\alpha q^{\beta-1} - 1)(p^{\alpha-1} q^{\beta-1} (p + q - 1) - 2) \right]^{q^{\beta-1}(q-1)(p^{\alpha-1} q^{\beta-1}-1)} \end{aligned}$$

$$\begin{aligned} & \cdot \left[(p^{\alpha-1}q^\beta - 1)(p^{\alpha-1}q^{\beta-1}(p+q-1) - 2) \right]^{p^{\alpha-1}(p-1)(p^{\alpha-1}q^{\beta-1}-1)} \\ & \cdot \left[p^{\alpha-1}q^{\beta-1}(p+q-1) - 2 \right]^{(p^{\alpha-1}q^{\beta-1}-1)(p^{\alpha-1}q^{\beta-1}-2)}. \end{aligned} \quad (1)$$

Proof. Let p, q be distinct primes, $\alpha > 1, \beta > 1$ be integers, and $n = p^\alpha q^\beta$. We partition the vertex set of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ into five sets in terms of zero-divisors of $\mathbb{Z}_n$:

$$\begin{aligned} X_1 &= \{x \in \mathbb{Z}_n : \gcd(x, n) = p^i, i = 1, 2, \dots, \alpha - 1\}, \\ X_2 &= \{x \in \mathbb{Z}_n : \gcd(x, n) = q^i, i = 1, 2, \dots, \beta - 1\}, \\ X_3 &= \{x \in \mathbb{Z}_n : \gcd(x, n) = p^\alpha\}, \quad X_4 = \{x \in \mathbb{Z}_n : \gcd(x, n) = q^\beta\}, \\ X_5 &= \{x \in \mathbb{Z}_n \setminus \{0\} : x = kp^i q^j, i = 1, 2, \dots, \alpha, j = 1, 2, \dots, \beta\}. \end{aligned}$$

Note that X_5 is the set of all nonzero nilpotent elements in $\mathbb{Z}_{p^\alpha q^\beta}$. Observe that all the sets X_1, X_2, X_3, X_4, X_5 are mutually disjoint. Thus, $\mathcal{P} = \{X_1, X_2, X_3, X_4, X_5\}$ forms a partition of the vertex set of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$.

Next, we determine the cardinalities of each set. Let $x \in X_1$. Then $\gcd(x, n) = p^i$ for some $i \in \{1, 2, \dots, \alpha - 1\}$. The number of elements in X_1 with $\gcd(x, n) = p^i$ is

$$\varphi\left(\frac{p^\alpha q^\beta}{p^i}\right) = \varphi(p^{\alpha-i} q^\beta) = p^{\alpha-i-1} q^{\beta-1} (p-1)(q-1), \quad i = 1, 2, \dots, \alpha - 1.$$

Therefore,

$$\begin{aligned} n_1 = |X_1| &= \varphi\left(\frac{p^\alpha q^\beta}{p^1}\right) + \varphi\left(\frac{p^\alpha q^\beta}{p^2}\right) + \dots + \varphi\left(\frac{p^\alpha q^\beta}{p^{\alpha-1}}\right) \\ &= p^{\alpha-2} q^{\beta-1} (p-1)(q-1) + p^{\alpha-3} q^{\beta-1} (p-1)(q-1) + \dots + q^{\beta-1} (p-1)(q-1) \\ &= q^{\beta-1} (q-1) (p^{\alpha-1} - 1). \end{aligned}$$

$$n_2 = |X_2| = \varphi\left(\frac{p^\alpha q^\beta}{q^1}\right) + \varphi\left(\frac{p^\alpha q^\beta}{q^2}\right) + \dots + \varphi\left(\frac{p^\alpha q^\beta}{q^{\beta-1}}\right) = p^{\alpha-1} (p-1) (q^{\beta-1} - 1).$$

$$n_3 = |X_3| = \varphi\left(\frac{p^\alpha q^\beta}{p^\alpha}\right) = \varphi(q^\beta) = q^{\beta-1} (q-1); \quad n_4 = |X_4| = \varphi\left(\frac{p^\alpha q^\beta}{q^\beta}\right) = p^{\alpha-1} (p-1).$$

Clearly, the number of multiples of pq in $\mathbb{Z}_{p^\alpha q^\beta}$ is $p^{\alpha-1} q^{\beta-1}$. Therefore,

$$n_5 = |X_5| = p^{\alpha-1} q^{\beta-1} - 1.$$

Since $\varphi(p^\alpha q^\beta) = p^{\alpha-1} q^{\beta-1} (p-1)(q-1)$, the total number of nonzero zero-divisors in $\mathbb{Z}_{p^\alpha q^\beta}$ is

$$n - \varphi(n) - 1 = p^\alpha q^\beta - p^{\alpha-1} q^{\beta-1} (p-1)(q-1) - 1 = p^{\alpha-1} q^{\beta-1} (p+q-1) - 1.$$

Similar to the proof of [Theorem 3.2](#), we find the adjacency relationship among the vertices.

1. Since X_5 is the set of all nonzero nilpotent elements, by [Definition 2.1](#), each element in X_5 is adjacent to all the remaining vertices. Thus, $X_5 \sim X_1, X_5 \sim X_2, X_5 \sim X_3, X_5 \sim X_4, X_5 \sim X_5$.

2. Similarly, $X_1 \sim X_4$, $X_1 \not\sim X_1$, $X_1 \not\sim X_2$, $X_1 \not\sim X_3$.
3. $X_2 \sim X_3$, $X_2 \not\sim X_1$, $X_2 \not\sim X_2$, $X_2 \not\sim X_4$.
4. $X_3 \sim X_2$, $X_3 \sim X_4$, $X_3 \not\sim X_1$, $X_3 \not\sim X_3$.
5. $X_4 \sim X_1$, $X_4 \sim X_3$, $X_4 \not\sim X_2$, $X_4 \not\sim X_4$.

The adjacency structure of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ is depicted in Fig. 4.

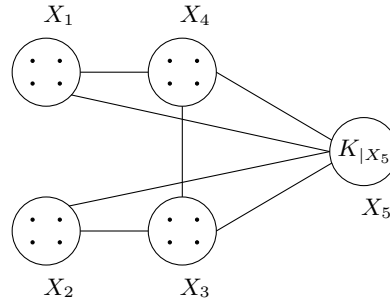


Figure 4. $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$

Next, we determine the vertex degrees.

Let $u \in X_1$: u is adjacent to all vertices in X_4 and X_5 only.

$$d_1 = \deg(u \in X_1) = n_4 + n_5 = p^{\alpha-1}(p-1) + (p^{\alpha-1}q^{\beta-1} - 1) = p^{\alpha-1}(p + q^{\beta-1} - 1) - 1,$$

Let $u \in X_2$: u is adjacent to all vertices in X_3 and X_5 only.

$$d_2 = \deg(u \in X_2) = n_3 + n_5 = q^{\beta-1}(q-1) + (p^{\alpha-1}q^{\beta-1} - 1) = q^{\beta-1}(p^{\alpha-1} + q - 1) - 1,$$

Let $u \in X_3$: u is adjacent to all vertices in X_2 , X_4 , and X_5 .

$$d_3 = \deg(u \in X_3) = n_2 + n_4 + n_5 = p^{\alpha-1}(p-1)(q^{\beta-1}) + (p^{\alpha-1}q^{\beta-1} - 1) = p^\alpha q^{\beta-1} - 1,$$

Let $u \in X_4$: u is adjacent to all vertices in X_1 , X_3 , and X_5 .

$$d_4 = \deg(u \in X_4) = n_1 + n_3 + n_5 = q^{\beta-1}(q-1)(p^{\alpha-1}) + (p^{\alpha-1}q^{\beta-1} - 1) = p^{\alpha-1}q^\beta - 1,$$

Let $u \in X_5$: u is adjacent to all vertices in X_1 , X_2 , X_3 , X_4 , and all other vertices in X_5 . Therefore,

$$d_5 = \deg(u \in X_5) = n_1 + n_2 + n_3 + n_4 + (n_5 - 1) = p^{\alpha-1}q^{\beta-1}(p + q - 1) - 2.$$

We now compute the Zagreb indices.

(1) *First Zagreb Index:*

$$\begin{aligned} M_1(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})) &= \sum_{v \in X_1} \deg(v)^2 + \sum_{v \in X_2} \deg(v)^2 + \sum_{v \in X_3} \deg(v)^2 + \sum_{v \in X_4} \deg(v)^2 + \sum_{v \in X_5} \deg(v)^2 \\ &= \sum_{v \in X_1} d_1^2 + \sum_{v \in X_2} d_2^2 + \sum_{v \in X_3} d_3^2 + \sum_{v \in X_4} d_4^2 + \sum_{v \in X_5} d_5^2 \\ &= n_1 d_1^2 + n_2 d_2^2 + n_3 d_3^2 + n_4 d_4^2 + n_5 d_5^2 \end{aligned}$$

$$\begin{aligned}
 &= q^{\beta-1}(q-1)(p^{\alpha-1}-1)[p^{\alpha-1}(p+q^{\beta-1}-1)-1]^2 \\
 &\quad + p^{\alpha-1}(p-1)(q^{\beta-1}-1)[q^{\beta-1}(p^{\alpha-1}+q-1)-1]^2 \\
 &\quad + q^{\beta-1}(q-1)(p^{\alpha}q^{\beta-1}-1)^2 + p^{\alpha-1}(p-1)(p^{\alpha-1}q^{\beta}-1)^2 \\
 &\quad + (p^{\alpha-1}q^{\beta-1}-1)[p^{\alpha-1}q^{\beta-1}(p+q-1)-2]^2.
 \end{aligned}$$

(2) *Second Zagreb Index:* The edge classes of $\Gamma'(\mathbb{Z}_{p^{\alpha}q^{\beta}})$ correspond to adjacent pairs among $\{X_1, \dots, X_5\}$, namely: X_1-X_4 , X_2-X_3 , X_3-X_4 , X_1-X_5 , X_2-X_5 , X_3-X_5 , X_4-X_5 , and X_5-X_5 . Therefore:

$$\begin{aligned}
 M_2(\Gamma'(\mathbb{Z}_{p^{\alpha}q^{\beta}})) &= \sum_{\substack{u \in X_1 \\ v \in X_4}} d_1d_4 + \sum_{\substack{u \in X_2 \\ v \in X_3}} d_2d_3 + \sum_{\substack{u \in X_3 \\ v \in X_4}} d_3d_4 + \sum_{\substack{u \in X_1 \\ v \in X_5}} d_1d_5 + \sum_{\substack{u \in X_2 \\ v \in X_5}} d_2d_5 + \sum_{\substack{u \in X_3 \\ v \in X_5}} d_3d_5 \\
 &\quad + \sum_{\substack{u \in X_4 \\ v \in X_5}} d_4d_5 + \frac{1}{2} \sum_{\substack{u, v \in X_5 \\ u \neq v}} d_5^2 \\
 &= n_1n_4d_1d_4 + n_2n_3d_2d_3 + n_3n_4d_3d_4 + n_1n_5d_1d_5 + n_2n_5d_2d_5 \\
 &\quad + n_3n_5d_3d_5 + n_4n_5d_4d_5 + \frac{n_5(n_5-1)}{2}d_5^2 \\
 &= p^{\alpha-1}q^{\beta-1}(p-1)(q-1)(p^{\alpha-1}-1)[p^{\alpha-1}(p+q^{\beta-1}-1)-1](p^{\alpha-1}q^{\beta}-1) \\
 &\quad + p^{\alpha-1}q^{\beta-1}(p-1)(q-1)(q^{\beta-1}-1)[q^{\beta-1}(p^{\alpha-1}+q-1)-1](p^{\alpha}q^{\beta-1}-1) \\
 &\quad + p^{\alpha-1}q^{\beta-1}(p-1)(q-1)(p^{\alpha}q^{\beta-1}-1)(p^{\alpha-1}q^{\beta}-1) \\
 &\quad + q^{\beta-1}(q-1)(p^{\alpha-1}-1)(p^{\alpha-1}q^{\beta-1}-1)[p^{\alpha-1}(p+q^{\beta-1}-1)-1] \\
 &\quad \quad [p^{\alpha-1}q^{\beta-1}(p+q-1)-2] \\
 &\quad + p^{\alpha-1}(p-1)(q^{\beta-1}-1)(p^{\alpha-1}q^{\beta-1}-1)[q^{\beta-1}(p^{\alpha-1}+q-1)-1] \\
 &\quad \quad [p^{\alpha-1}q^{\beta-1}(p+q-1)-2] \\
 &\quad + q^{\beta-1}(q-1)(p^{\alpha-1}q^{\beta-1}-1)(p^{\alpha}q^{\beta-1}-1)[p^{\alpha-1}q^{\beta-1}(p+q-1)-2] \\
 &\quad + p^{\alpha-1}(p-1)(p^{\alpha-1}q^{\beta-1}-1)(p^{\alpha-1}q^{\beta}-1)[p^{\alpha-1}q^{\beta-1}(p+q-1)-2] \\
 &\quad + \frac{1}{2}(p^{\alpha-1}q^{\beta-1}-1)(p^{\alpha-1}q^{\beta-1}-2)[p^{\alpha-1}q^{\beta-1}(p+q-1)-2]^2.
 \end{aligned}$$

Factoring $p^{\alpha-1}q^{\beta-1}(p-1)(q-1)$ from the first three terms and $(p^{\alpha-1}q^{\beta-1}-1)[p^{\alpha-1}q^{\beta-1}(p+q-1)-2]$ from the next four.

$$\begin{aligned}
 M_2(\Gamma'(\mathbb{Z}_{p^{\alpha}q^{\beta}})) &= p^{\alpha-1}q^{\beta-1}(p-1)(q-1)\left[(p^{\alpha-1}-1)(p^{\alpha-1}(p+q^{\beta-1}-1)-1)(p^{\alpha-1}q^{\beta}-1)\right. \\
 &\quad \left. + (q^{\beta-1}-1)(q^{\beta-1}(p^{\alpha-1}+q-1)-1)(p^{\alpha}q^{\beta-1}-1)\right. \\
 &\quad \left. + (p^{\alpha}q^{\beta-1}-1)(p^{\alpha-1}q^{\beta}-1)\right] + (p^{\alpha-1}q^{\beta-1}-1)[p^{\alpha-1}q^{\beta-1}(p+q-1)-2] \\
 &\quad \left[q^{\beta-1}(q-1)(p^{\alpha-1}-1)(p^{\alpha-1}(p+q^{\beta-1}-1)-1) + p^{\alpha-1}(p-1)(q^{\beta-1}-1)\right. \\
 &\quad \left.(q^{\beta-1}(p^{\alpha-1}+q-1)-1) + q^{\beta-1}(q-1)(p^{\alpha}q^{\beta-1}-1) + p^{\alpha-1}(p-1)\right. \\
 &\quad \left.(p^{\alpha-1}q^{\beta}-1)\right] + \frac{1}{2}(p^{\alpha-1}q^{\beta-1}-1)(p^{\alpha-1}q^{\beta-1}-2)[p^{\alpha-1}q^{\beta-1}(p+q-1)-2]^2.
 \end{aligned}$$

(3) *First Multiplicative Zagreb Index:*

$$\begin{aligned}\Pi_1(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})) &= \prod_{v \in X_1} d_1^2 \cdot \prod_{v \in X_2} d_2^2 \cdot \prod_{v \in X_3} d_3^2 \cdot \prod_{v \in X_4} d_4^2 \cdot \prod_{v \in X_5} d_5^2 = d_1^{2n_1} \cdot d_2^{2n_2} \cdot d_3^{2n_3} \cdot d_4^{2n_4} \cdot d_5^{2n_5} \\ &= [p^{\alpha-1}(p+q^{\beta-1}-1)-1]^{2q^{\beta-1}(q-1)(p^{\alpha-1}-1)} \\ &\quad \cdot [q^{\beta-1}(p^{\alpha-1}+q-1)-1]^{2p^{\alpha-1}(p-1)(q^{\beta-1}-1)} \\ &\quad \cdot (p^\alpha q^{\beta-1}-1)^{2q^{\beta-1}(q-1)} \cdot (p^{\alpha-1}q^\beta-1)^{2p^{\alpha-1}(p-1)} \\ &\quad \cdot [p^{\alpha-1}q^{\beta-1}(p+q-1)-2]^{2(p^{\alpha-1}q^{\beta-1}-1)}.\end{aligned}$$

(4) *Second Multiplicative Zagreb Index:* Following the calculations used in the proof of [Theorem 3.2](#), the second multiplicative Zagreb index is obtained as given in (1). $\square$

3.2. Structural and extremal properties

In this subsection, we derive structural consequences of [Theorems 3.2](#) and [3.5](#) and [Remark 3.1](#): non-regularity of $\Gamma'(\mathbb{Z}_n)$ in all three cases, a sharp inequality between M_1 and Π_1 , monotonicity of all four indices in the exponents α, β , an extremal (minimizing) case, and the limiting behaviour of vertex degrees as $\alpha, \beta \rightarrow \infty$.

Theorem 3.6. *For distinct primes p, q , the graph $\Gamma'(\mathbb{Z}_n)$ is not regular for $n = pq$, $n = p^\alpha q$ ($\alpha > 1$), or $n = p^\alpha q^\beta$ ($\alpha, \beta > 1$).*

Proof. *Case $n = pq$.* By [Remark 3.1](#), $\Gamma'(\mathbb{Z}_{pq}) \cong K_{p-1, q-1}$, which is regular if and only if $p-1 = q-1$, i.e., $p = q$. Since $p \neq q$, $K_{p-1, q-1}$ is not regular.

Case $n = p^\alpha q$. By [Theorem 3.2](#), $d_1 = p^\alpha - 1$ and $d_2 = p^{\alpha-1}q - 1$. If $d_1 = d_2$, then $p^\alpha = p^{\alpha-1}q$, and dividing by $p^{\alpha-1}$ gives $p = q$, a contradiction. Hence $d_1 \neq d_2$ and $\Gamma'(\mathbb{Z}_{p^\alpha q})$ is not regular.

Case $n = p^\alpha q^\beta$. By [Theorem 3.5](#), $d_3 = p^\alpha q^{\beta-1} - 1$ and $d_4 = p^{\alpha-1}q^\beta - 1$. If $d_3 = d_4$, then $p^\alpha q^{\beta-1} = p^{\alpha-1}q^\beta$, and dividing by $p^{\alpha-1}q^{\beta-1}$ gives $p = q$, again a contradiction. Hence $d_3 \neq d_4$ and $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ is not regular. $\square$

Recall the *AM–GM inequality*: For any finite multiset of N non-negative real numbers $x_1, \dots, x_N$,

$$\frac{1}{N} \sum_{i=1}^N x_i \geq \left(\prod_{i=1}^N x_i \right)^{1/N},$$

with equality if and only if $x_1 = x_2 = \dots = x_N$.

Corollary 3.7. *Let N denote the number of vertices of $\Gamma'(\mathbb{Z}_n)$ in any of the three cases above. Then*

$$M_1(\Gamma'(\mathbb{Z}_n)) > N \cdot [\Pi_1(\Gamma'(\mathbb{Z}_n))]^{1/N},$$

with strict inequality in every case.

Proof. Apply the AM–GM inequality above to the multiset of N values $\{\deg(v)^2 : v \in V(G)\}$:

$$\frac{1}{N} \sum_{v \in V(G)} \deg(v)^2 \geq \left(\prod_{v \in V(G)} \deg(v)^2 \right)^{1/N},$$

i.e., $M_1(\Gamma'(\mathbb{Z}_n))/N \geq [\Pi_1(\Gamma'(\mathbb{Z}_n))]^{1/N}$, with equality if and only if all $\deg(v)$ are equal, i.e., G is regular. By Theorem 3.6, none of $\Gamma'(\mathbb{Z}_{pq})$, $\Gamma'(\mathbb{Z}_{p^\alpha q})$, $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ is regular, so the inequality is strict in every case. $\square$

Theorem 3.8. Fix distinct primes p, q .

1. In the family $\Gamma'(\mathbb{Z}_{p^\alpha q})$, $\alpha > 1$, each of M_1, M_2, Π_1, Π_2 is a strictly increasing function of α .
2. In the family $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$, $\alpha, \beta > 1$, each of M_1, M_2, Π_1, Π_2 is strictly increasing in α for fixed β , and strictly increasing in β for fixed α .

Proof. (1) From Theorem 3.2, $n_1 = p^{\alpha-1}(q-1)$, $n_2 = p^{\alpha-1}(p-1)$, $n_3 = p^{\alpha-1} - 1$, and $d_1 = p^\alpha - 1$, $d_2 = p^{\alpha-1}q - 1$, $d_3 = p^{\alpha-1}(p+q-1) - 2$. Since $p \geq 2$, each of these six quantities is a strictly increasing function of the integer α . As $M_1 = n_1d_1^2 + n_2d_2^2 + n_3d_3^2$ is a sum of products of strictly increasing positive terms, M_1 is strictly increasing in α ; the same argument applies term-by-term to $M_2 = \sum n_i n_j d_i d_j$ (Theorem 3.2(2)). For $\Pi_1 = d_1^{2n_1} d_2^{2n_2} d_3^{2n_3}$ and Π_2 , note that if $f(\alpha)$ is a strictly increasing sequence of integers with $f(\alpha) > 1$ and $g(\alpha)$ is a non-decreasing sequence of positive integers, then $f(\alpha)^{g(\alpha)}$ is strictly increasing, since $f(\alpha+1)^{g(\alpha+1)} \geq f(\alpha+1)^{g(\alpha)} > f(\alpha)^{g(\alpha)}$. Applying this to each factor of Π_1, Π_2 (Theorem 3.2(3),(4)) gives strict monotonicity.

(2) From Theorem 3.5, $n_3 = q^{\beta-1}(q-1)$ and $n_4 = p^{\alpha-1}(p-1)$ depend on only one of α, β , while n_1, n_2, n_5 and $d_1, \dots, d_5$ each strictly increase in both α and β individually (each contains a factor $p^{\alpha-1}$ or p^α , and a factor $q^{\beta-1}$ or q^β , with $p, q \geq 2$). The same term-by-term argument as in (1), applied to the expressions for M_1, M_2, Π_1, Π_2 in Theorem 3.5, gives strict monotonicity in α (fixed β) and in β (fixed α). $\square$

Corollary 3.9. Among the family $\Gamma'(\mathbb{Z}_{p^\alpha q})$, $\alpha \geq 1$, each of M_1, M_2, Π_1, Π_2 attains its minimum uniquely at $\alpha = 1$, where the formulas of Theorem 3.2 reduce exactly to those of Remark 3.1 for $\Gamma'(\mathbb{Z}_{pq})$.

Proof. By Theorem 3.8(1), each index is strictly increasing on $\alpha > 1$, so it suffices to check that the formulas extend continuously to $\alpha = 1$ and agree with Remark 3.1 there. Substituting $\alpha = 1$ into Theorem 3.2(1),

$$M_1 = p^0(q-1)(p-1)^2 + p^0(p-1)(q-1)^2 + (p^0 - 1)[\dots]^2 = (q-1)(p-1)^2 + (p-1)(q-1)^2$$

since the coefficient $p^{\alpha-1} - 1 = 0$ annihilates the third term. This equals $(p-1)(q-1)[(p-1) + (q-1)] = (p-1)(q-1)(p+q-2)$, which is exactly $M_1(\Gamma'(\mathbb{Z}_{pq}))$ from Remark 3.1. The same substitution applied to Theorem 3.2(2)–(4) recovers the remaining three formulas of Remark 3.1 verbatim, using $p^{\alpha-1} - 1 = 0$ to eliminate every term carrying that factor. $\square$

Remark 3.10. Corollary 3.9 identifies Remark 3.1 as the extremal ($\alpha = 1$) boundary case of Theorem 3.2, rather than an independent result: the complete bipartite graph $K_{p-1, q-1}$ is the unique minimizer of all four Zagreb-type indices as n ranges over $\{p^\alpha q : \alpha \geq 1\}$, and the indices grow without bound as $\alpha \rightarrow \infty$.

Proposition 3.11. Fix distinct primes p, q .

1. In $\Gamma'(\mathbb{Z}_{p^\alpha q})$, as $\alpha \rightarrow \infty$,

$$\frac{d_2}{d_1} \rightarrow \frac{q}{p}, \quad \frac{d_3}{d_1} \rightarrow \frac{p+q-1}{p}.$$

2. In $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$, with β fixed, as $\alpha \rightarrow \infty$,

$$\frac{d_3}{d_5} \rightarrow \frac{p}{p+q-1}, \quad \frac{d_4}{d_5} \rightarrow \frac{q}{p+q-1}.$$

Proof. (1) From Theorem 3.2, $d_1 = p^\alpha - 1$, $d_2 = p^{\alpha-1}q - 1$, $d_3 = p^{\alpha-1}(p + q - 1) - 2$. Dividing numerator and denominator by $p^{\alpha-1}$ and letting $\alpha \rightarrow \infty$ gives $d_2/d_1 \rightarrow q/p$ and $d_3/d_1 \rightarrow (p + q - 1)/p$.

(2) From Theorem 3.5 with β fixed, $d_3 = p^\alpha q^{\beta-1} - 1$, $d_4 = p^{\alpha-1}q^\beta - 1$, $d_5 = p^{\alpha-1}q^{\beta-1}(p + q - 1) - 2$. Dividing by $p^{\alpha-1}q^{\beta-1}$ and letting $\alpha \rightarrow \infty$ gives $d_3/d_5 \rightarrow p/(p + q - 1)$ and $d_4/d_5 \rightarrow q/(p + q - 1)$. $\square$

Remark 3.12. Proposition 3.11 shows that although the graph $\Gamma'(\mathbb{Z}_n)$ grows without bound as $\alpha \rightarrow \infty$, the relative degree structure between vertex classes stabilizes to fixed rational ratios determined only by p and q – the local structure of the graph does not degenerate even as its order tends to infinity.

4. Discussion and analysis

The generalized zero-divisor graph provides a more dense adjacency structure than the classical zero-divisor graph because it accounts for nilpotency, and the closed-form expressions of Theorems 3.2 and 3.5 allow this density to be quantified precisely rather than only described qualitatively. We discuss this in five parts below.

4.1. Comparison with the classical zero-divisor graph

In the classical zero-divisor graph $\Gamma(R)$, two elements x, y are adjacent only if $xy = 0$. In the generalized zero-divisor graph $\Gamma'(R)$ studied here (Definition 2.1), adjacency instead requires only $x^n y = 0$ or $y^n x = 0$ for some positive integer n , a strictly weaker condition. This is precisely why the nilpotent elements of $\mathbb{Z}_n$ form a distinguished vertex class – X_3 in Theorem 3.2, X_5 in Theorem 3.5 – that is adjacent to every other vertex, a clique with no counterpart in the classical graph, since two elements can be adjacent in $\Gamma'(R)$ while satisfying $xy \neq 0$. Consequently, every degree formula in Theorems 3.2 and 3.5 is strictly larger than the corresponding degree would be in $\Gamma(\mathbb{Z}_n)$, and this gap is what drives all four Zagreb-type indices higher than the classical graph would give.

4.2. Influence of nilpotent elements on the indices

Because the nilpotent class (X_3 for $n = p^\alpha q$, X_5 for $n = p^\alpha q^\beta$) is adjacent to every other vertex, its growth raises the degree of every vertex in the graph, not only the vertices within the nilpotent class itself. Theorem 3.8 makes this precise: M_1, M_2, Π_1, Π_2 are strictly increasing functions of α and of β , since these exponents directly control $|X_3| = p^{\alpha-1} - 1$ and $|X_5| = p^{\alpha-1}q^{\beta-1} - 1$. The increase in every index is therefore strict, not merely a partial effect confined to the nilpotent vertices.

4.3. Comparison between additive and multiplicative indices

Corollary 3.7 gives the inequality $M_1 > N \cdot [\Pi_1]^{1/N}$, which already indicates that Π_1 cannot grow as fast as a fixed power of M_1 . More concretely, each vertex class of size n_i contributes to Π_1, Π_2 as an exponent of the corresponding degree ($d_i^{2n_i}$ in Theorem 3.2(3) and Theorem 3.5(3)), whereas the same class contributes to M_1, M_2 only linearly, weighted by n_i . This explains why the multiplicative indices grow so much faster: in Example 3.4, already for the small case $n = 12$, $M_2 = 252$ while $\Pi_2 = 242,137,805,625,000,000$ – a disparity of many orders of magnitude that becomes still more pronounced as α, β increase, since the class sizes n_i appearing as exponents in Π_1, Π_2 themselves grow with α, β .

4.4. Asymptotic behaviour as α, β increase

By Theorem 3.8, all four indices grow without bound as $\alpha, \beta \rightarrow \infty$, with Π_1, Π_2 growing substantially faster than M_1, M_2 for the reason given above. Despite this unbounded growth, Proposition 3.11 shows

that the *relative* degree structure between vertex classes stabilizes: as $\alpha \rightarrow \infty$, the ratios d_2/d_1 and d_3/d_1 (in the $p^\alpha q$ case) and d_3/d_5 , d_4/d_5 (in the $p^\alpha q^\beta$ case) all converge to fixed rational constants depending only on p and q . Thus the graph's absolute size grows without bound while its local shape does not degenerate.

4.5. Graph-theoretic and algebraic interpretation

Graph-theoretically, [Theorems 3.6](#) and [3.8](#) together track how increasingly clique-dominated and dense $\Gamma'(\mathbb{Z}_n)$ becomes as α, β grow, while remaining permanently irregular – the asymmetry between the classes built on p and on q is never resolved, regardless of how large α, β become. Algebraically, this same growth tracks how deep the prime-power factorization of n is: [Corollary 3.9](#) shows that the least-factored case $n = pq$ is the unique minimizer of every index within the family $\{p^\alpha q\}$, so increasing the multiplicity of either prime factor strictly increases every Zagreb-type invariant of the graph. Taken together with the inequality of [Corollary 3.7](#) and the ratio limits of [Proposition 3.11](#), these results show that the four Zagreb-type indices depend on the arithmetic structure of n in a strict, fully quantified way, rather than only through the qualitative density argument given at the start of this section.

5. Conclusion

In this research paper, we have derived closed formulas for the first and second Zagreb indices, M_1 and M_2 , along with the first and second multiplicative Zagreb indices, Π_1 and Π_2 , of the generalized zero-divisor graph $\Gamma'(\mathbb{Z}_n)$ when $n = pq$, $n = p^\alpha q$, and $n = p^\alpha q^\beta$, where p, q are distinct primes and α, β be positive integers. In each of the above cases, our derivation process involved classifying the vertices of $\Gamma'(\mathbb{Z}_n)$ according to the zero-divisors of $\mathbb{Z}_n$, determining the degree of each of these classes of vertices from [Definition 2.1](#), and using these in the formulas for the four indices. The resulting formulas are new in the literature for this graph family.

In addition to the formulas, the explicit formula for the degrees led us to obtain five results of independent structural significance. First, $\Gamma'(\mathbb{Z}_n)$ is not a regular graph in any of the three families ([Theorem 3.6](#)), which in turn implies the inequality between M_1 and Π_1 using the AM-GM inequality ([Corollary 3.7](#)). Second, all four indices are strictly increasing functions of the nilpotency parameters α and β ([Theorem 3.8](#)), due to the fact that the size of the nilpotent vertex set depends on the exponents α, β and is adjacent to all other vertices. Third, the special case when $\alpha = 1$ is an extremal, minimally factored and dense case, and it gives rise to the complete bipartite structure of $\Gamma'(\mathbb{Z}_{pq})$ ([Corollary 3.9](#)). This means that the case $n = pq$ represents the most minimally factored and least dense element in every family. Fourth, while the nilpotency exponents α and β tend to infinity, the ratio of the degrees of different vertices becomes stable, that is, it is determined only by p and q ([Proposition 3.11](#)), showing that the graph does not degenerate locally when its order goes to infinity. Fifth, comparison of M_1, M_2 and Π_1, Π_2 indices shows that the multiplicative indices grow substantially faster than the additive indices as the nilpotency parameters increase. Taken together, these results show that the additional adjacency permitted by the generalized zero-divisor graph, captured entirely by the nilpotent vertex class X_3 (for $n = p^\alpha q$) or X_5 (for $n = p^\alpha q^\beta$), is not merely a qualitative enrichment of the classical zero-divisor graph, but one that can be quantified precisely through the closed-form degree expressions derived here.

The prime-power factorization of n is thus shown to control the graph's topological indices in a strict and fully determined way, rather than only through the qualitative adjacency structure established in earlier work.

One possible direction for future research could involve the analysis of $n = p^\alpha q^\beta r^\gamma$ and similar multi-prime factorizations, since in this case a finer lattice of gcd classes would have to be considered within the vertex partition. Also, it could be worth calculating other topological indices based on the degree of the vertices, for example, the hyper-Zagreb index, the Randić index, and the sum-connectivity index, and studying their monotonicity properties and extremal values as in [Theorem 3.8](#) and [Corollary 3.9](#).

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Author contributions

All authors contributed to the manuscript and approved the final version.

Conflict of interest

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Data availability

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